Project JRP — Lattice QCD Calculations

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FOR5237 Workshop
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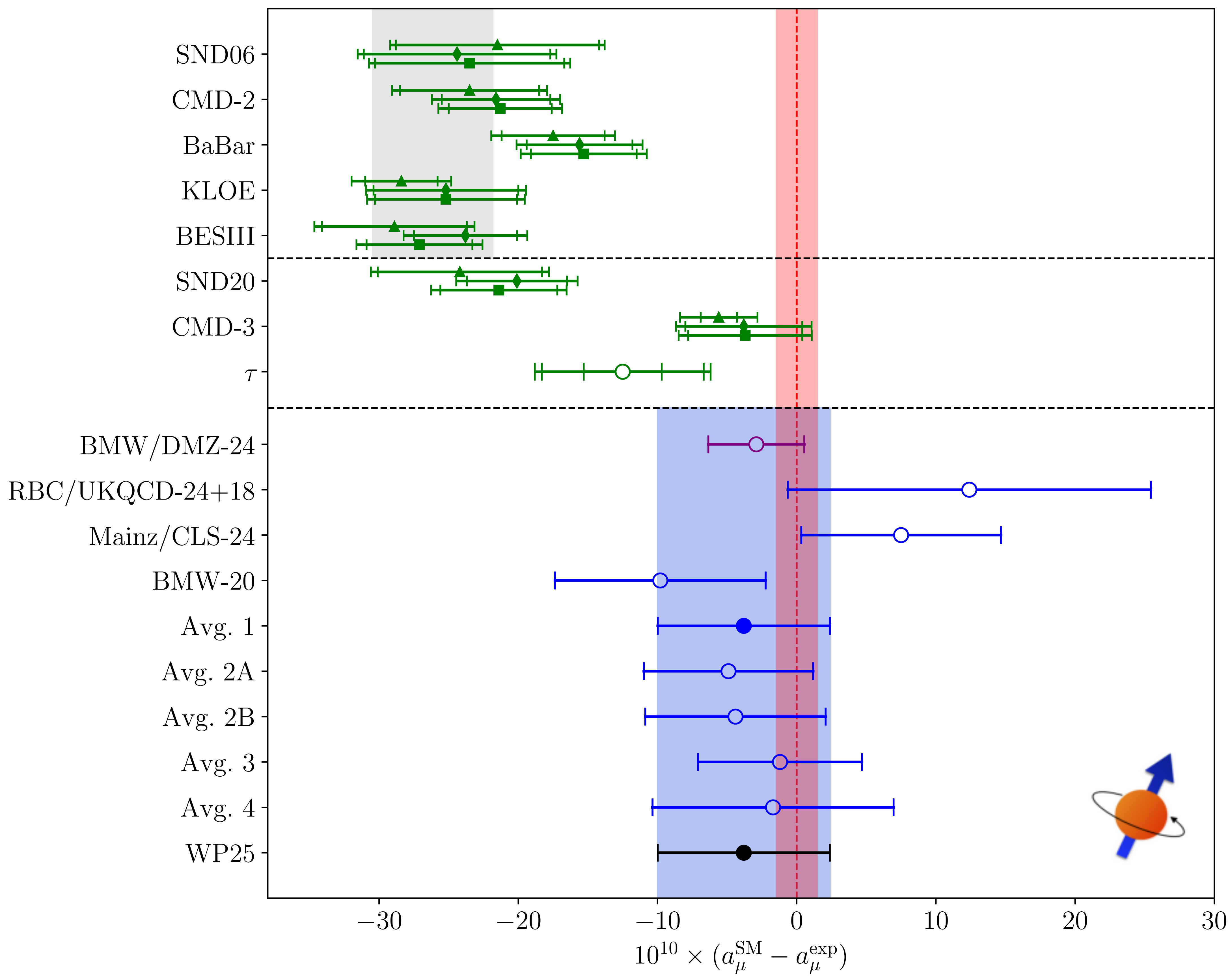
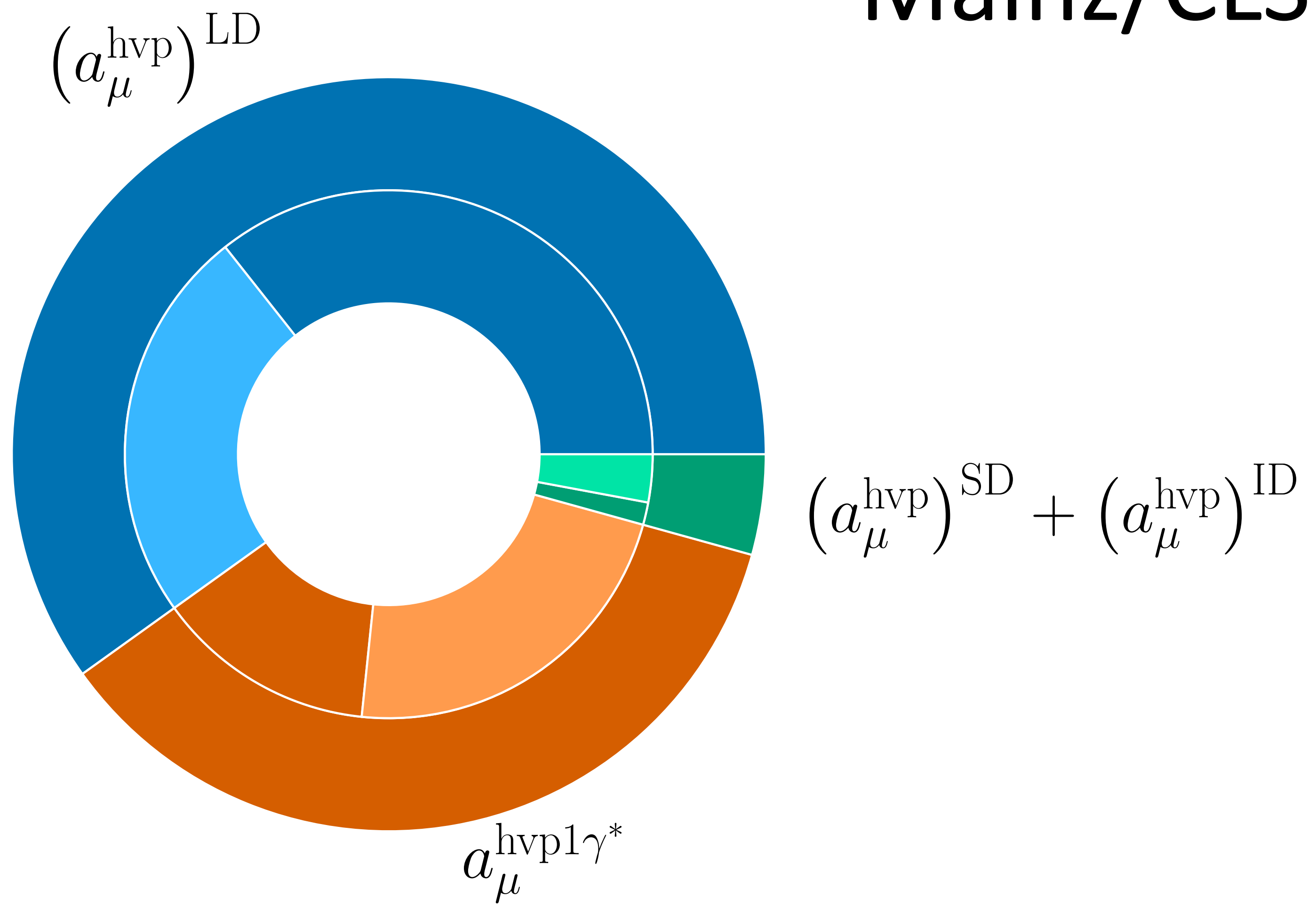
11–13 June 2024

Muon $g-2$ after WP25 and E989



Mainz/CLS-24 error budget



$$a_\mu^{\rm hvp}\Big|_{\rm MZ/CLS-24} = (7245 \pm 49 \pm 52) \cdot 10^{-11} \quad [0.99\%]$$

$$a_\mu^{\rm hvp}\Big|_{\rm WP25} = (7132 \pm 61) \cdot 10^{-11} \quad [0.86\%]$$

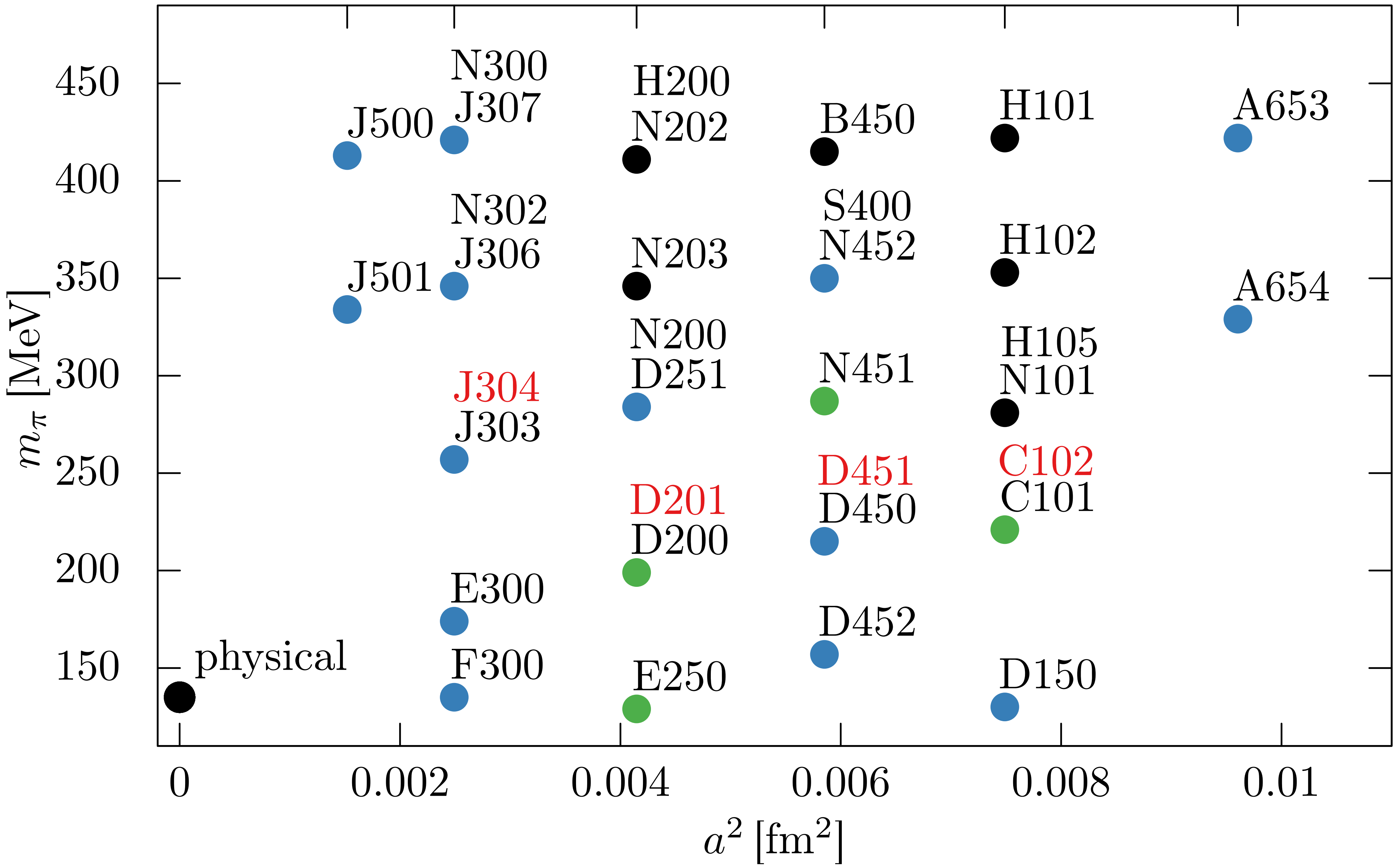
Statistical precision of isovector contribution and isospin-breaking effects dominate error

[Aliberti et al., arXiv:2505.21476; Djukanovic et al., JHEP 04 (2025) 098]

Lattice QCD at Mainz

Ensembles with $N_f = 2+1$ flavours of $\mathrm{O}(a)$ improved Wilson fermions generated by CLS effort

Six lattice spacings: $a = 0.099 - 0.035\,\mathrm{fm}$; Pion masses: $m_\pi = 130 - 420\,\mathrm{MeV}$



Computational cost: ≈ 400 M core-hrs

Ensemble "F300":

$$m_\pi \approx 135\,\mathrm{MeV}\,, \quad a = 0.050\,\mathrm{fm}\,, \quad 128^3 \cdot 256$$

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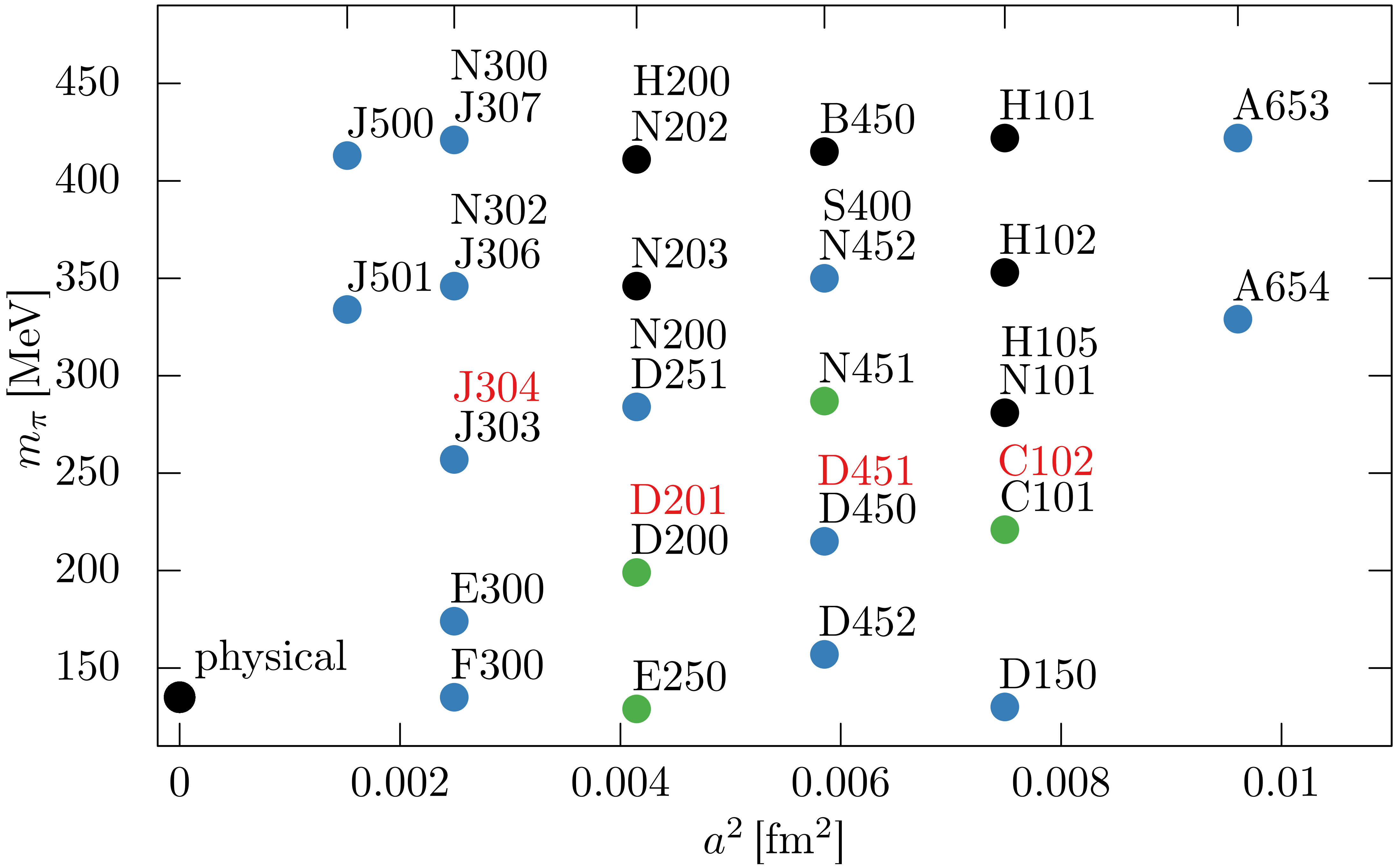

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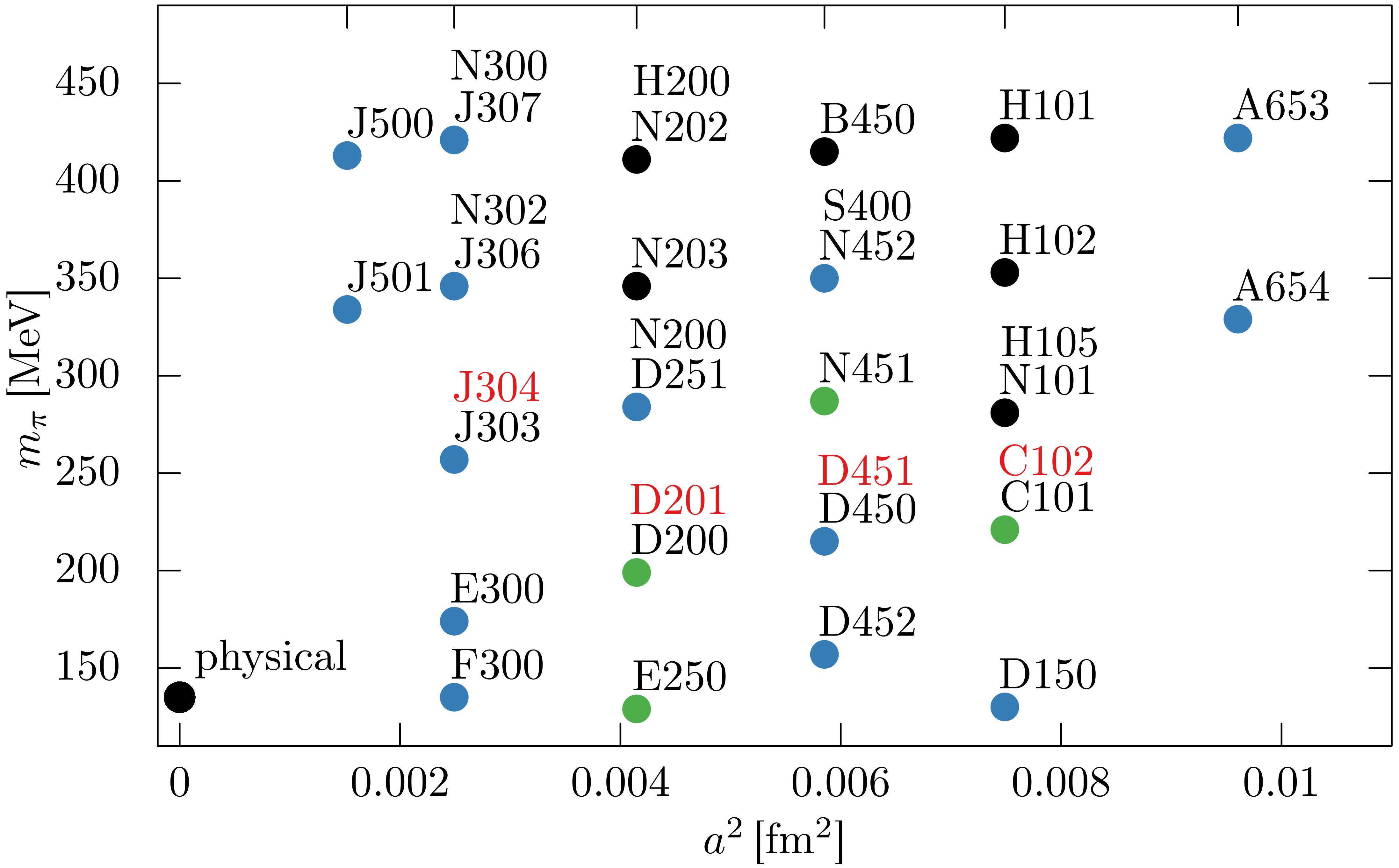

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Intrinsic numerical cost

Wilson fermions

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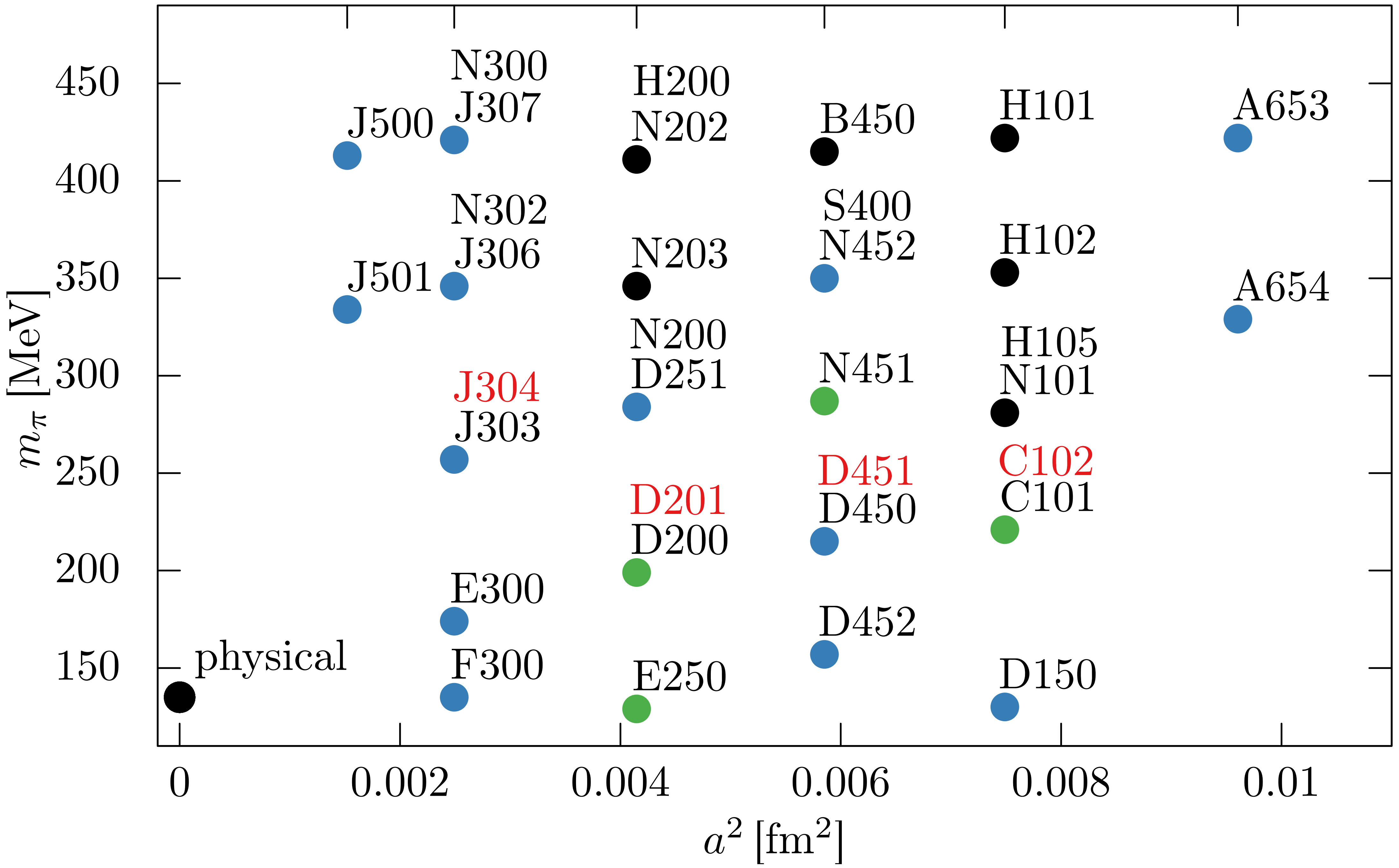

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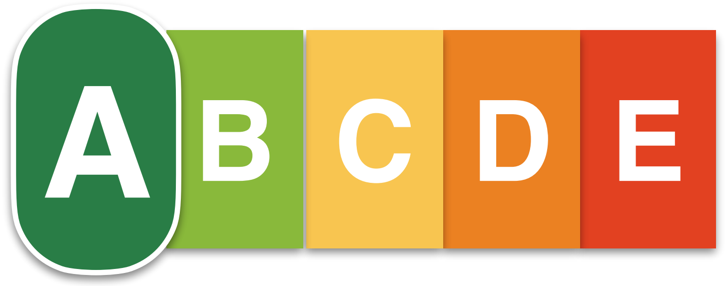
Staggered fermions



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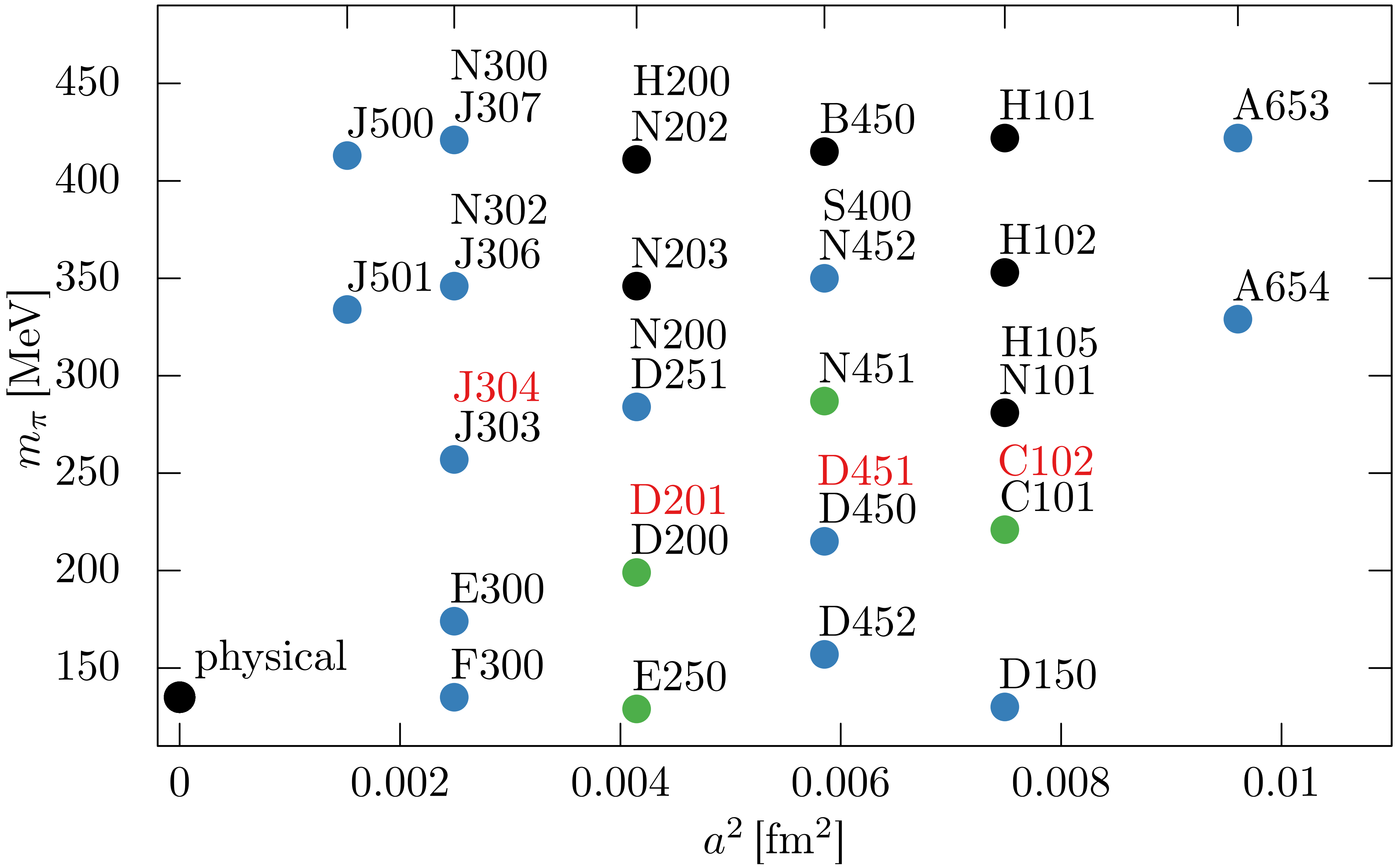

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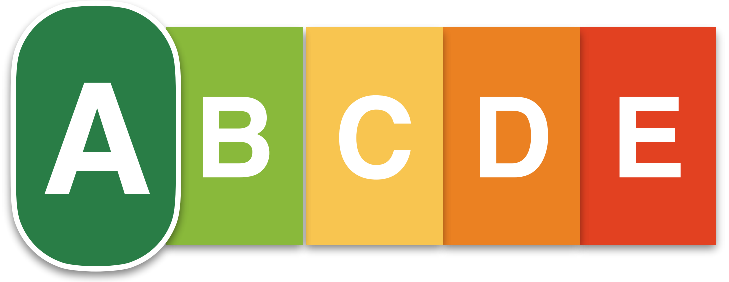
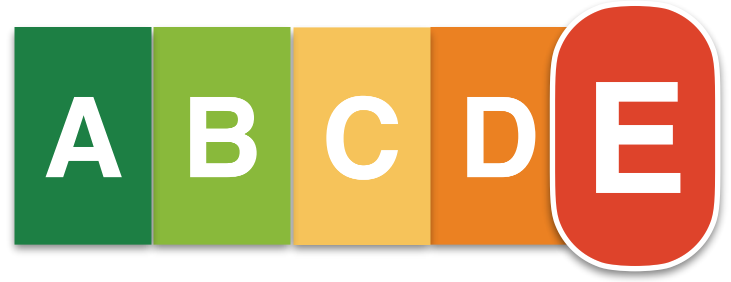


Domain-wall fermions



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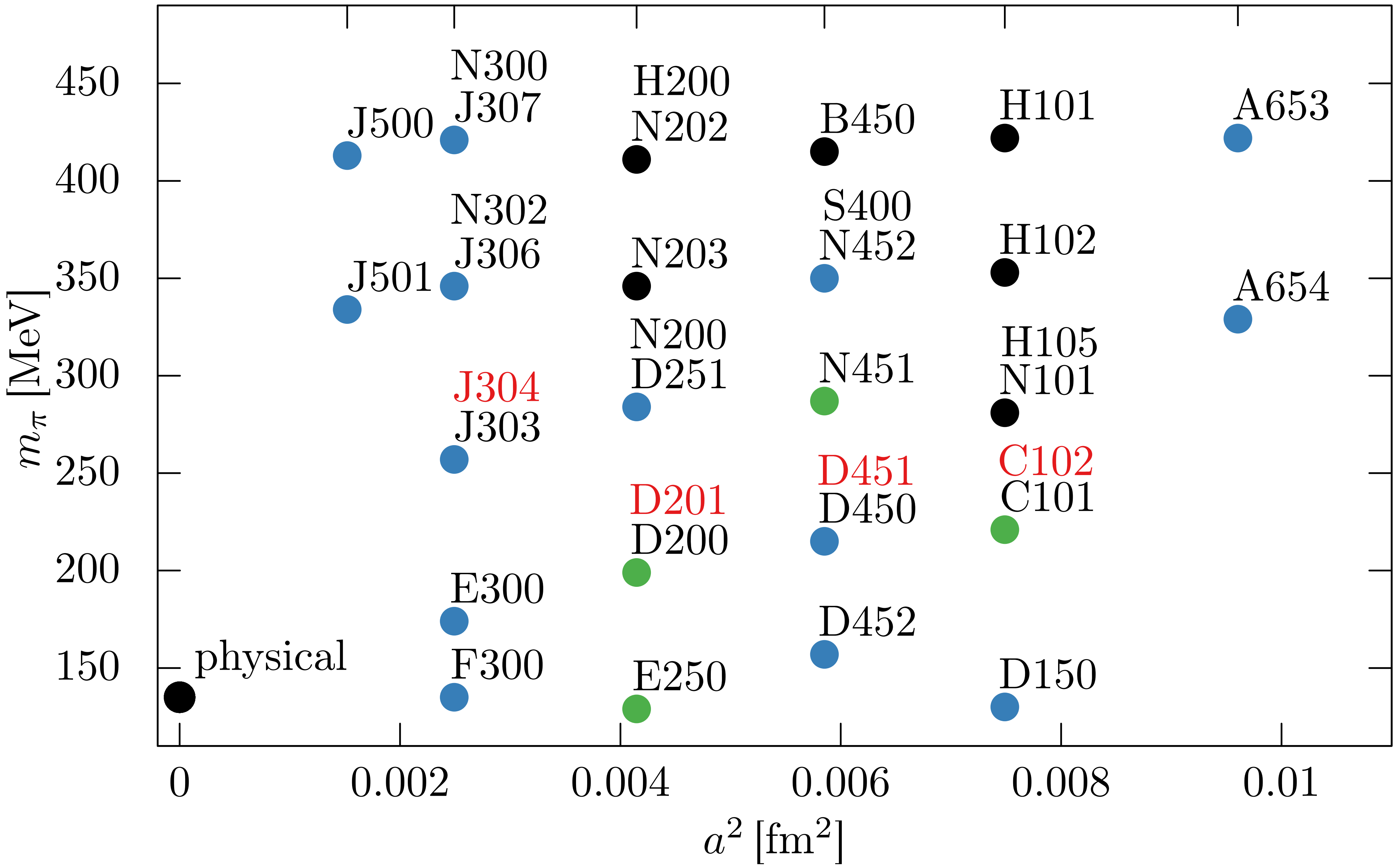

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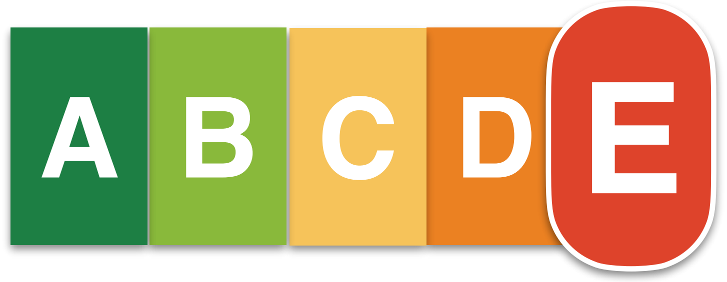
Conceptual issues / fermion doubling problem

Wilson fermions

Staggered fermions



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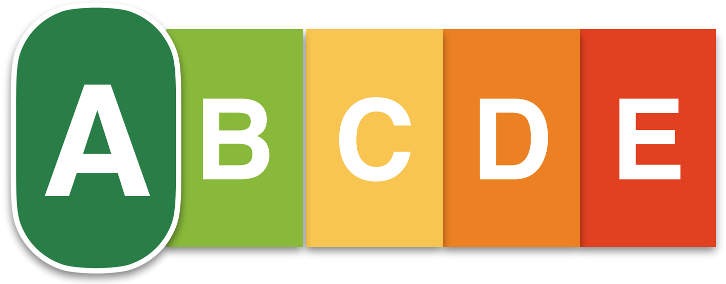

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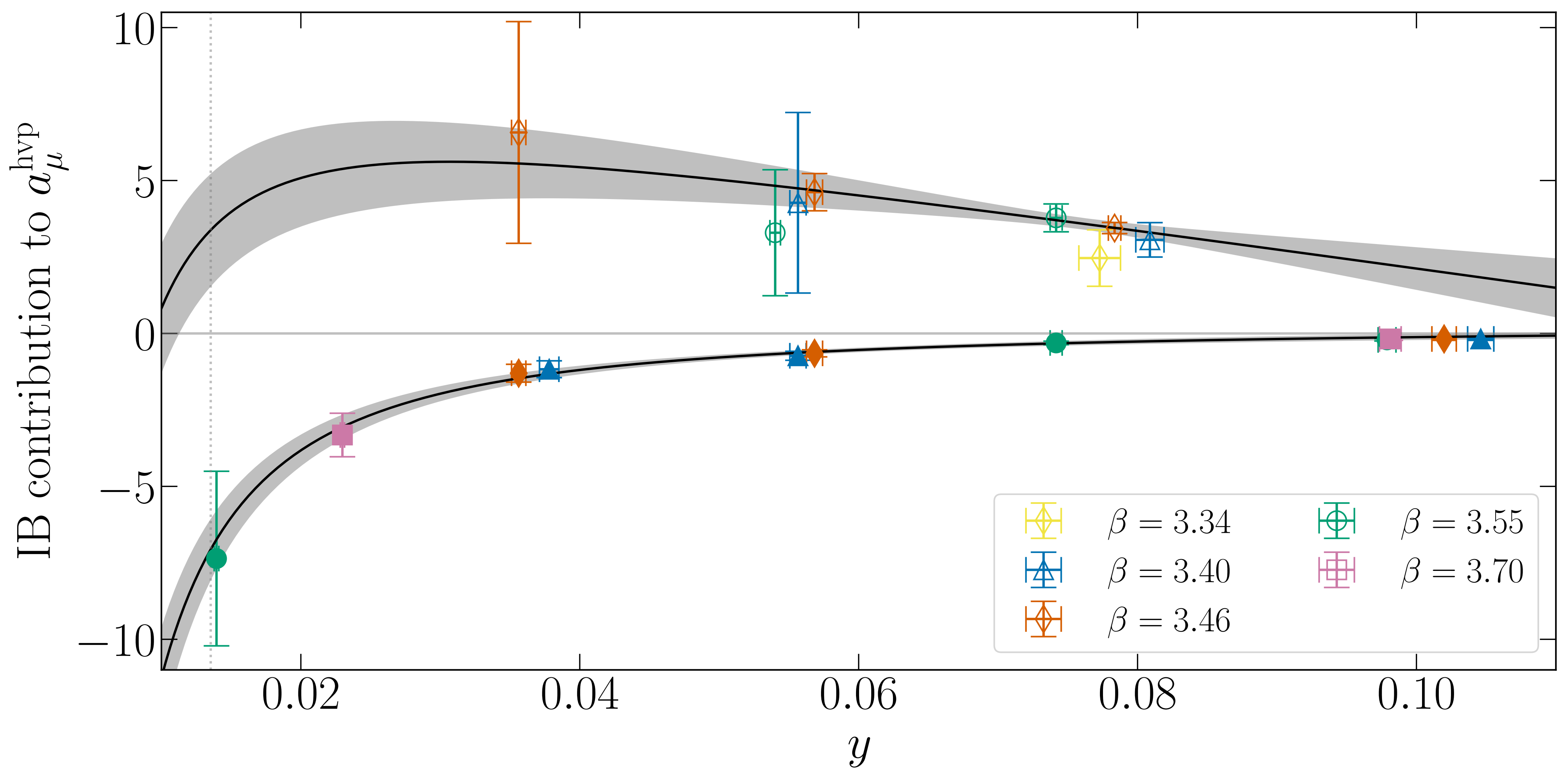
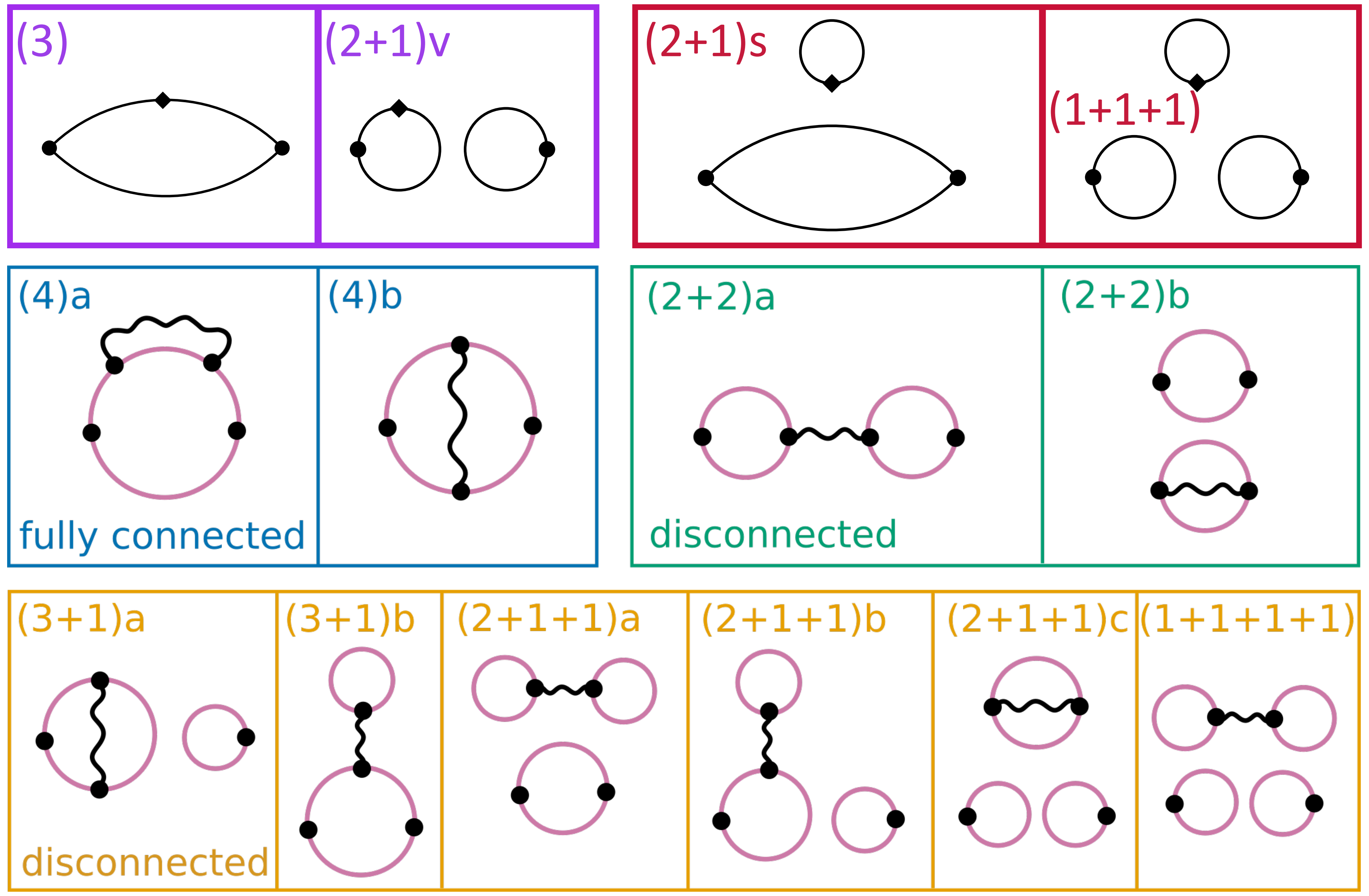
Isospin-breaking corrections in Mainz/CLS-24





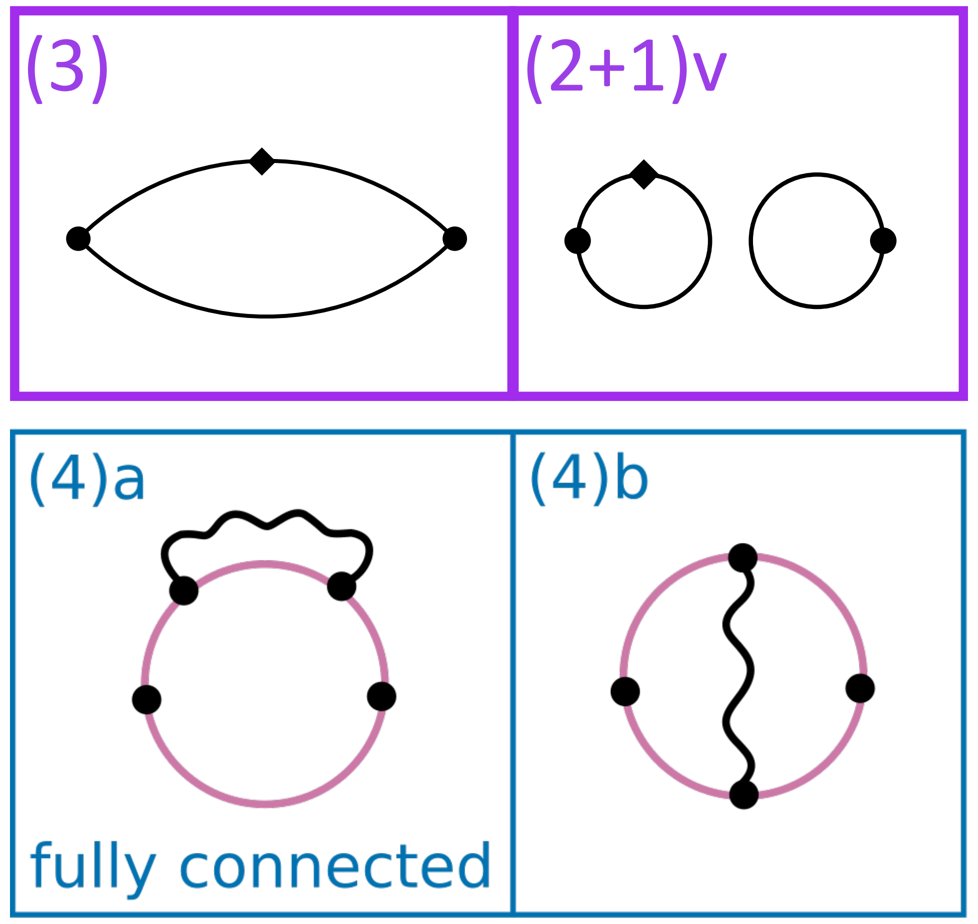
Upper curve: (3) + (4)a + (4)b + ChPT-estimate of (2+1)v; "Rome approach" and $\mathrm{QED_L}$

Lower curve: (2+2)a; photon propagator in continuum and infinite volume; remaining diagrams estimated via charged pion loop

$$a_\mu^{\rm hvp}\Big|_{\rm IB} \cdot 10^{11} = -(41 \pm 24 \pm 9 \pm 35)[\pm 43]$$

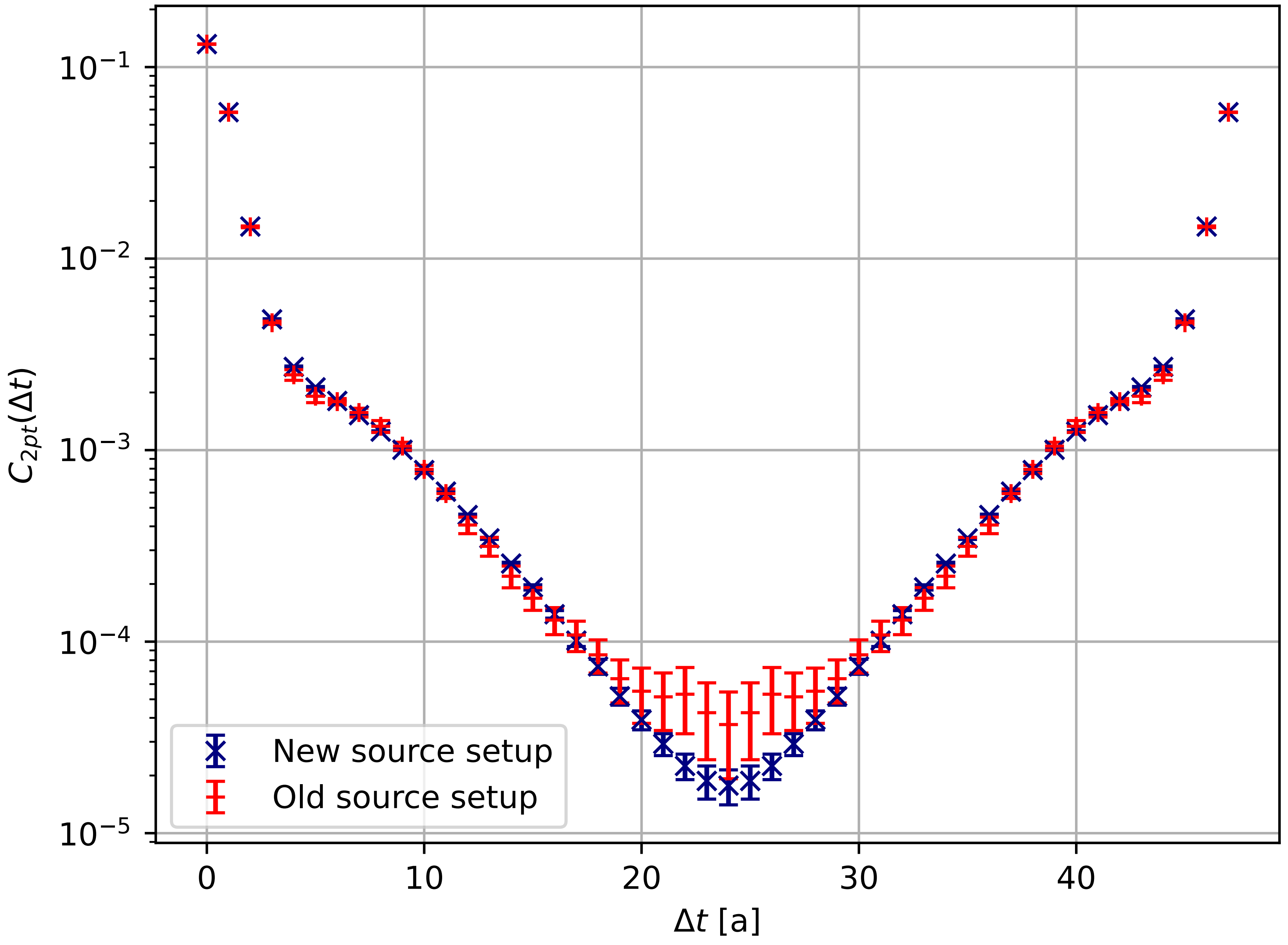
[Djukanovic et al., JHEP 04 (2025) 098; Parrino et al., arXiv:2501.03192]

Isospin-breaking effects



Combat high levels of statistical noise in evaluation of mass insertion and fully connected QED diagrams using "Rome approach" and QED_L

- Low-mode averaging — indispensable for long-distance part in isoQCD
- Four-dimensional stochastic sources at mass insertion



[Sebastian Lahrtz, PhD project]

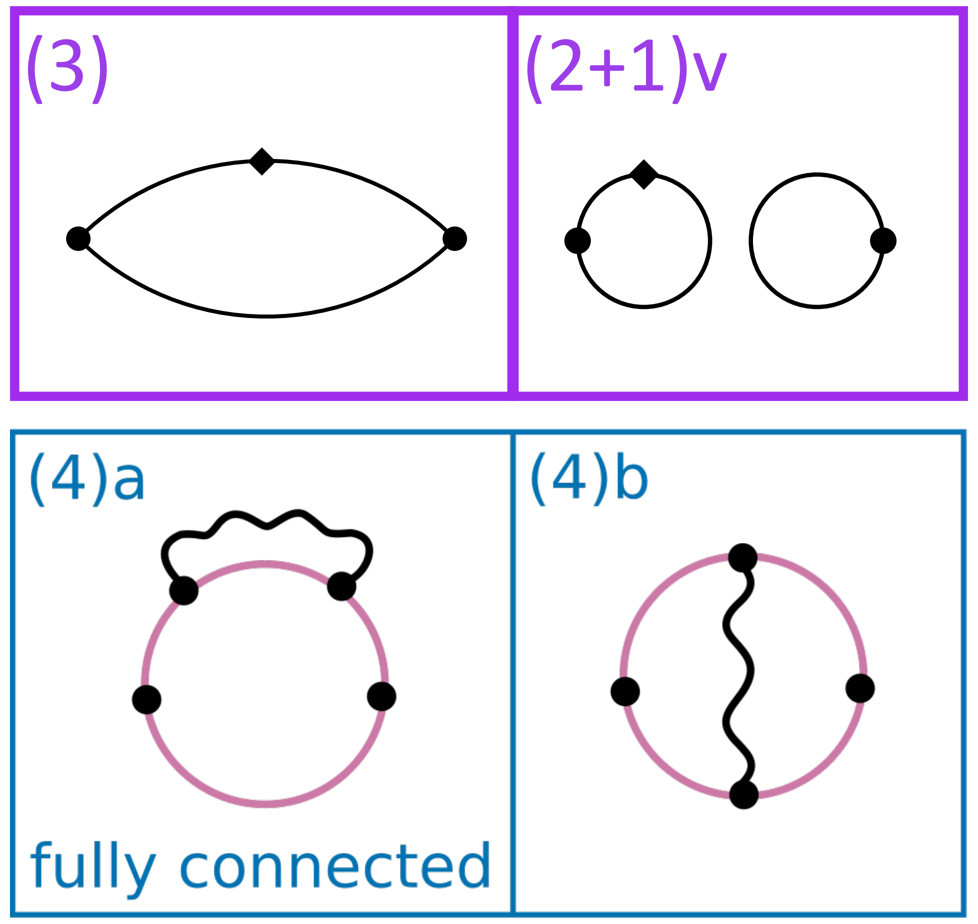
Ensemble A654, 5000 configs, $m_\pi \simeq 330\,\text{MeV}$ $a \simeq 0.1\,\text{fm}$

Old setup: sequential propagator across mass insertion;
"spin-diluted" time slice sources (3D)

New setup: sequential propagator across vector current;
stochastic volume sources (4D);
compute (2+1)v diagram directly

Comparison for same numerical cost (# of inversions)

Isospin-breaking effects



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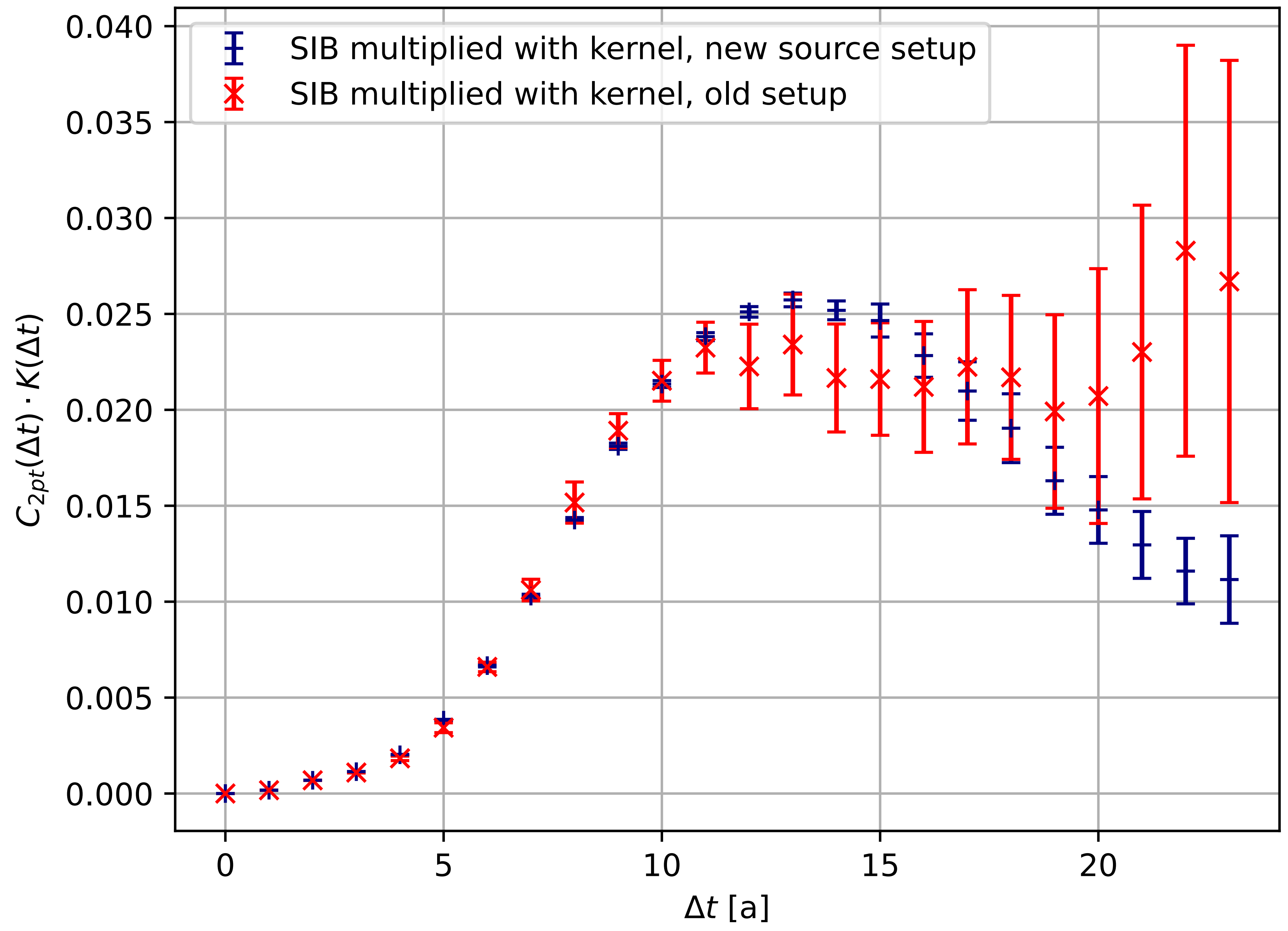

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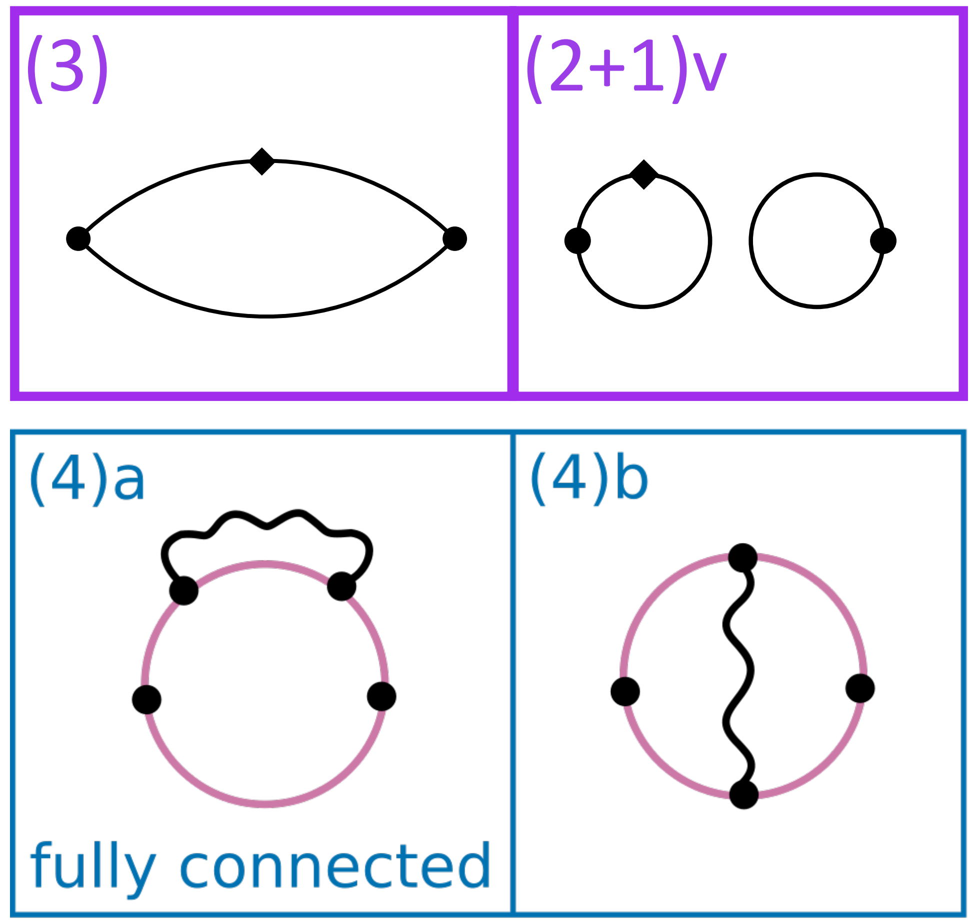

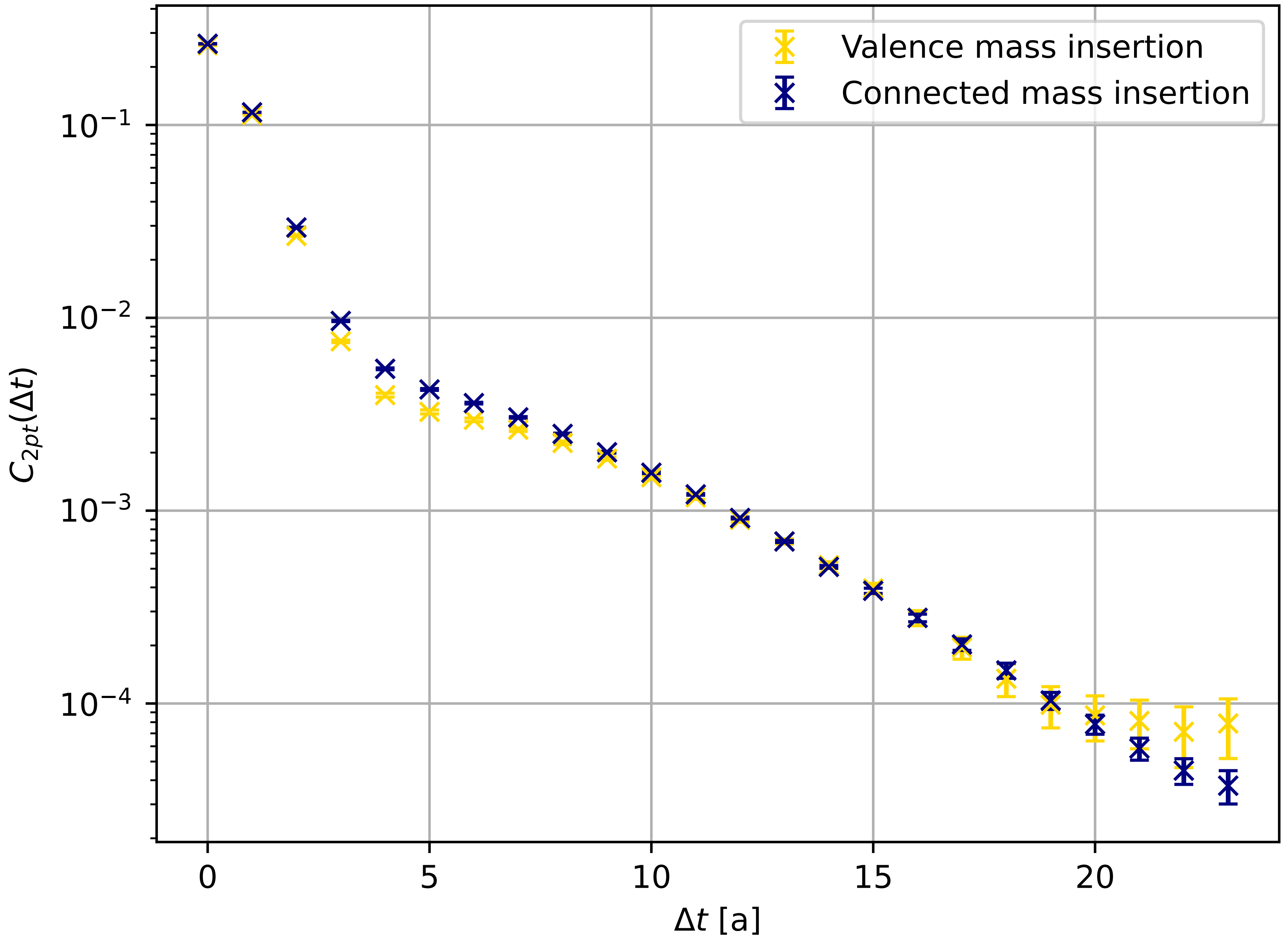

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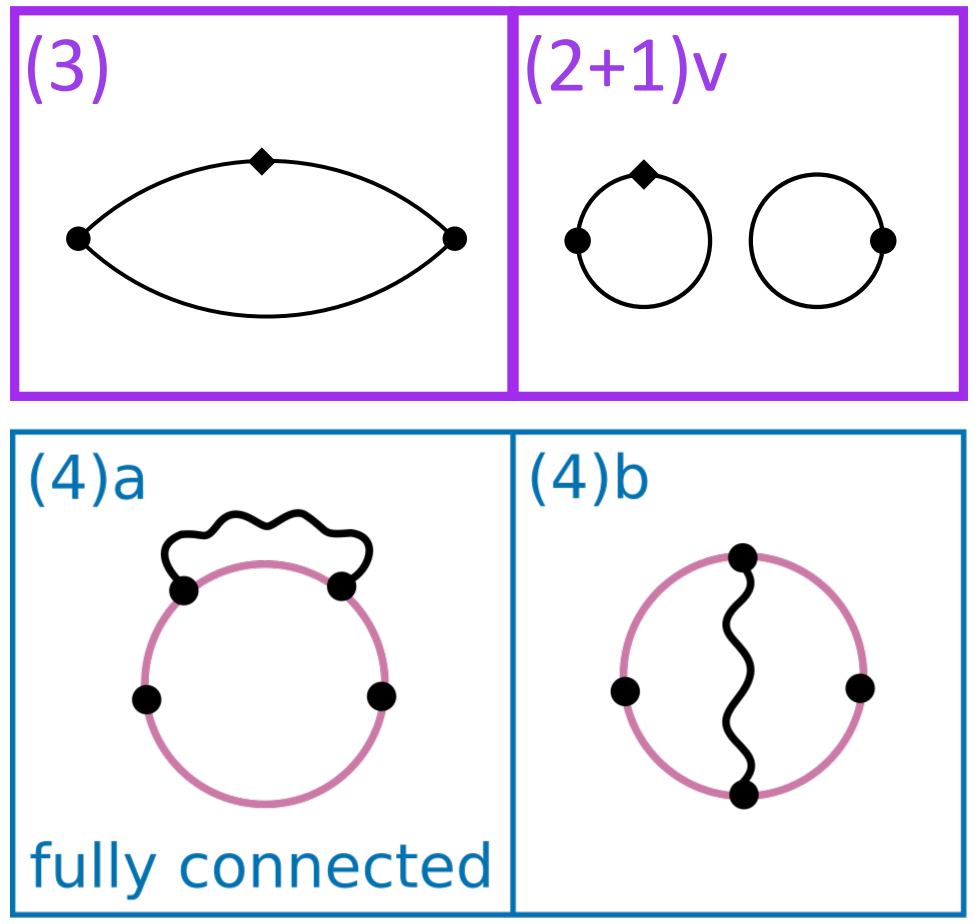

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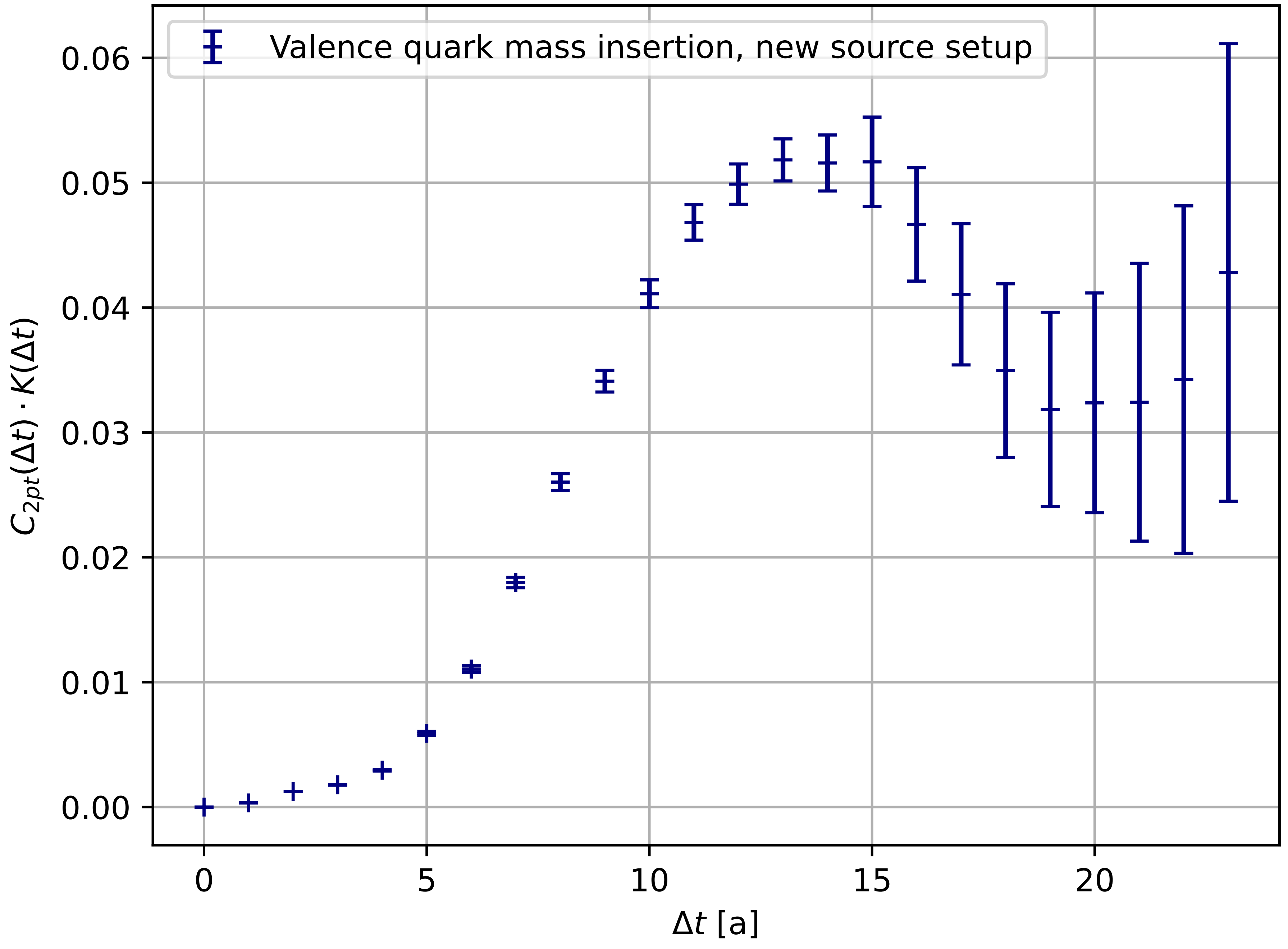

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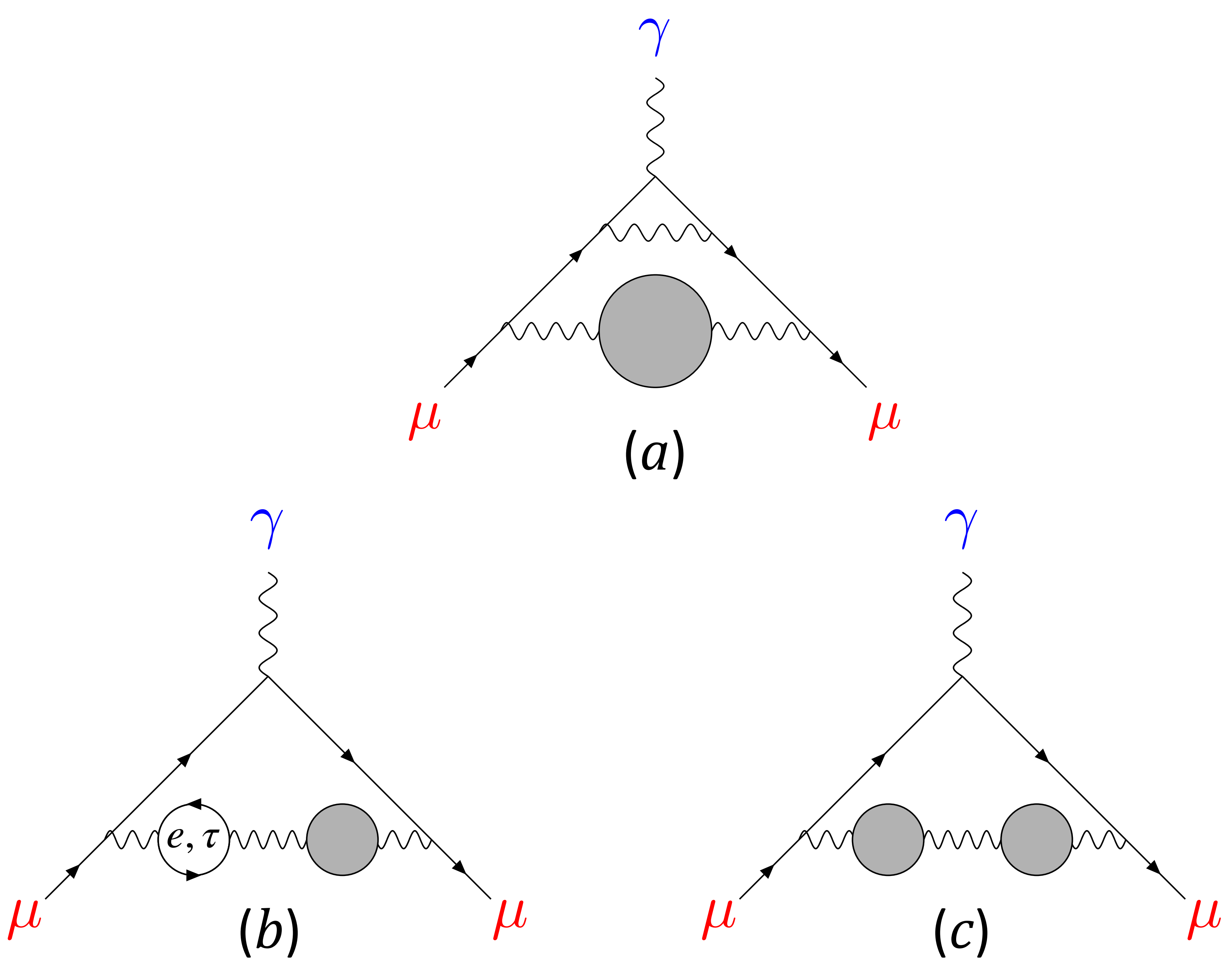
Higher-order hadronic vacuum polarisation

Contribution	Section	Equation	Value $\times 10^{11}$	References
Experiment (E989)		Eq. (9.5)	116 592 059(22)	Refs. [5–7, 9–12]
HVP LO (lattice)	Sec. 3.6.1	Eq. (3.37)	7132(61)	Refs. [13–29]
HVP LO (e^+e^-, τ)	Sec. 2	Table 5	Estimates not provided at this point	
HVP NLO (e^+e^-)	Sec. 2.9	Eq. (2.47)	−99.6(1.3)	Refs. [30, 31]
HVP NNLO (e^+e^-)	Sec. 2.9	Eq. (2.48)	12.4(1)	Ref. [32]
HLbL (phenomenology)	Sec. 5.10	Eq. (5.69)	103.3(8.8)	Refs. [33–56]
HLbL NLO (phenomenology)	Sec. 5.10	Eq. (5.70)	2.6(6)	Ref. [57]
HLbL (lattice)	Sec. 6.2.8	Eq. (6.34)	122.5(9.0)	Refs. [58–62]
HLbL (phenomenology + lattice)	Sec. 9	Eq. (9.2)	112.6(9.6)	Refs. [33–56, 58–62]
QED	Sec. 7.5	Eq. (7.27)	116 584 718.8(2)	Refs. [63–69]
EW	Sec. 8	Eq. (8.12)	154.4(4)	Refs. [50, 70–72]
HVP LO (lattice) + HVP N(N)LO (e^+e^-)	Sec. 9	Eq. (9.1)	7045(61)	Refs. [13–32]
HLbL (phenomenology + lattice + NLO)	Sec. 9	Eq. (9.3)	115.5(9.9)	Refs. [33–62]
Total SM Value	Sec. 9	Eq. (9.4)	116 592 033(62)	Refs. [13–72]
Difference: $\Delta a_\mu \equiv a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}}$	Sec. 9	Eq. (9.6)	26(66)	



$$-(a) \approx 2(b) \gg (c)$$

NLO HVP arises at $\mathrm{O}(\alpha^3)$; estimated using the data-driven method

Lattice calculation uses the same vector correlator $G(t)$ as input

Technical challenges similar to the LO HVP:

- statistical noise, disconnected diagrams, continuum limit, finite-volume effects, isospin breaking

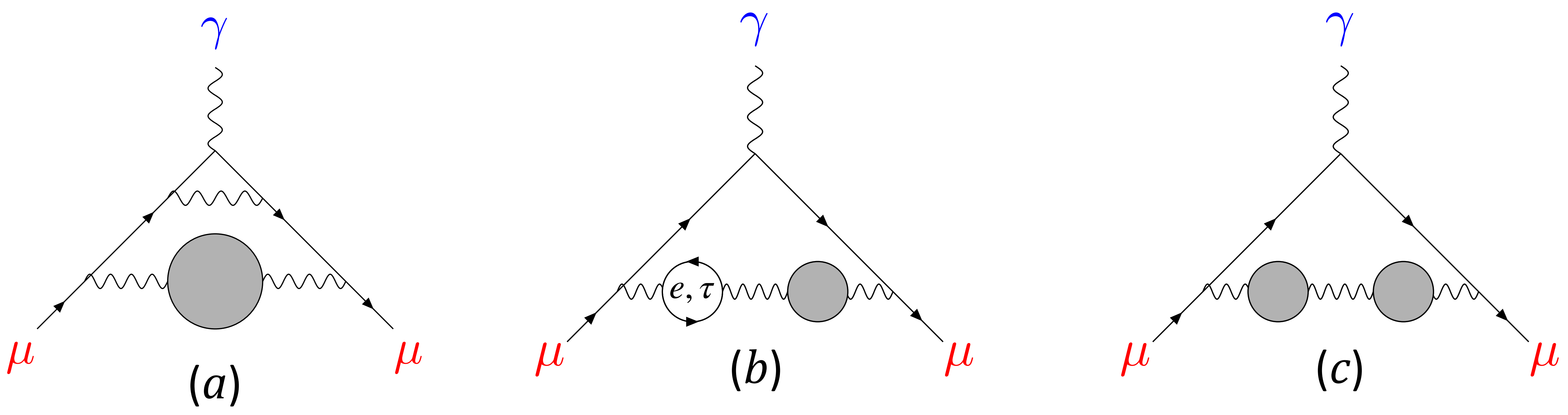
Lattice approach to NLO HVP

Adopt "Time-momentum representation" (TMR) *[Bernecker & Meyer, 2011]*

Express NLO contributions in terms of convolution integrals of the spatially summed vector correlator

$$a_\mu^{\mathrm{LO\,hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{f}(t)\, G(t), \quad G(t) = -\frac{a^3}{3} \sum_k \sum_{\vec{x}} \left\langle j_k^{\mathrm{em}}(\vec{x}, t)\, j_k^{\mathrm{em}}(0) \right\rangle$$

$$a_\mu^{\mathrm{NLO\,hvp}}[a,b] = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dt\, \tilde{f}[a,b](t)\, G(t), \quad a_\mu^{\mathrm{NLO\,hvp}}[c] = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dt \int_0^\infty d\tau\, \tilde{f}[c](t,\tau)\, G(t) G(\tau)$$



Work out kernel functions and find numerical representation with sufficient numerical precision

[Passera et al., PLB 834 (2022) 137462; Arnau Beltran Martínez, PhD project]

TMR kernel functions

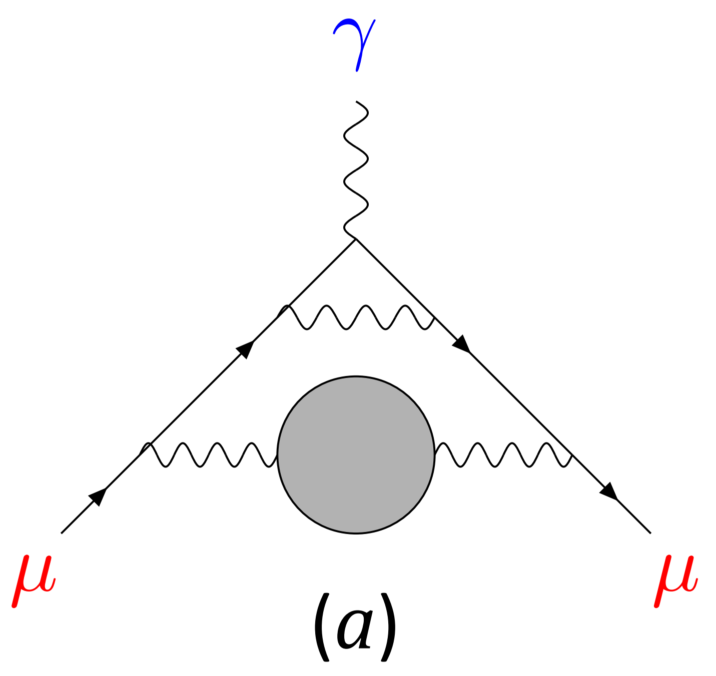


Diagram (a)

NO closed-form solution.

$\hat{t} \ll 1$:

$$\int_0^\infty \frac{d\hat{\omega}^2}{\hat{\omega}^2} \hat{f}^{(\mathrm{NLO_a})}(\hat{\omega}^2)\left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right] = \int_0^{\frac{1-\hat{t}}{\sqrt{\hat{t}}}} \frac{d\hat{\omega}^2}{\hat{\omega}^2} \hat{f}^{(\mathrm{NLO_a})}(\hat{\omega}^2)\underbrace{\left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right]}_{\hat{\omega}\hat{t}\sim 0} + \int_{\frac{1-\hat{t}}{\sqrt{\hat{t}}}}^\infty \frac{d\hat{\omega}^2}{\hat{\omega}^2} \underbrace{\hat{f}^{(\mathrm{NLO_a})}(\hat{\omega}^2)}_{\hat{\omega}\sim\infty}\left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right]$$

For $\hat{t} \gg 1$ one solves analytically as much as possible and rotates the cosine to the complex plane to make it real and thus make explicit the suppressed contribution. Then, one numerically expands around $\hat{t}_0$ by studying the asymptotic behavior of each piece.

Asymptotic expansions:

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}^{\mathrm{NLO_a}}(t) = \begin{cases} \sum_{n=4,n\in\mathrm{even}}^{N} \frac{a_n + b_n\pi^2 + c_n(\gamma_E + \ln\hat{t}) + d_n(\gamma_E + \ln\hat{t})^2}{n!}\hat{t}^n & \hat{t} \le \hat{t}^* \\ \text{Dominant}[\propto \hat{t}^2] + \sum_{p=0}^{P}\left[\left(\frac{a_p^{(b;1;1)}}{\hat{t}} + \frac{a_p^{(b;1;2)}}{\hat{t}^2}\right)\left(\frac{\hat{t}_0^2}{\hat{t}^2} - 1\right)^p + e^{-2\hat{t}}\left(a_p^{(b;2;1)} + \frac{a_p^{(b;2;2)}\ln\hat{t} + a_p^{(b;2;3)}}{\sqrt{\hat{t}}}\right)\left(\frac{\hat{t}_0}{\hat{t}} - 1\right)^p\right] & \hat{t} > \hat{t}^* \end{cases}$$

Ensure that numerical precision in representation of kernel function is better than $\lesssim 10^{-8}$

[Balzani, Laporta, Passera, PLB 858 (2024) 139040, Arnau Beltran Martínez, PhD project]

TMR kernel functions

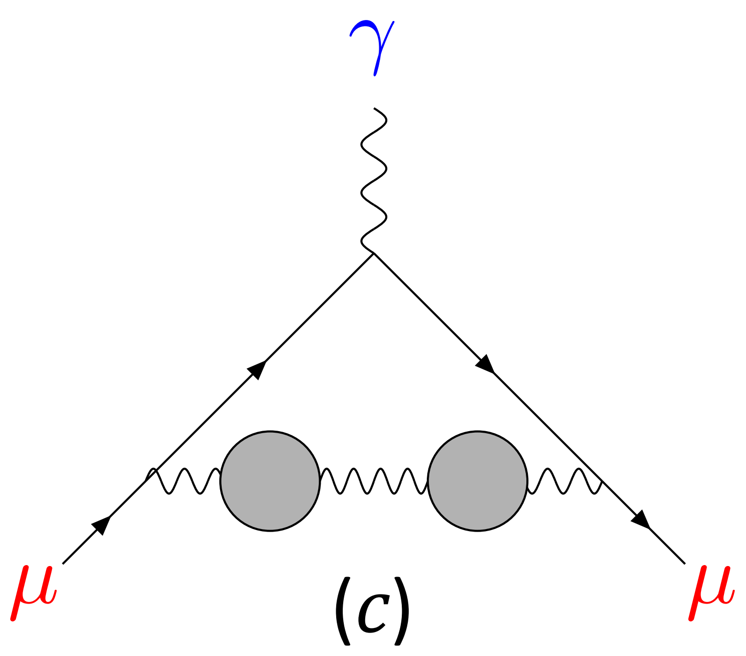


Diagram (c)

Has a closed-form solution.

$$
\begin{aligned}
\frac{m_\mu^4}{32\pi^4}\tilde{f}^{(4c)}\left(\hat{t},\hat{\tau}\right) = {} & \frac{\hat{\tau}^2\hat{t}^2}{4} + \frac{\hat{t}^2}{\hat{\tau}^2} + \frac{\hat{\tau}^2}{\hat{t}^2} - \frac{1}{2}\left(\hat{t}^2+\hat{\tau}^2\right) + \frac{1}{6} - 2(1+\gamma_E) + 2\hat{t}^2(\ln\hat{\tau}+\gamma_E) + 2\hat{\tau}^2(\ln\hat{t}+\gamma_E) + 2\left(\hat{t}^2-1\right)\ln\hat{t} + 2\left(\hat{\tau}^2-1\right)\ln\hat{\tau} \\
& + \left[1-(\hat{t}+\hat{\tau})^2\right]\ln(\hat{t}+\hat{\tau}) + \left[1-(\hat{t}-\hat{\tau})^2\right]\ln|\hat{t}-\hat{\tau}| + \left(\frac{\hat{t}^2}{6}-2\right)K_0(2t) + \left(\frac{\hat{\tau}^2}{6}-2\right)K_0(2\tau) + \left(1-\frac{1}{12}(\hat{t}+\hat{\tau})^2\right)K_0(2(\hat{t}+\hat{\tau})) + \left(1-\frac{1}{12}(\hat{t}-\hat{\tau})^2\right)K_0(2|\hat{t}-\hat{\tau}|) \\
& - \left(\frac{2\hat{t}^2}{\hat{\tau}}+\frac{\hat{\tau}}{12}\right)K_1(2\hat{\tau}) - \left(\frac{2\hat{\tau}^2}{\hat{t}}+\frac{\hat{t}}{12}\right)K_1(2\hat{t}) + \frac{1}{24}|\hat{t}-\hat{\tau}|K_1(2|\hat{t}-\hat{\tau}|) + \frac{1}{24}(\hat{t}+\hat{\tau})K_1(2(\hat{t}+\hat{\tau})) + \left(\frac{\hat{t}^2}{12}+\frac{\hat{\tau}^2}{4}-\frac{15}{16}\right)G^{2,1}_{1,3}\left(\hat{t}^2\,\middle|\,\begin{matrix}\frac{3}{2}\\ 0,1,\frac{1}{2}\end{matrix}\right) \\
& + \left(\frac{\hat{\tau}^2}{12}+\frac{\hat{t}^2}{4}-\frac{15}{16}\right)G^{2,1}_{1,3}\left(\hat{\tau}^2\,\middle|\,\begin{matrix}\frac{3}{2}\\ 0,1,\frac{1}{2}\end{matrix}\right) + \left(\frac{15}{32}-\frac{1}{24}(\hat{t}+\hat{\tau})^2\right)G^{2,1}_{1,3}\left((\hat{t}+\hat{\tau})^2\,\middle|\,\begin{matrix}\frac{3}{2}\\ 0,1,\frac{1}{2}\end{matrix}\right) + \left(\frac{15}{32}-\frac{1}{24}(\hat{t}-\hat{\tau})^2\right)G^{2,1}_{1,3}\left((\hat{t}-\hat{\tau})^2\,\middle|\,\begin{matrix}\frac{3}{2}\\ 0,1,\frac{1}{2}\end{matrix}\right)
\end{aligned}
$$

Perform different asymptotic expansions depending on relative size of $\hat{t},\hat{\tau}$

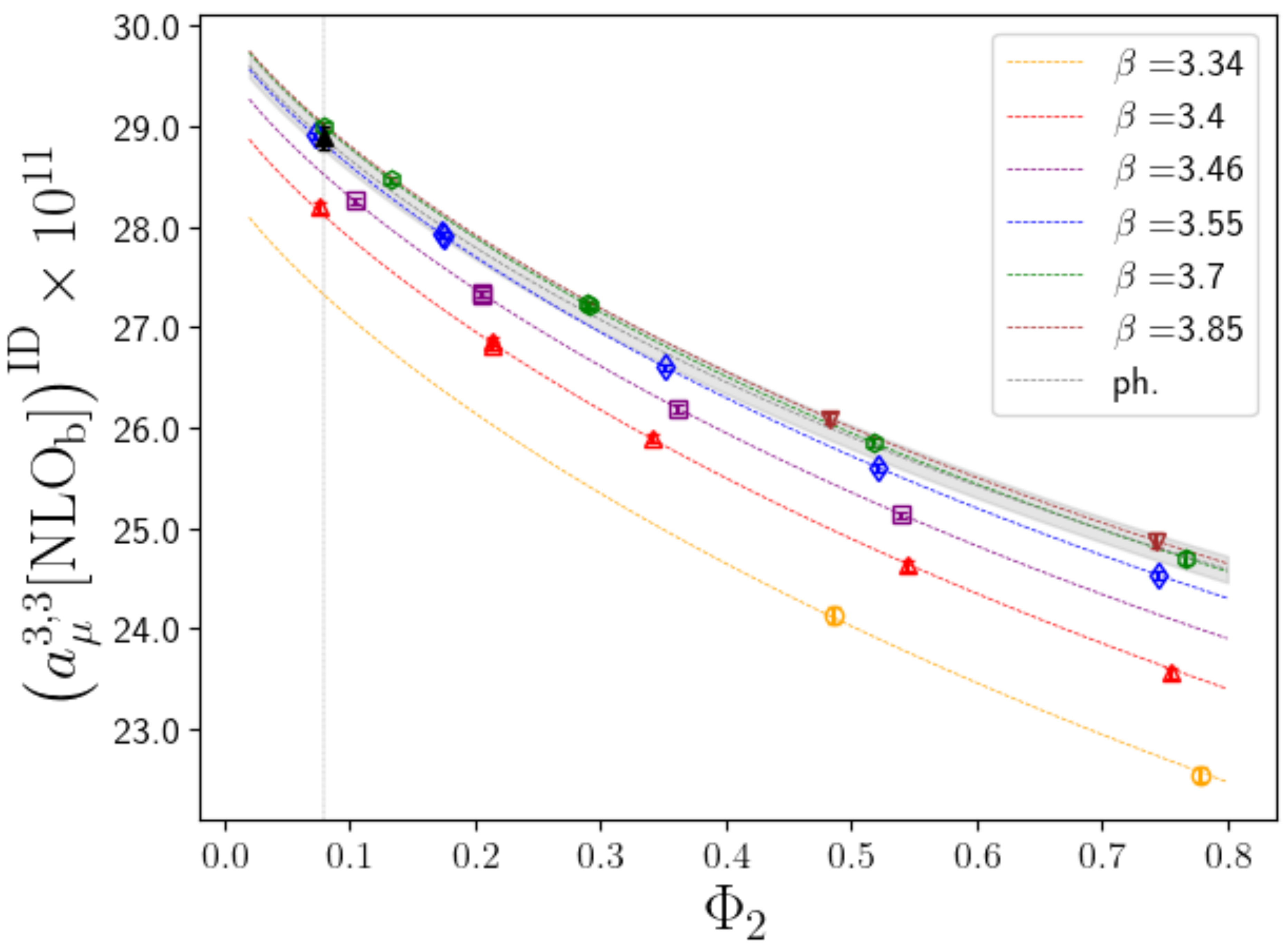
[Balzani, Laporta, Passera, PLB 858 (2024) 139040, Arnau Beltran Martínez, PhD project]

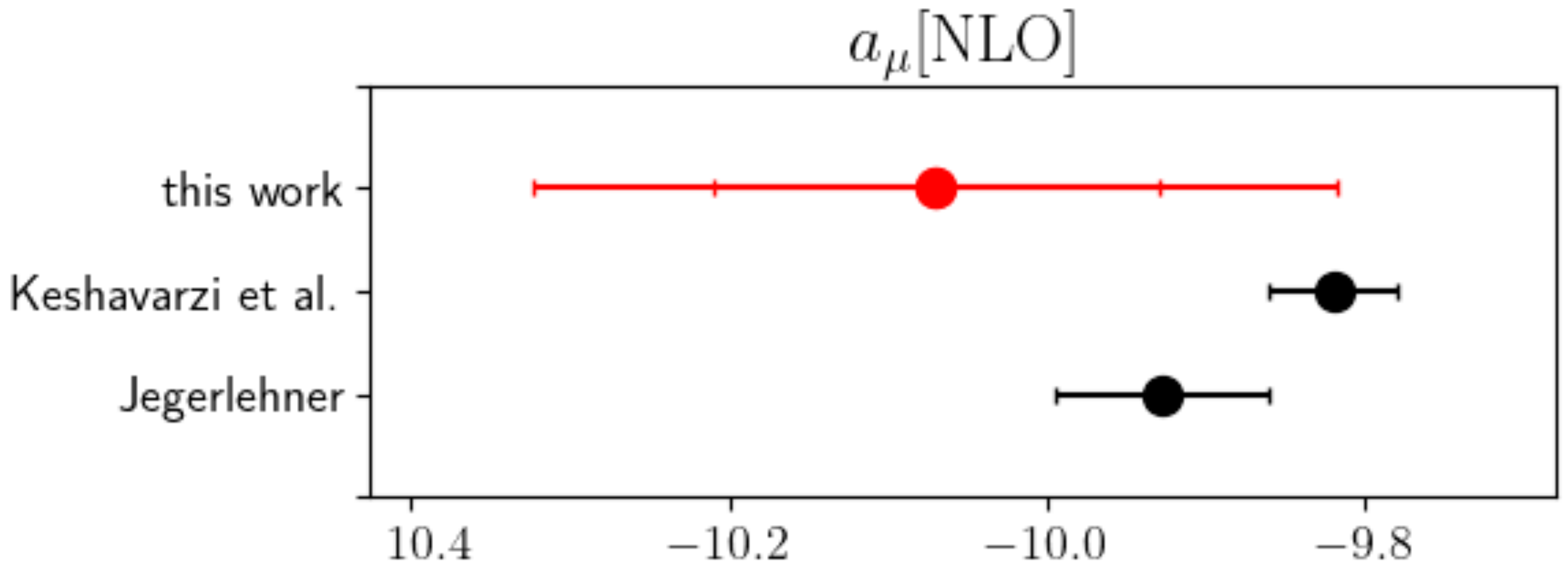
Preliminary results for NLO HVP contribution

Use subset of CLS ensembles

Compute diagrams (a)–(c) on each ensemble; correct for finite-volume effects

Perform chiral+continuum extrapolation; final result from model average





- Fully blinded analysis in progress
- Expect much higher statistical precision
- Include isospin-breaking corrections

[Balzani, Laporta, Passera, PLB 858 (2024) 139040, Arnau Beltran Martínez, PhD project]

(Interim) Conclusions

Strong and electromagnetic isospin breaking corrections crucial for improving precision of lattice-QCD calculations of the (LO) HVP contribution

Many technical challenges

- Statistical noise in diagrams containing mass insertions
- Complexity of set of diagrams

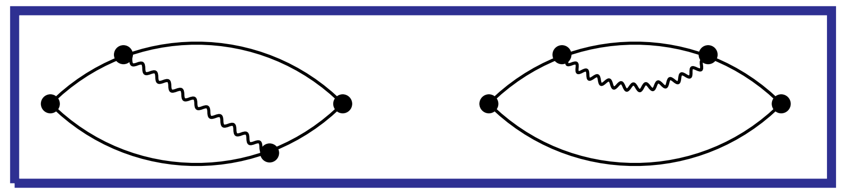
Mix of complementary methods crucial:

- Coordinate-space methods vs. $\mathrm{QED_L}$
- Simulations of QCD with $N_f = 1 + 1 + 1$

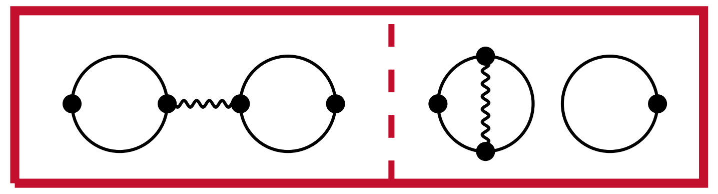
Good progress in calculating NLO HVP contribution in lattice QCD with competitive precision

Expect more precise updates on hadronic running of α and $\sin^2\theta_W$

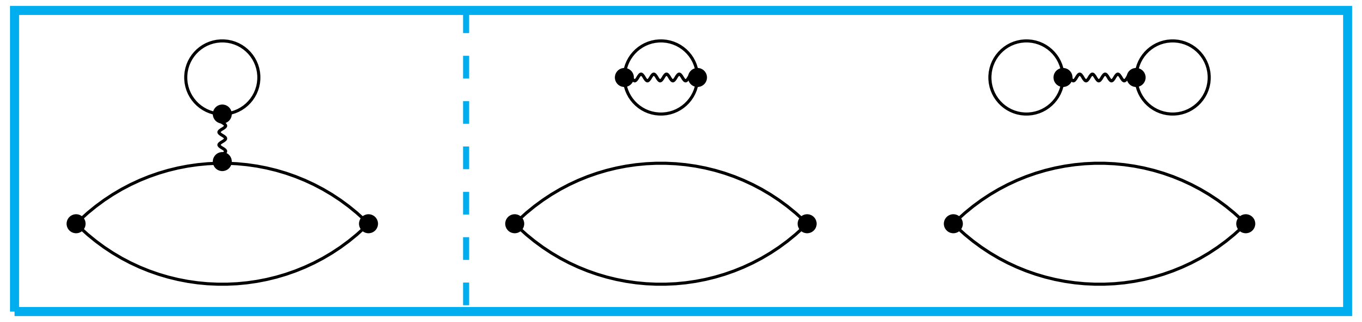
Isospin Breaking



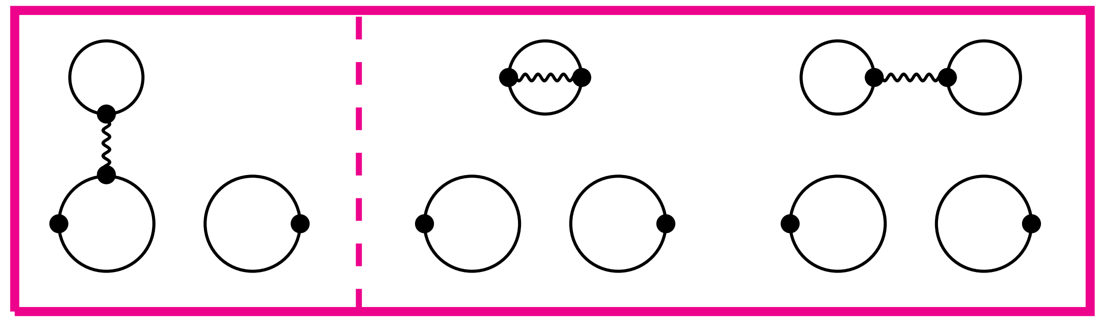
BMW $-1.27(40)(33)$
RBC/UKQCD $5.9(5.7)(1.7)$
ETM $1.1(1.0)$



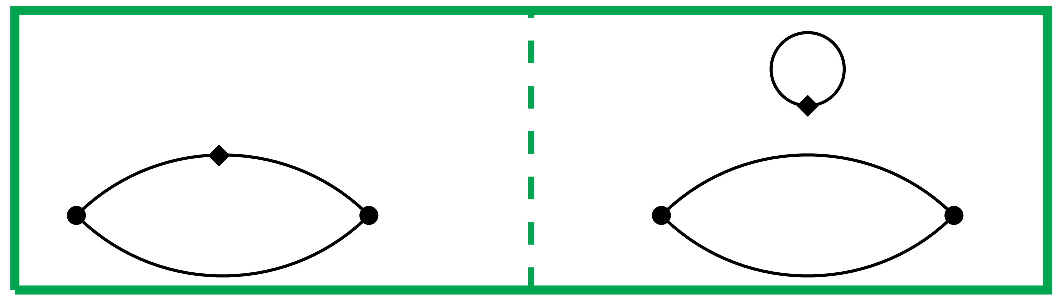
$-0.55(15)(11)$ BMW
$-6.9(2.1)(2.0)$ RBC/UKQCD



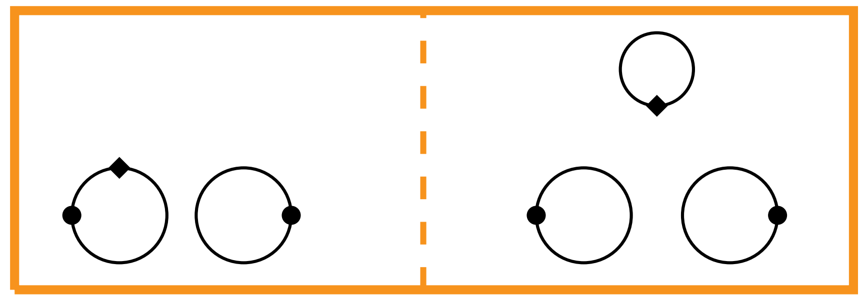
$-0.0095(86)(99)$ $0.42(20)(19)$ BMW



$0.011(24)(14)$ $-0.047(33)(23)$ BMW

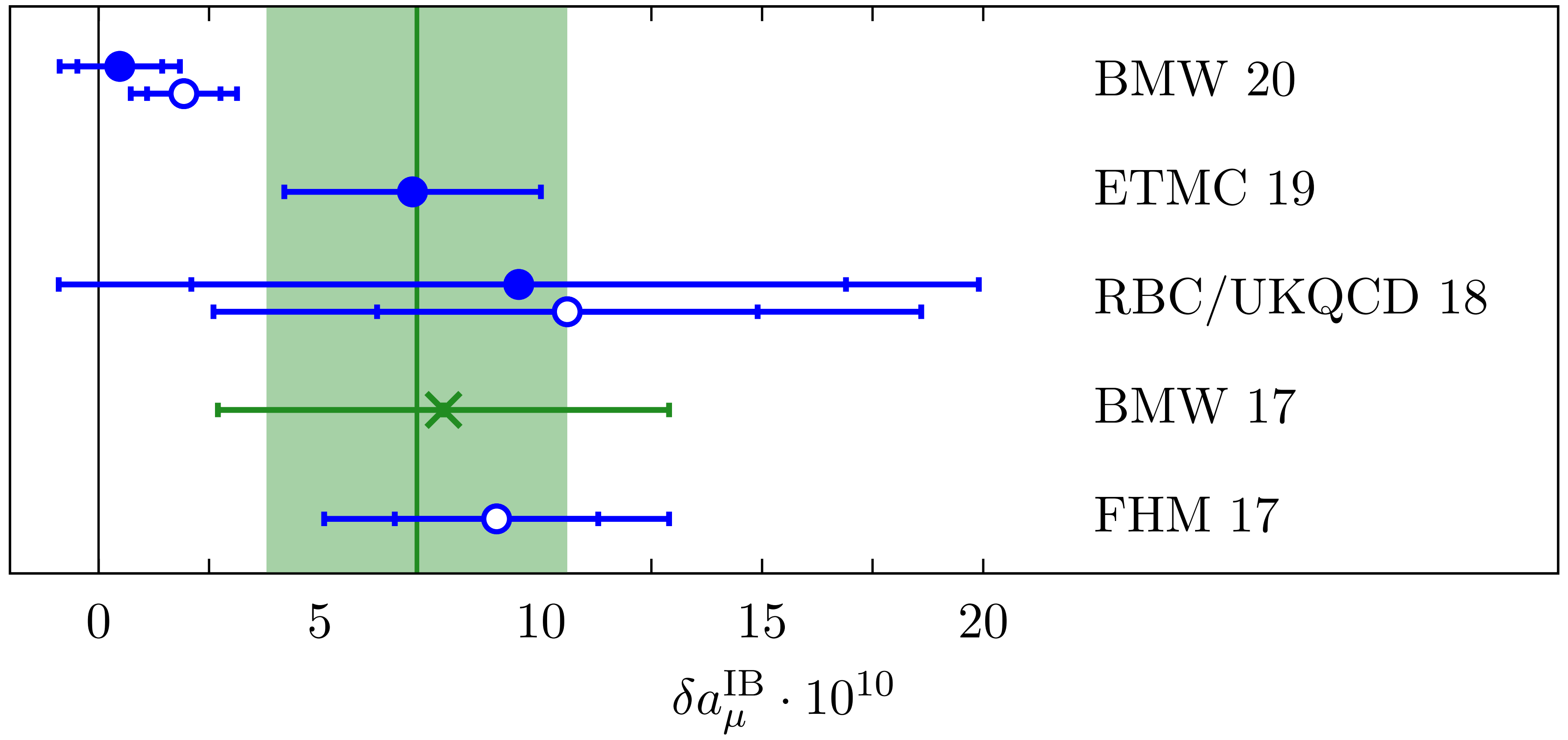


$6.59(63)(53)$ BMW
$10.6(4.3)(6.8)$ RBC/UKQCD
$6.0(2.3)$ ETM
$7.7(3.7)$ $9.0(2.3)$ FHM
$9.0(0.8)(1.2)$ LM



$-4.63(54)(69)$ BMW

Collection of published results:



BMW [arXiv:2002.12347]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

- Small overall value result of cancellations
- Large statistical uncertainties:

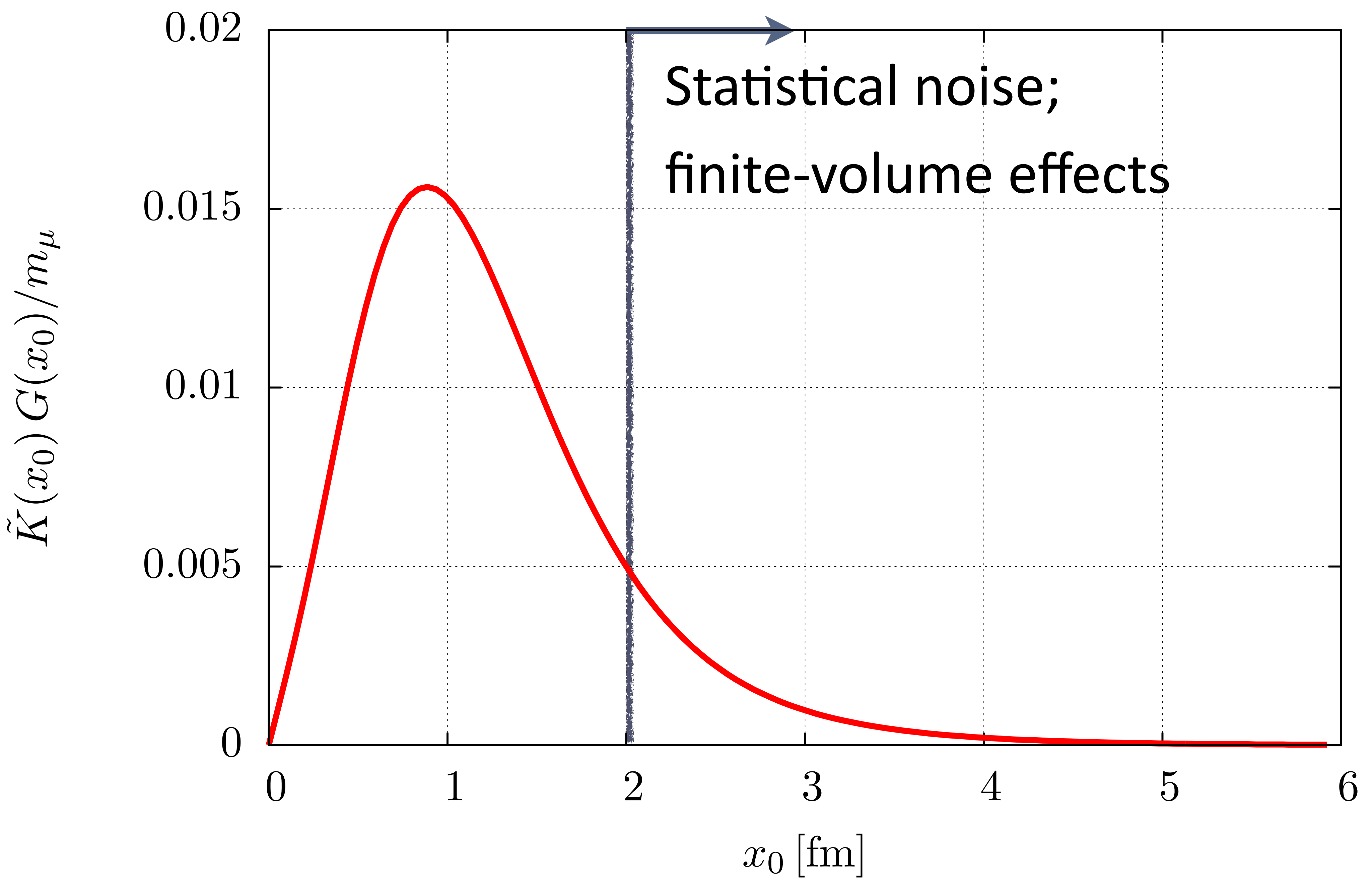
$$a_\mu^{\mathrm{IB}} \lesssim 1\,\%\,, \quad \delta a_\mu^{\mathrm{IB}} \lesssim 100\,\%$$

- More precise calculations required

(Compilation by Vera Gülpers, Lattice-HVP Workshop Nov 2020)

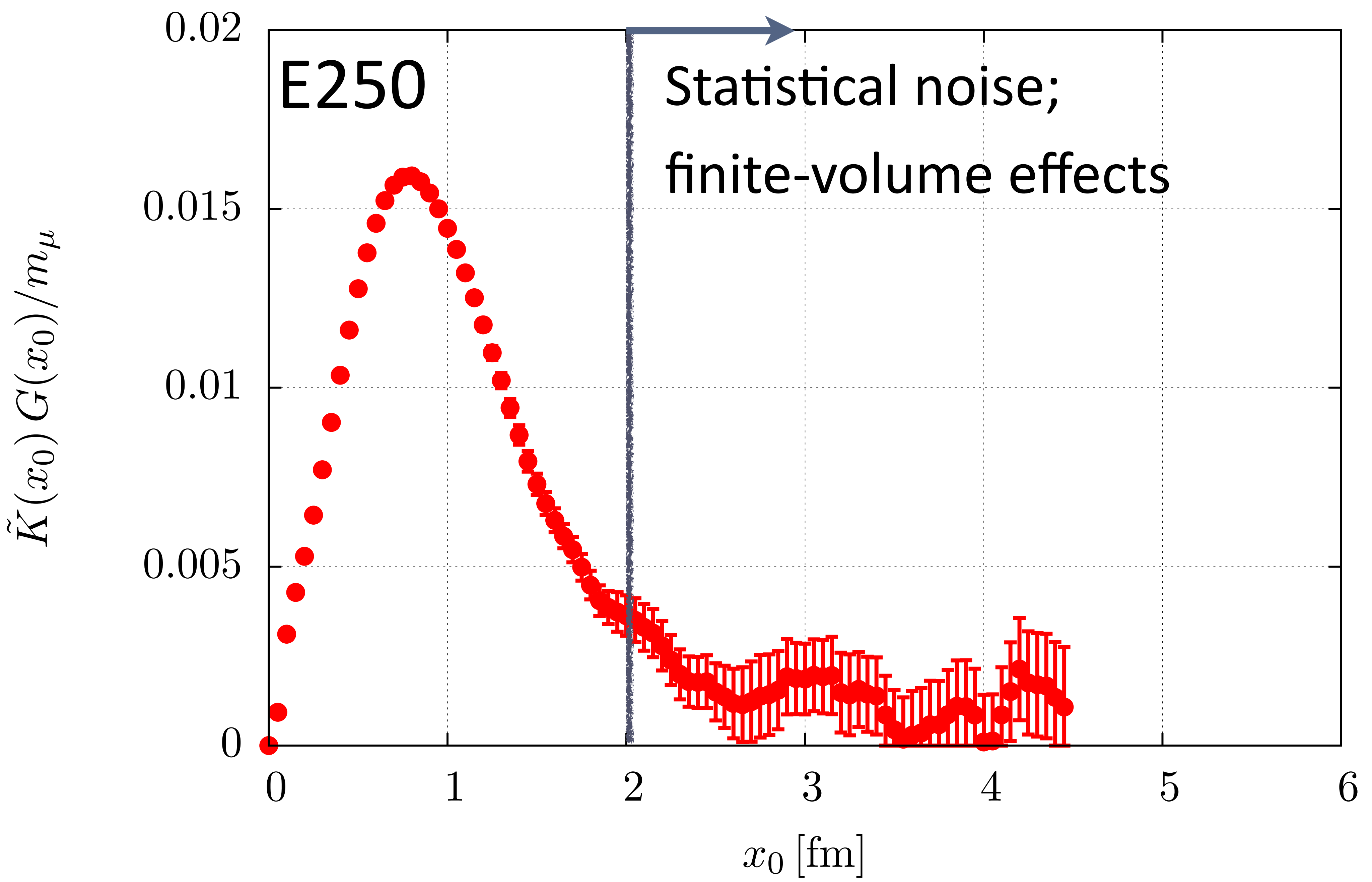
Noise-reduction strategies

Problem: exponential growth of signal-to-noise ratio in $G(t)$ for large t



Noise-reduction strategies

Problem: exponential growth of signal-to-noise ratio in $G(t)$ for large t



Noise-reduction technique: "Low-mode averaging" (LMA)

- Express quark propagator in terms of eigenmodes of the Wilson-Dirac operator

$$S(y,x) = S_{\text{eigen}}(y,x) + S_{\text{rest}}(y,x)$$

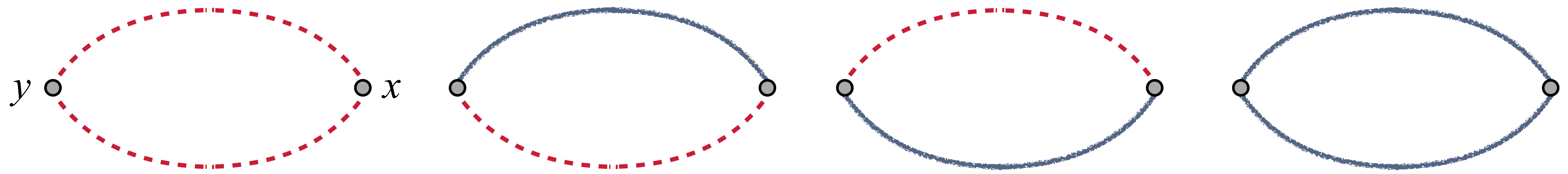
[Giusti, Hernández, Laine, Weisz, HW 2004; DeGrand & Schaefer 2004]

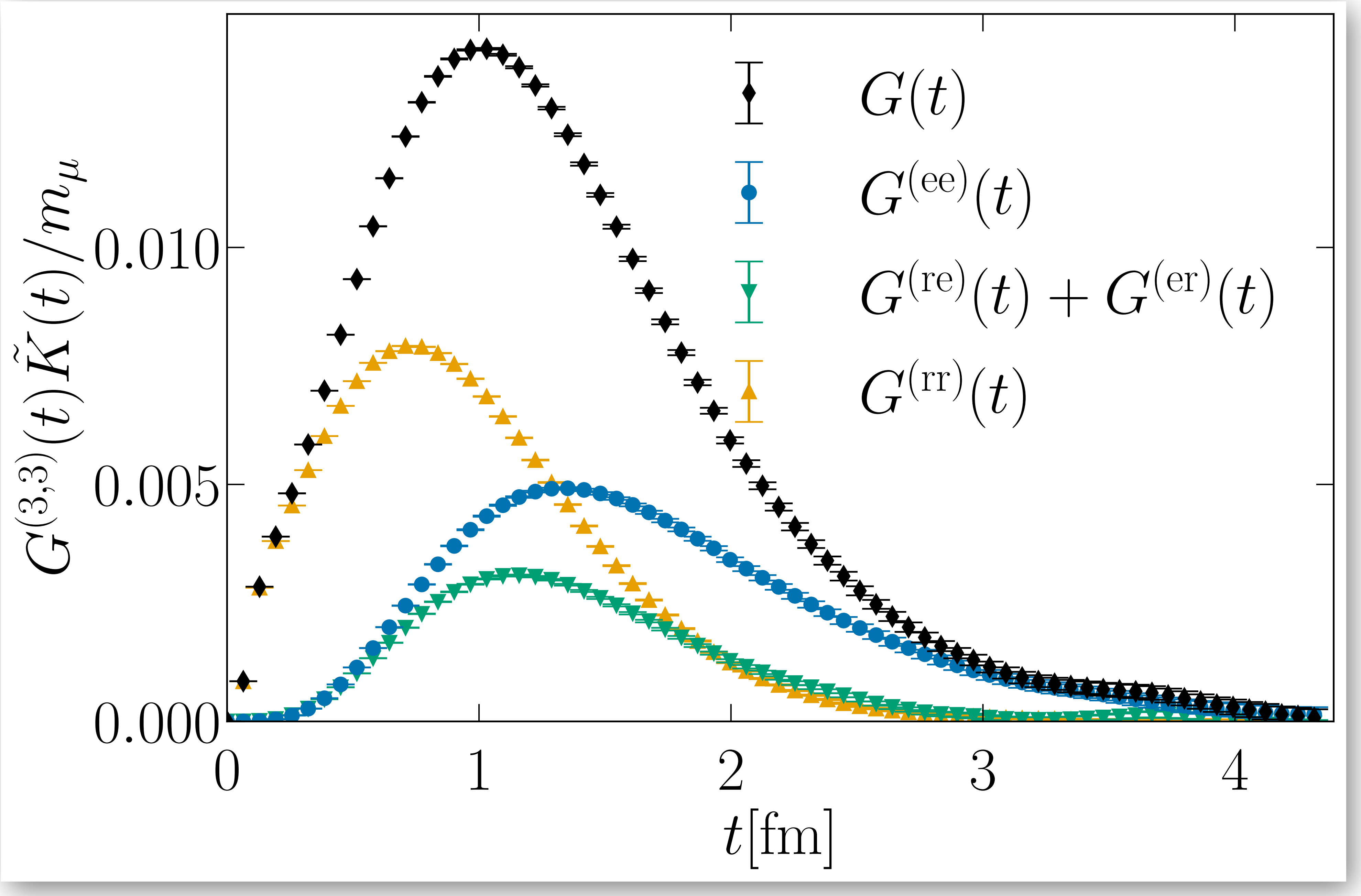
$$S_{\text{eigen}}(y,x) = \sum_{i=1}^{N_{\text{low}}} \lambda_i^{-1}\, v_i(x) \otimes (\gamma_5 v_i(y))^\dagger\,, \qquad (\gamma_5 D_{\text{w}})\, v_i(x) = \lambda_i\, v_i(x)\,, \qquad N_{\text{low}} \lesssim 1000$$

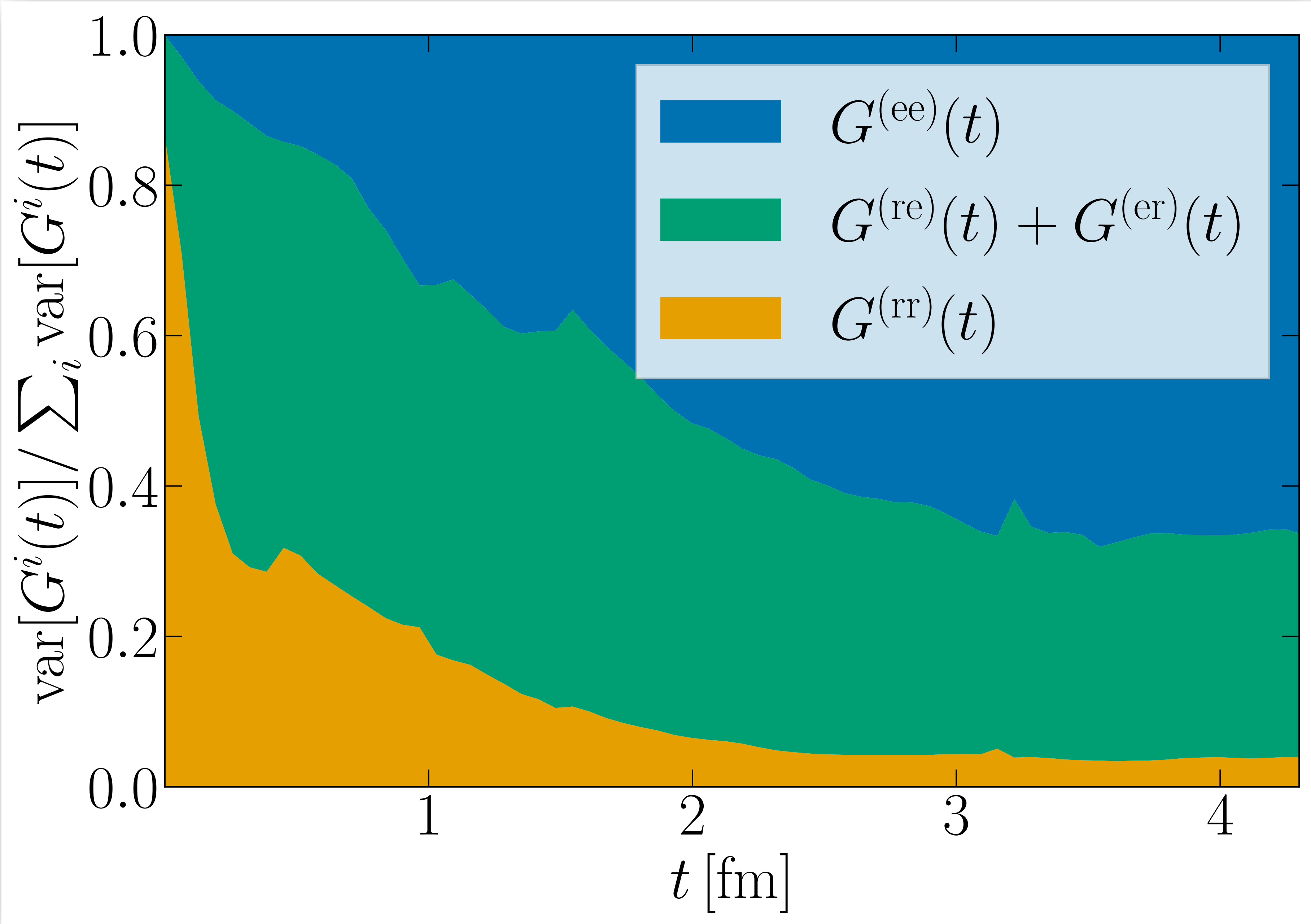
Low modes responsible for statistical fluctuations — LMA leads to better sampling of the correlator

Low-mode averaging

Compute and sum all combinations of low and high mode contributions to $G(t)$







Spectral reconstruction

Observation: long-distance regime of $G(t)$ dominated by two-pion states with isospin one

$$G(x_0) \overset{x_0\to\infty}{=} \sum_n |A_n|^2\, e^{-\omega_n t}$$

Strategy B:

- Perform a dedicated calculation of the spectrum of pion-pion states in the isovector channel
- Accumulate contributions from states $n = 0, 1, 2, \ldots$ until saturation is observed