Hadronic light-by-light amplitude and e.m. correction to hadronic vacuum polarization

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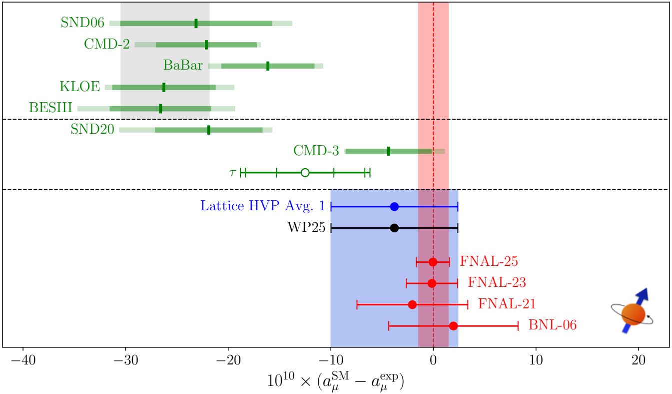
Papers with FOR5327 acknowledgment

1. V. Biloshytskyi et al, Forward light-by-light scattering and electromagnetic correction to hadronic vacuum polarization, JHEP 03 (2023) 194 [2209.02149]
2. N. Asmussen et al, Hadronic light-by-light scattering contribution to the muon $g-2$ from lattice QCD: semi-analytical calculation of the QED kernel, JHEP 04 (2023) 040 [2210.12263]
3. H.B. Meyer, Low-energy matrix elements of heavy-quark currents, Eur.Phys.J.C 83 (2023) 12, 1134 [2310.09085]
4. J. Parrino et al, Computing the UV-finite electromagnetic corrections to the hadronic vacuum polarization in the muon $(g-2)$ from lattice QCD, [2501.03192], under review at JHEP
5. D. Erb et al, Isospin-violating vacuum polarization in the muon $(g-2)$ with SU(3) flavour symmetry from lattice QCD, [2505.24344], under review at JHEP

Lattice conference proceedings:

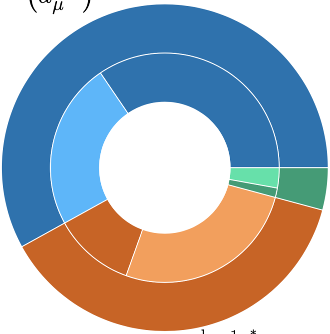
1. J. Koponen et al, The $\pi^0 \to \gamma^*\gamma^*$ transition form factor and the pion pole contribution to a_μ on CLS ensembles, 2503.11428
2. D. Erb et al, The isospin-violating part of the hadronic vacuum polarisation, 2412.14760
3. J. Koponen et al, Status update: $\pi^0 \to \gamma^*\gamma^*$ transition form factor on CLS ensembles, 2311.07330
4. J. Parrino et al, Coordinate-space calculation of QED corrections to the hadronic vacuum polarization contribution to $(g-2)_\mu$, 2310.20556

Status a_μ: SM vs. direct measurement (WP'25)



- a_μ^{hvp} has an uncertainty of 6.1×10^{-10}, totally dominating the SM error of 6.2×10^{-10};
- a_μ^{HLbL} has an uncertainty of 0.96×10^{-10};
- the experimental error of a_μ is 1.45×10^{-10}.

Error budget of Mainz/CLS calculation of a_μ^{hvp} 2411.07969 (JHEP)



(dark shade: statistical error; light shade: systematic error)

- Clearly, improving our control over the correction due to IB effects is imperative: its uncertainty is 4.4×10^{-10}.
- In the WP'25 average, the uncertainty of the correction is 3.4×10^{-10}.

Conceptual/methodological developments

A. We have proposed and implemented two main ideas for handling 'dynamical' photons in lattice QCD:

I. use coordinate-space methods

⋆ motivation: keep the observable *local*, not spread over the entire volume

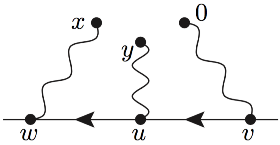
II. where needed, use a Pauli-Villars UV cutoff $\Lambda \ll a^{-1}$, propagator $\frac{1}{k^2} - \frac{1}{k^2+\Lambda^2}$

⋆ motivation: bare e.m. effects are re-usable by a different lattice collaboration and can be compared to continuum calculations.

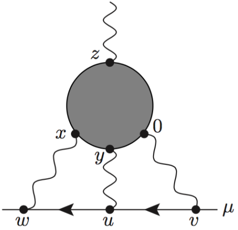
B. We have provided a rigorous starting point (Cottingham-like formula) for computing the e.m. correction to HVP with continuum methods.

Coordinate-space approach to a_μ^{HLbL}

QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$



$\Rightarrow$



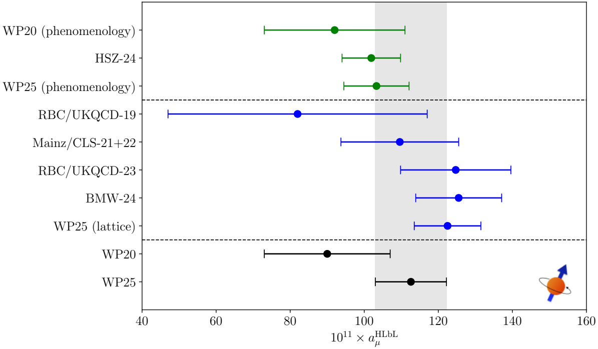
$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \left[\int d^4x \, \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\mathrm{QED}} \; \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{=\mathrm{QCD\ blob}} \right].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z \; z_\rho \left\langle \, j_\mu(x) \, j_\nu(y) \, j_\sigma(z) \, j_\lambda(0) \right\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume

[Asmussen, Gérardin, Green, HM, Nyffeler 1510.08384; 2210.12263 (JHEP).]

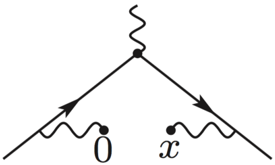
WP'25: status of $a_\mu^{\rm HLbL}$



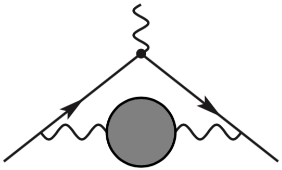
- The three most recent lattice calculations use coordinate-space methods in slightly different variations; BMW employed the Mainz QED kernel.
- $a_\mu^{\rm HLbL} = 11.26(0.96) \times 10^{-10}$ moved up by about two units as compared to WP'20; scale factor of 1.5 was applied in the pheno/lattice average.

Hadronic vacuum polarization in $(g-2)_\mu$

HM 1706.01139 (EPJC)



$\Rightarrow$



QED kernel $H_{\mu\nu}(x)$

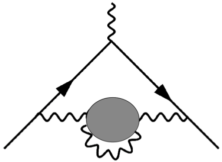
$a_\mu^{\rm hvp}$

$$a_\mu^{\rm hvp} = \int d^4x \; H_{\lambda\sigma}(x) \, \Big\langle j_\lambda(x) j_\sigma(0) \Big\rangle_{\rm QCD},$$

$$j_\lambda = \frac{2}{3}\bar u \gamma_\lambda u - \frac{1}{3}\bar d \gamma_\lambda d - \frac{1}{3}\bar s \gamma_\lambda s + \dots; \qquad H_{\lambda\sigma}(x) = -\delta_{\lambda\sigma}\mathcal{H}_1(|x|) + \frac{x_\lambda x_\sigma}{x^2}\,\mathcal{H}_2(|x|)$$

Weight functions $\mathcal{H}_i$ are linear combinations of Meijer's functions.

Kernel $H_{\lambda\sigma}(x)$ also applicable
to the e.m. corrections to HVP



Recipe for correction to HVP due to IB effects

First, compute $a_\mu^{\rm hvp}$ in pure, isospin-symmetric QCD, with 'reference' hadron masses (e.g. π, K, Ω) close to their PDG values.
These masses serve as renormalized parameters of the theory.

Then compute the additive correction as (Eq. (3.45) in WP'25)

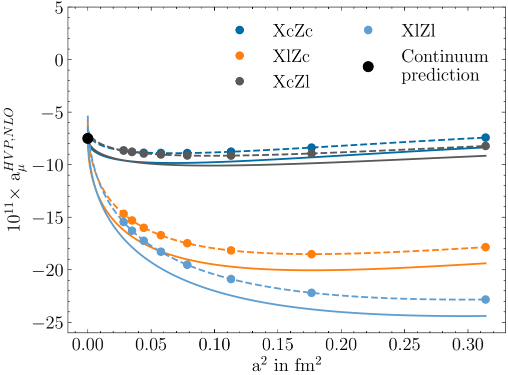
$$\delta a_\mu^{\rm hvp} = \lim_{\Lambda\to\infty} \left\{ a_\mu^{{\rm hvp}1\gamma^*}(\Lambda) - \nabla_{\overrightarrow{M}} a_\mu^{\rm hvp} \cdot \overrightarrow{M}^{\rm self}(\Lambda) \right\} + \nabla_{\overrightarrow{M}} a_\mu^{\rm hvp} \cdot \delta\overrightarrow{M}.$$

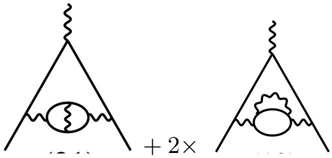
The term $-\nabla_{\overrightarrow{M}} a_\mu^{\rm hvp} \cdot \overrightarrow{M}^{\rm self}(\Lambda)$ corresponds to the fact that we want to subtract from the bare the e.m. correction $a_\mu^{{\rm hvp}1\gamma^*}(\Lambda)$ the effect that merely comes from a shift of the reference hadron masses.

The term $+\nabla_{\overrightarrow{M}} a_\mu^{\rm hvp} \cdot \delta\overrightarrow{M}$ corresponds to a shift in $a_\mu^{\rm hvp}$ due to the fact that at the expansion point, not all reference hadron masses have their physical values.

NB. The procedure above takes into account 'strong isospin breaking'.

Lattice calculations reproduce two-loop QED vacuum polarization





$$m_{\ell\text{oop}} = m_\mu$$
$$\Lambda = 3m_\mu$$

D. Erb, V. Biloshytskyi et al 2505.24344

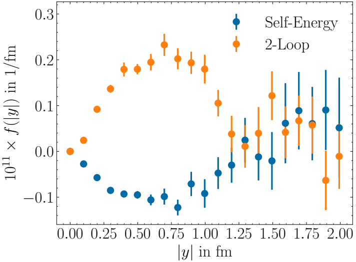
$$a_\mu^{\text{2loop vp}} = -\frac{e^2}{2}\delta_{\mu\nu}\int_{x,y,z} H_{\lambda\sigma}(z)[G_0]_\Lambda(y-x)\,\left\langle V_\sigma^{\text{em}}(z)V_\nu^{\text{em}}(y)V_\mu^{\text{em}}(x)V_\lambda^{\text{em}}(0)\right\rangle,$$

$$[G_0]_\Lambda(x) = G_0(x) - 2G_{\frac{\Lambda}{\sqrt{2}}}(x) + G_\Lambda(x), \qquad G_m(x) \equiv \int \frac{d^4q}{(2\pi)^4}\,\frac{e^{iq\cdot x}}{q^2+m^2}.$$

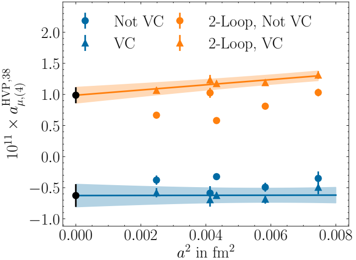
The ‘continuum prediction’ was obtained with the help of dispersive techniques.

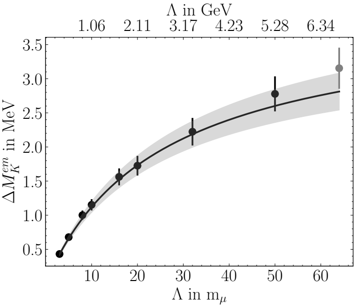
Idem, in QCD ($m_\pi = m_K = 415$ MeV; D. Erb et al 2505.24344)

Last integrand: (y is one of the internal vertices)

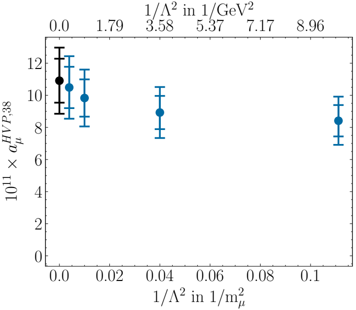


Continuum extrapolation $\Lambda = 16 m_\mu$



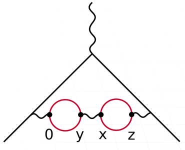


Λ dependence of $(M^{\mathrm{self}}_{K^+} - M^{\mathrm{self}}_{K^0})$:
we see the log. behaviour predicted by OPE

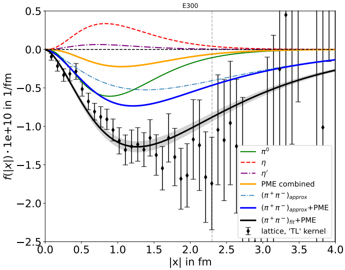


Extrapolation $\Lambda \to \infty$ of δa_μ^{38}

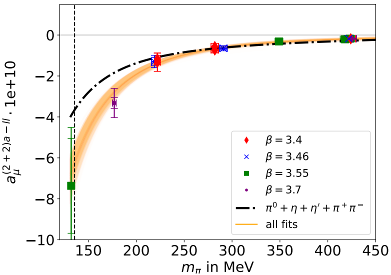
Calculation of one of the largest diagrams in $a_\mu^{\mathrm{hvp}1\gamma^*}$



One-photon irreducible diagram: perturbatively, at least two gluons must be exchanged between the two quark loops



Final 1d integral ($m_\pi = 170\,\mathrm{MeV}$)

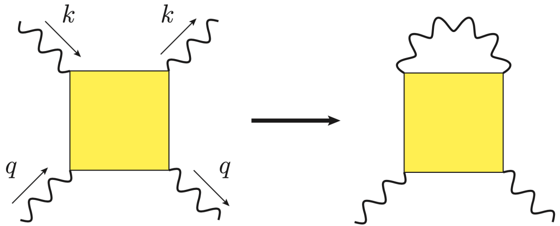


Chiral extrapolation

- Chiral perturbation theory helps control the noisy tail of the correlator.
- This calculation entered our a_μ^{hvp} result Kuberski et al, JHEP 04 (2025) 098 .

J. Parrino et al, 2501.03192

Correction to hadronic vacuum polarization: continuum point of view



$= \Pi_{1\gamma^*}(q^2, \Lambda)$

- the leading correction to HVP is expressible in terms of the forward HLbL amplitude;
- including the counterterms: $\overline{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$

$$\delta\overline{\Pi}(q^2) = \lim_{\Lambda\to\infty} \left\{ \overline{\Pi}_{1\gamma^*}(q^2, \Lambda) + \left(\delta g(\Lambda) \frac{\partial}{\partial g} + \sum_f \delta m_f(\Lambda) \frac{\partial}{\partial m_f} \right) \overline{\Pi}(q^2) \right\}_{\alpha=0, m_u=m_d}.$$

E.g. get $(m_u - m_d)(\Lambda)$ by requiring the PDG kaon mass splitting be reproduced:

$$\Delta M_K^{phys} = \underbrace{\Delta M_K^{em}(\Lambda)}_{\text{Cottingham}} + (m_u - m_d)(\Lambda) \frac{\partial \, \Delta M_K}{\partial (m_u - m_d)}.$$

Analogue of the Nyffeler formula for pseudoscalar pole contribution: [2209.02149]

$$\Pi_{1\gamma^*}^{\mathrm{PS\,pole}}(q^2,\Lambda) = \frac{-e^4}{16\pi^2|q|}\int_0^\infty d|k|\,|k|^4\left[\frac{1}{k^2}\right]_\Lambda \mathcal{F}(-q^2,-k^2)^2\, Z^{m_\pi}_{|q|,|k|}\left(1-\tfrac{1}{3}(Z^{m_\pi}_{|q|,|k|})^2\right),$$

$$Z^m_{|q|,|k|} = \frac{1}{2|q||k|}\left(q^2+k^2+m^2-\sqrt{(q^2+k^2+m^2)^2-4q^2k^2}\right).$$

- The double-virtual transition form factor enters.
- Note that a_μ^{hvp} requires $\Pi_{1\gamma^*}(q^2,\Lambda)-\Pi_{1\gamma^*}(0,\Lambda)$.

We have provided a phenomenological estimate for $a_\mu^{\mathrm{hvp}1\gamma^*}$: [2501.03192]

$$a_\mu^{\mathrm{hvp}1\gamma^*} = -4.91(2.46)\times 10^{-10}.$$

Summary

Photons present particular challenges for lattice QCD:

- Position-space methods help handle the long-distance effects.
- Take the continuum limit at fixed cutoff Λ on the photon virtuality.
- This allows for comparisons with phenomenological treatment, for which we have provided a rigorous starting point.

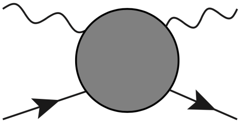
Determining the counterterms induced by the photons

Determine the isoscalar counterterms from three conditions such as

$$M_N^{\rm phys} - M_N^{\rm isoQCD} \stackrel{!}{=} M_N^{\rm self}(\Lambda) + \tfrac{1}{6}\delta(m_u + m_d - 2m_s)(\Lambda)\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle$$
$$+\tfrac{1}{3}\delta(m_u + m_d + m_s)(\Lambda)\langle N|\bar{u}u + \bar{d}d + \bar{s}s|N\rangle + \delta g^{-2}(\Lambda)\ \langle N|\tfrac{1}{2}{\rm Tr}\{G_{\mu\nu}G_{\mu\nu}\}|N\rangle$$

for the average nucleon mass, and $(m_u - m_d)(\Lambda)$ from the mass splitting.

Forward Compton amplitude on hadron H
$T_i(\nu = q\cdot p/m, q^2),\ i = 1, 2$:



$$M_H^{\rm self}(\Lambda) = \frac{e^2}{2M_H}\int \frac{d^4Q}{(2\pi)^4}\left[\frac{1}{Q^2}\right]_\Lambda \big(3Q^2T_1(iQ_0, -Q^2) + (2Q_0^2 + Q^2)T_2(iQ_0, -Q^2)\big)$$

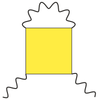
...followed by a dispersive representation of the T_i
via the hadron's structure functions $F_i(x = Q^2/(2M_H\nu), Q^2)$.

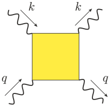
Cottingham, Ann.Phy. **25**, 424 (1963); [...]; Gasser, Leutwyler, Rusetsky PLB 814 (2021) 136087.

Electromagnetic correction to HVP from forward HLbL amplitude

Master formula:

$$\Pi_{1\gamma^*}(Q^2,\Lambda) = \frac{1}{6Q^4(2\pi)^3}\int_0^\infty dK^2 \underbrace{\left[\frac{1}{K^2}\right]_\Lambda}_{\frac{1}{K^2}-\frac{1}{K^2+\Lambda^2}} \int_0^{K^2Q^2} d\nu^2 \left(\frac{K^2Q^2}{\nu^2}-1\right)^{1/2} \mathcal{M}(\nu,\, K^2,\, Q^2)$$





... the relevant forward hadronic light-by-light amplitude being

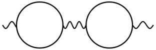
$$\mathcal{M}(\nu,\, K^2,\, Q^2) = g_{\mu_1\mu_3} g_{\mu_2\mu_4} \mathcal{M}^{\mu_1\mu_2\mu_3\mu_4}(k,q) = 4\mathcal{M}_{TT} - 2\mathcal{M}_{LT} - 2\mathcal{M}_{TL} + \mathcal{M}_{LL}.$$

NB. $\mathcal{M}$ admits a once-subtracted dispersion relation in the variable $\nu = k\cdot q$, in terms of $\gamma^*\gamma^* \to$ hadrons fusion cross-sections.

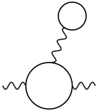
Biloshytskyi et al 2209.02149 (JHEP)

The subset of UV-finite diagrams

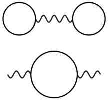
Operator-product expansion and power-counting $\Rightarrow$
about half of the diagrams are UV-finite diagrams.



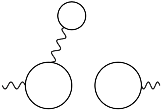
(2+2)a



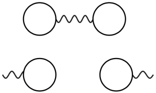
(3+1)b



(2+1+1)a



(2+1+1)b



(1+1+1+1)

$\rightsquigarrow$ For these, the internal photon propagator does not need to be regulated.

A test in QED: two-loop VP from one-loop forward LbL amplitude

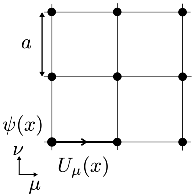
$$
\begin{aligned}
\mathcal{M}(\nu, K^2, Q^2) = 16\alpha^2 \Bigg(6 - \Bigg\{ & \frac{2\log\left[\frac{1}{2} Q\left(\sqrt{Q^2+4}+Q\right)+1\right]}{\sqrt{Q^2+4}} \\
\times\Big(& -4\nu^2 Q^2 \left[\left(K^2-2\right)\left(K^2+1\right)Q^4+\left(K^2+2\right)\left(7K^2-2\right)Q^2+6K^4+52K^2+16\right] \\
& +K^2Q^4\left(K^2+Q^2+4\right)^2\left[K^2\left(Q^2+4\right)-2Q^2+4\right]+96\nu^4\Big)\Big/\Big(K^4Q^5\left(K^2+Q^2+4\right)^2 \\
& +16\nu^4 Q-4K^2\nu^2Q^3\left[K^2\left(Q^2+2\right)+2\left(Q^2+4\right)\right]\Big)+\{K\leftrightarrow Q\}\Bigg\} \\
+\Bigg\{ & \frac{2\sqrt{1+\frac{4}{K^2+2\nu+Q^2}}\log\left[\frac{1}{2}\left(\sqrt{\left(K^2+2\nu+Q^2\right)\left(K^2+2\nu+Q^2+4\right)}+K^2+2\nu+Q^2+2\right)\right]}{K^2Q^2\left(K^2+Q^2+2\nu+4\right)-4\nu^2} \\
& \times\left(K^2Q^2(K^2+Q^2+2\nu)-2(K^2+Q^2)(\nu-1)-(K^4+Q^4)-2\nu(\nu+2)\right) \\
& +\frac{(K^2+Q^2)^2+2\nu(K^2+Q^2)+2\nu(\nu-2)-4}{\nu}\,C_0\left(-K^2,-Q^2,-K^2-2\nu-Q^2;1,1,1\right) \\
& +\{\nu\to-\nu\}\Bigg\}\Bigg), \qquad \text{(lepton mass set to unity)}
\end{aligned}
$$

where $C_0(p_1^2, p_2^2, (p_1+p_2)^2; m_1^2, m_2^2, m_3^2)$ is the scalar one-loop integral [hep-ph/9807565].
Inserting this expression into the master formula gives the same result for $\overline{\Pi}^{(2)}(Q^2)$ as

$$\overline{\Pi}(Q^2) = -\frac{Q^2}{\pi}\int_{4m_\ell^2}^{\infty}\frac{dt}{t(t+Q^2)}\,\mathrm{Im}\Pi(t)$$

using the 1955 Källen-Sabry next-to-leading-order spectral function $\frac{1}{\pi}\mathrm{Im}\Pi(t)$.

The lattice regularization of QCD   K.G. Wilson 1974



Gluon 'link' variables:

$$U_\mu(x) = e^{iag_0 A_\mu(x)} \in SU(3)$$

Quarks: on-site Grassmann variables,

$$\psi_1\psi_2 = -\psi_2\psi_1$$

Action: has exact gauge invariance.

Finite volume: work on $L \times L \times L$ torus – periodic boundary conditions.

Euclidean path integral: finite number of compact degrees of freedom

$$Z = \int \mathcal{D}U_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-S_G[U]-\bar{\psi}D[U]\psi} = \int \mathcal{D}U_\mu \, \det D[U] e^{-S_G[U]}$$

QCD $\leftrightarrow$ 4d statistical mechanics system $\Rightarrow$ importance sampling Monte-Carlo

Continuum limit: $g_0^2 \sim 1/\log(1/a)$ (asymptotic freedom)

Correlation functions and parameters of lattice QCD

$$\left\langle j_\mu(x) j_\nu(0) \right\rangle \overset{x_0>0}{=} \left\langle \mathrm{vac} \middle| j_\mu(0) e^{-Hx_0 + i\vec{P}\cdot\vec{x}} j_\nu(0) \middle| \mathrm{vac} \right\rangle$$

$$= \left\langle -\mathrm{Tr}\{S(x,0)\gamma_\nu S(0,x)\gamma_\mu\} + \mathrm{Tr}\{S(x,x)\}\,\mathrm{Tr}\{S(0,0)\} \right\rangle_{\mathrm{SU(3)\ gauge\ field}}$$

Connected   Disconnected diagram

- bare parameters: $m_u = m_d$, m_s and g_0
- fix their values by computing am_π, am_K and (typically) calibrate the lattice spacing via $a = (am_\Omega)/m_\Omega^{\mathrm{PDG}}$.
- electromagnetic effects are usually included as a correction: 1st order expansion around isosymmetric QCD [de Divitiis et al 1303.4896 (PRD)].