Dispersive analysis of $e^+e^- \to \gamma^* \to \gamma P_1 P_2$ processes near vector resonances ϕ and J/ψ

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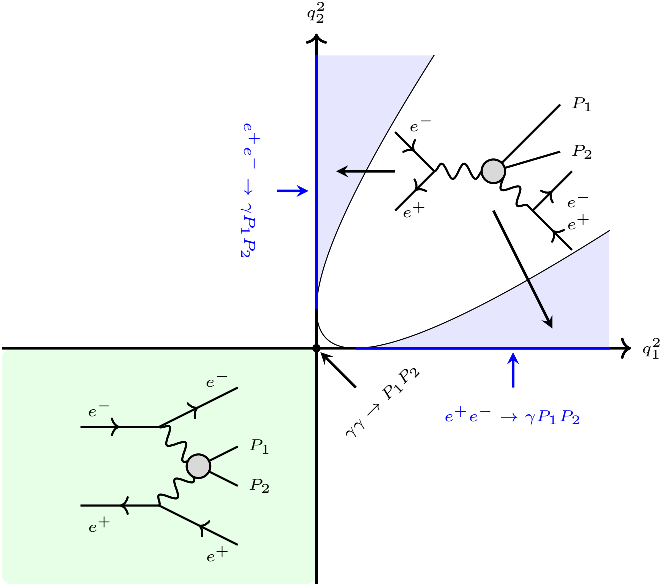
Outline

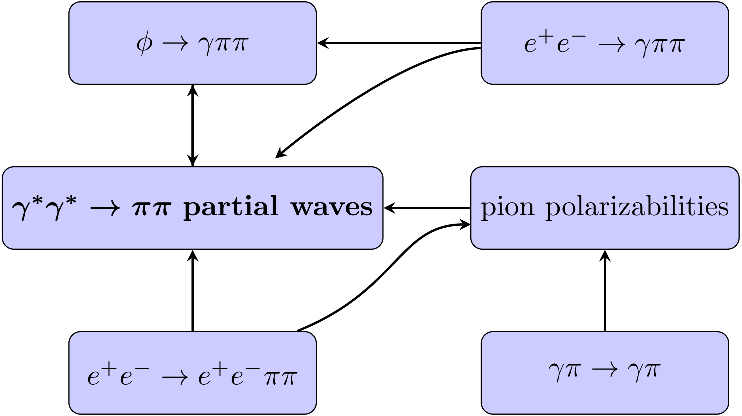
Introduction & motivation

Kinematic regions of $\gamma^*(q_2) \to \gamma^*(q_1) P_1 P_2$ processes on q_1^2-q_2^2 plane:



Introduction & motivation

Connections between different processes for $P_1P_2 = \pi\pi$:



Introduction & motivation

ϕ radiative decay modes of interest: PDG, 2024

Decay modes	$\phi \to \gamma\pi^0\pi^0$	$\phi \to \gamma\pi^0\eta$	$\phi \to \gamma K^0\bar{K}^0$
Branching ratios	$1.13(6) \times 10^{-4}$	$7.27(3) \times 10^{-5}$	$< 1.9 \times 10^{-8}$

- Determined by SND & KLEO

J/ψ radiative decay modes of interest: PDG, 2024

Decay modes	$J/\psi \to \gamma\pi^0\pi^0$	$J/\psi \to \gamma K_S^0 K_S^0$	$J/\psi \to \gamma\pi^0\eta$
Branching ratios	$1.15(5) \times 10^{-3}$	$8.1(4) \times 10^{-4}$	$2.14(31) \times 10^{-5}$

- Determined by BESIII

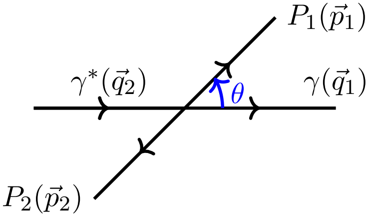
$\Rightarrow$ All listed decays undergo strong final-state interactions!

Kinematics & helicity amplitudes

$\gamma^*(q_2) \to \gamma(q_1)P_1(p_1)P_2(p_2)$:
Kinematics with $q_1^2 = 0$, $q_2^2 = q^2$

$s = (p_1+p_2)^2, t = (p_1+q_1)^2, u = (p_2+q_1)^2$

$s + t + u = q^2 + m_1^2 + m_2^2$



$$H^{\mu\nu} = A(s,t,u)H_1^{\mu\nu} + B(s,t,u)H_2^{\mu\nu} + C(s,t,u)H_3^{\mu\nu}$$
$$e^{i(\lambda_2-\lambda_1)\phi}H_{\lambda_1\lambda_2}(s,t) = -e_{\gamma}^{*\mu}(q_1,\lambda_1)e_{\gamma^*}^{\nu}(q_2,\lambda_2)H_{\mu\nu}$$

- Three independent helicity amplitudes: H_{++}, H_{+-}, H_{+0}

Helicity partial waves

Partial-wave projection in terms of t-variable:

$$h_{J,\lambda_1\lambda_2}(s) = \frac{1}{2}\int_{-1}^{1} H_{\lambda_1\lambda_2}(s,t) d^J_{\lambda_1-\lambda_2}(\theta)\,\mathsf{d}\cos\theta$$

$$= \frac{s}{(q^2-s)\lambda_{12}(s)}\int_{t_-(s)}^{t_+(s)} \mathsf{d}t H_{\lambda_1\lambda_2}(s,t) d^J_{\lambda_1-\lambda_2}(z(t))$$

$$t_\pm(s) = m_1^2 + \frac{q^2-s}{2s}(s+m_1^2-m_2^2 \pm \lambda_{12}(s))$$

$$\lambda_{12}(s) = \sqrt{(s-(m_1+m_2)^2)\,(s-(m_1-m_2)^2)}$$

- Deal with potential kinematic singularities!
- Comply with low-energy theorems:
  - Soft-photon theorem — Low, 1958
  - Soft-pion theorem — Adler, 1965

Dispersive analysis

Coupled-channel Muskhelishvili–Omnès (MO) dispersive representation incorporating constraints from analyticity, unitarity, and crossing:

Muskhelishvili, 1953, Omnès, 1958

- Final-state interactions via $\pi\pi/K\bar{K}$ & $\pi\eta/K\bar{K}$ T-matrices
  - Provide the coupled-channel Omnès matrices $\mathbf{\Omega}(s)$
  - Unitarity right-hand cuts from $s \geq (m_1 + m_2)^2$

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  - Unitarity right-hand cuts from $s \geq (m_1 + m_2)^2$
- Contributions from left-hand cuts via Born term and vector-meson exchanges García-Martín, Moussallam, 2010, Hoferichter, Stoffer, 2019 Danilkin, Deineka, Vanderhaeghen, 2019
  - Real and complex cuts for time-like q^2!
  - Overlap of left- and right-hand cuts! Moussallam, 2021
  - Violation of Watson's theorem! Watson, 1954

Partial-wave dispersion relations

Final representations for S-wave:

- Standard MO representation:

$$\begin{pmatrix} h^I_{0,\lambda\lambda'}(s) \\ k^I_{0,\lambda\lambda'}(s) \end{pmatrix} = \begin{pmatrix} h^{I,\text{Born}}_{0,\lambda\lambda'}(s) \\ k^{I,\text{Born}}_{0,\lambda\lambda'}(s) \end{pmatrix} + \begin{pmatrix} h^{I,V}_{0,\lambda\lambda'}(s) \\ k^{I,V}_{0,\lambda\lambda'}(s) \end{pmatrix} + (s-q^2)\,\boldsymbol{\Omega}^I_0(s)\Bigg[- \int_{s_{\text{thr}}}^{\infty} \frac{\mathrm{d}s'}{\pi} \frac{\mathrm{Im}\left(\boldsymbol{\Omega}^I_0(s')^{-1}\right)}{(s'-s)(s'-q^2)} \left(\begin{pmatrix} h^{I,\text{Born}}_{0,\lambda\lambda'}(s') \\ k^{I,\text{Born}}_{0,\lambda\lambda'}(s') \end{pmatrix} + \begin{pmatrix} h^{I,V}_{0,\lambda\lambda'}(s') \\ k^{I,V}_{0,\lambda\lambda'}(s') \end{pmatrix} \right) \Bigg]$$

- Modified MO representation:

$$\begin{pmatrix} h^I_{0,\lambda\lambda'}(s) \\ k^I_{0,\lambda\lambda'}(s) \end{pmatrix} = \begin{pmatrix} h^{I,\text{Born}}_{J,\lambda\lambda'}(s) \\ k^{I,\text{Born}}_{0,\lambda\lambda'}(s) \end{pmatrix} + (s-q^2)\,\boldsymbol{\Omega}^I_J(s)\Bigg[- \int_{s_{\text{thr}}}^{\infty} \frac{\mathrm{d}s'}{\pi} \frac{\mathrm{Im}\left(\boldsymbol{\Omega}^I_J(s')^{-1}\right)}{(s'-s)(s'-q^2)} \begin{pmatrix} h^{I,\text{Born}}_{0,\lambda\lambda'}(s') \\ k^{I,\text{Born}}_{0,\lambda\lambda'}(s') \end{pmatrix} + \int_{\mathcal{C}_L} \frac{\mathrm{d}s'}{\pi} \frac{\boldsymbol{\Omega}^I_J(s')^{-1}}{(s'-s)(s'-q^2)} \mathrm{Im} \begin{pmatrix} h^{I,V}_{0,\lambda\lambda'}(s') \\ k^{I,V}_{0,\lambda\lambda'}(s') \end{pmatrix} \Bigg]$$

- Benefit from equivalence between two!

Status

Data availability:

- ϕ radiative decays: ✓
- J/ψ radiative decays: managed to obtain normalized data ✓

Omnès Input for formalism:

- $\pi\pi/K\bar{K}$: ✓ Danilkin, Deineka, Vanderhaeghen, 2021
- $\pi\eta/K\bar{K}$: ✓ Deineka, Danilkin, Vanderhaeghen, 2024

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Left-hand cuts for formalism:

- Born contribution: ✓̶
- Vector exchanges with narrow or finite widths: ✓
- Vector exchanges with dispersively improved Breit–Wigner: ✓̶
- Full treatment of 3π: ✗

Constraints on scenarios beyond the Standard Model: ✗

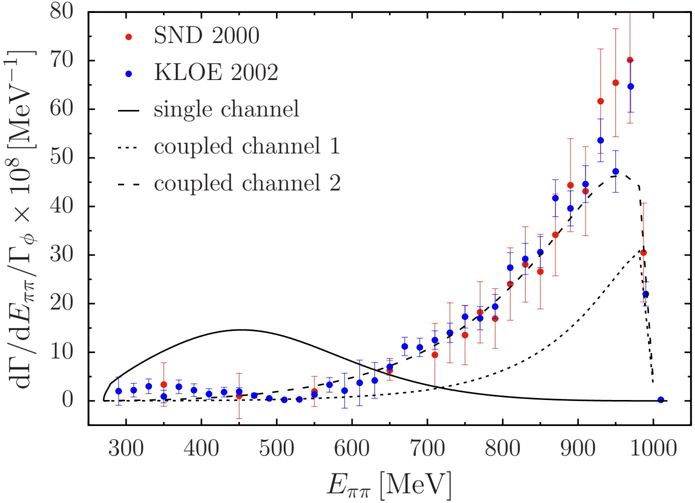
Status

Decay modes	$\phi \to \gamma\pi^0\pi^0$	$\phi \to \gamma\pi^0\eta$	$\phi \to \gamma K^0\bar{K}^0$
Status	ongoing	ongoing	prediction

Decay modes	$J/\psi \to \gamma\pi^0\pi^0$	$J/\psi \to \gamma K_S^0 K_S^0$	$J/\psi \to \gamma\pi^0\eta$
Status	ongoing	2nd?	2nd?

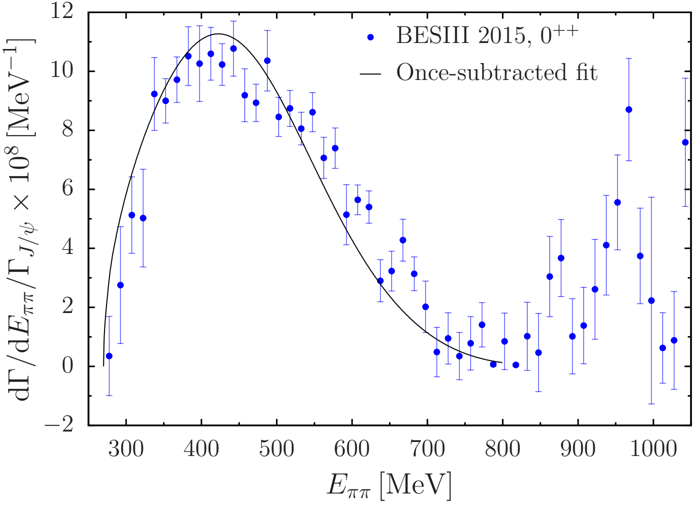
Status

Preliminary prediction for the $\phi \to \gamma\pi^0\pi^0$ spectrum:



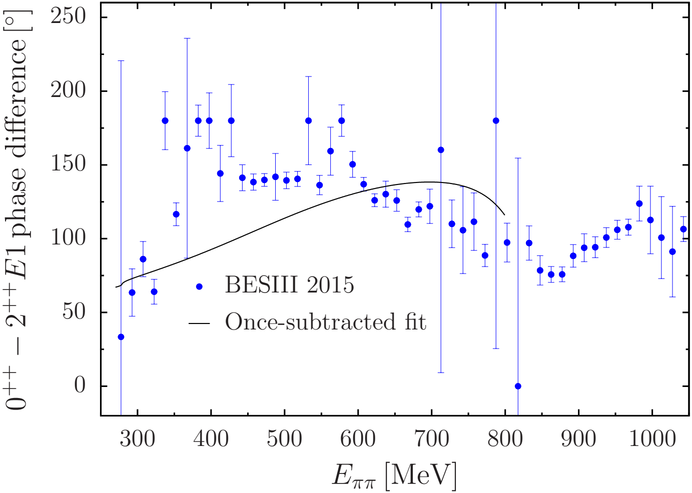
Status

Preliminary fit to the $J/\psi \to \gamma\pi^0\pi^0$ spectrum with one subtraction constant:



Status

Preliminary fit to the $J/\psi \to \gamma\pi^0\pi^0$ phase with one subtraction constant:



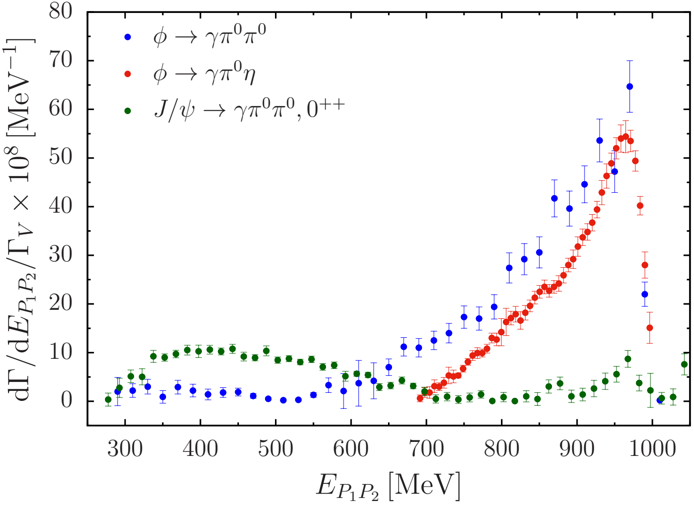
- First observation of strong deviation of phase shift from that of $\pi\pi$!

Conclusion & outlook

- Dispersive formalism for time-like processes $\gamma^* \to \gamma P_1 P_2$
- Numerical treatment using standard and modified MO approaches
- Correct implementation of helicity amplitudes and helicity sums in line with experiments
- Improved analysis of left-hand cuts
- Constraints on Leptophobic B-bosons
- Threshold J/ψ photoproduction on the nucleon

Status

Comparison of different decays:



Fit to data

Differential decay width:

$$\frac{\mathrm{d}\Gamma_{\phi\to\gamma\pi\eta}}{\mathrm{d}\sqrt{s}} = \frac{(q^2-s)\lambda_{\pi\eta}(s)}{384\pi^3 q^3\sqrt{s}} \int_{-1}^{1} \mathrm{d}z \left(\left| l_{0++}(s) + \sum_V \tilde{L}^V_{++}(s,z) \right|^2 + \left| \sum_V L^V_{+-}(s,z) \right|^2 + \left| \sum_V L^V_{+0}(s,z) \right|^2 \right)$$

$$\chi^2(\phi) = \sum_{i=1}^{47} \frac{1}{\sigma_i^2} \left(\frac{1}{\Gamma_\phi} \frac{\mathrm{d}\Gamma_{exp}}{\mathrm{d}E_i} \Delta E - \frac{1}{\Gamma_\phi} \int_{E_i-\Delta E/2}^{E_i+\Delta E/2} \mathrm{d}\sqrt{s'} \frac{\mathrm{d}\Gamma_{th}}{\mathrm{d}\sqrt{s'}} \right)^2$$

Data fits

Unsubtracted representation as phase prediction:

$$h^0_{0,++}(s) = N\left(s-q^2\right)\Omega^0_0(s)\left(-\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}s'}{\pi}\frac{\mathrm{Im}\left(\Omega^0_0\left(s'\right)^{-1}\right)}{s'-s}\frac{h^{0,\omega}_{0,++}\left(s'\right)}{s'-q^2}\right)$$

- $\mathrm{Arg}[h^0_{0,++}(4M^2_{\pi^0})] = -0.86(x)$

Once-subtracted representation, fit normalization and phase:

$$h^0_{0,++}(s) = N\left(s-q^2\right)\Omega^0_0(s)\left(e^{i\varphi}-\frac{s-s_0}{\pi}\int_{4M_\pi^2}^{\infty}\mathrm{d}s'\frac{\mathrm{Im}\left(\Omega^0_0\left(s'\right)^{-1}\right)}{(s'-s_0)(s'-s)}\frac{h^{0,\omega}_{0,++}\left(s'\right)}{s'-q^2}\right)$$

- Choose $s_0 = 4M^2_{\pi^0}$
- $\mathrm{Arg}\left[h^0_{0,++}\left(4M^2_{\pi^0}\right)\right] = -\pi+\varphi$ for $\varphi > 0$
- $\varphi = 1.17(2)$
- $\arg(z+w) = \arctan\frac{z_I+w_I}{z_R+w_R}$

Partial-wave dispersion relations

$\pi\eta$–$K\bar{K}$ final-state interactions in terms of Omnès matrix $\mathbf{\Omega}(s)$:

$$\mathrm{Im}\,\mathbf{\Omega}(\mathbf{s}) = \mathbf{T}^*(s)\mathbf{\Sigma}(s)\mathbf{\Omega}(s)$$

$$\mathbf{\Omega}(s) = \frac{1}{\pi}\int_{m_+^2}^{\infty} \frac{\mathbf{T}^*(s')\mathbf{\Sigma}(s')\mathbf{\Omega}(s')}{s'-s}\,\mathrm{d}s'$$

- Integral equation of Fredholm type
- $\mathbf{\Omega}(0) = 1$
- $\delta_{\pi\eta} + \delta_{K\bar{K}} \to 2\pi$ at infinity
- Numerical solution
- $\det\mathbf{\Omega}(s) = \exp\left[\frac{s}{\pi}\int_{m_+^2}^{\infty} \frac{\delta_{\pi\eta}(s') + \delta_{K\bar{K}}(s')}{s'(s'-s)}\,\mathrm{d}s'\right]$ as cross-check
- $\mathbf{D}(s) = \mathbf{\Omega}(s)^{-1}$

Chew, Mandelstam, 1960

Partial-wave dispersion relations

Born term:

$$k_{0++}^{1,\text{Born}}(s) = \alpha_B \frac{8M_{K^+}^2 I_{K^+}(s) - 2q^2}{s - q^2}, \quad \alpha_B = -\frac{e\, g_{\phi KK}}{\sqrt{2}}$$

- Cut on the negative real axis and pole at $s = q^2$

$$\text{Im}\, I_K(s + i\epsilon) = \frac{\pi}{2\sqrt{1 - 4M_{K^+}^2/s}}\theta(-s)$$

- Dispersive representation

$$k_{0++}^{1,\text{Born}}(s) = \alpha_B \Big[\frac{\beta(q^2)}{s - q^2} + \gamma(q^2) + (s - q^2)\int_{-\infty}^{0} \frac{4M_{K^+}^2}{(s' - s)(s' - q^2)^2\sqrt{1 - 4M_{K^+}^2/s'}}\text{d}s'\Big]$$

$$\beta(q^2) = 8M_{K^+}^2 I_{K^+}(q^2) - 2q^2, \quad \gamma(q^2) = 8M_{K^+}^2 I'_{K^+}(q^2)$$

Partial-wave dispersion relations

Vector-exchange $\rho, \omega, \phi,\ K^*$:

- ω-contribution as an example

$$l^{\omega}_{0++}(s) = \frac{e\, C_{\omega\pi\gamma} g_{\phi\omega\eta}}{2}\, \mathcal{F}_V(s, q^2, M_\omega^2, M_\pi^2, M_\eta)$$

$$\mathcal{F}_V(s, q^2, M_V^2, m_1^2, m_2^2) = \left[-M_V^2 + q^2 \left(\frac{M_V^2 - m_1^2}{q^2 - s} \right)^2 \right] L_V(s, q^2, M_V^2, m_1^2, m_2^2) + \frac{q^2}{2}\left(1 + \frac{m_1^2 - m_2^2}{s} + 2\,\frac{M_V^2 - m_1^2}{q^2 - s}\right) \pm (s - q^2)$$

$$L_V(s, q^2, M_V^2, m_1^2, m_2^2) = \frac{s}{\lambda_{12}(s)} \left[\log(M_V^2 - t_+(s, q^2, m_1^2, m_2^2)) - \log\big(M_V^2 - t_-(s, q^2, m_1^2, m_2^2)\big) \right]$$

- Pole only at $s = 0$ and logarithmic branch points!

Partial-wave dispersion relations

$$s_{\pm}(t,q^2,m_1^2,m_2^2) = q^2 - \frac{t-m_1^2}{2t}\left(q^2+t-m_2^2 \pm \sqrt{\lambda(t,q^2,m_2^2)}\right):$$

- $q \leq M_V - m_2$:
  - two cuts at $(-\infty, s_+]$ and $[s_-, 0]$
  - Reproduces the case of $\gamma^*(-Q^2)\gamma(0) \to P_1 P_2$
- $M_V - m_2 \leq q \leq M_V + m_2$:
  - Complex cuts
- $q \geq M_V + m_2$:
  - Left-hand cuts overlapping with unitarity cut
  - Violation of Watson's theorem!