Progress report on TFF-2,4 and JRP-1 (phenomenology)

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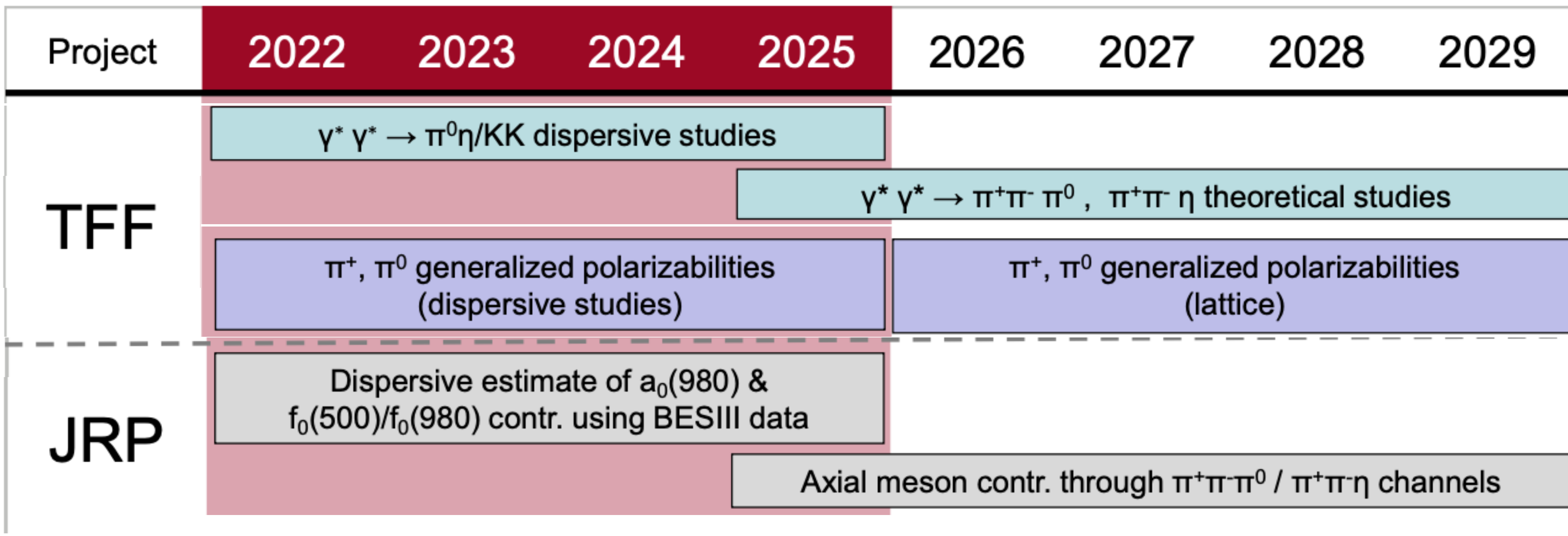
Overview

- TFF-2 - Extension of the dispersive formalism to $\gamma^*\gamma^* \to \pi\eta/K\bar{K}$
- TFF-4 - Pion generalized polarizabilities as a function of one virtuality
- JRP-1 - First dispersive estimate of $a_0(980)$ to $(g-2)_\mu$ and updated $f_0(500)/f_0(980)$

} 1 postdoc

- JRP-1 - Axial-vector meson contributions to a_μ
- TFF-2 - Extension of the dispersive formalism to $\gamma^*\gamma^* \to \pi^+\pi^-\pi^0$

} (no funding) talk by Xiu-Lei Ren



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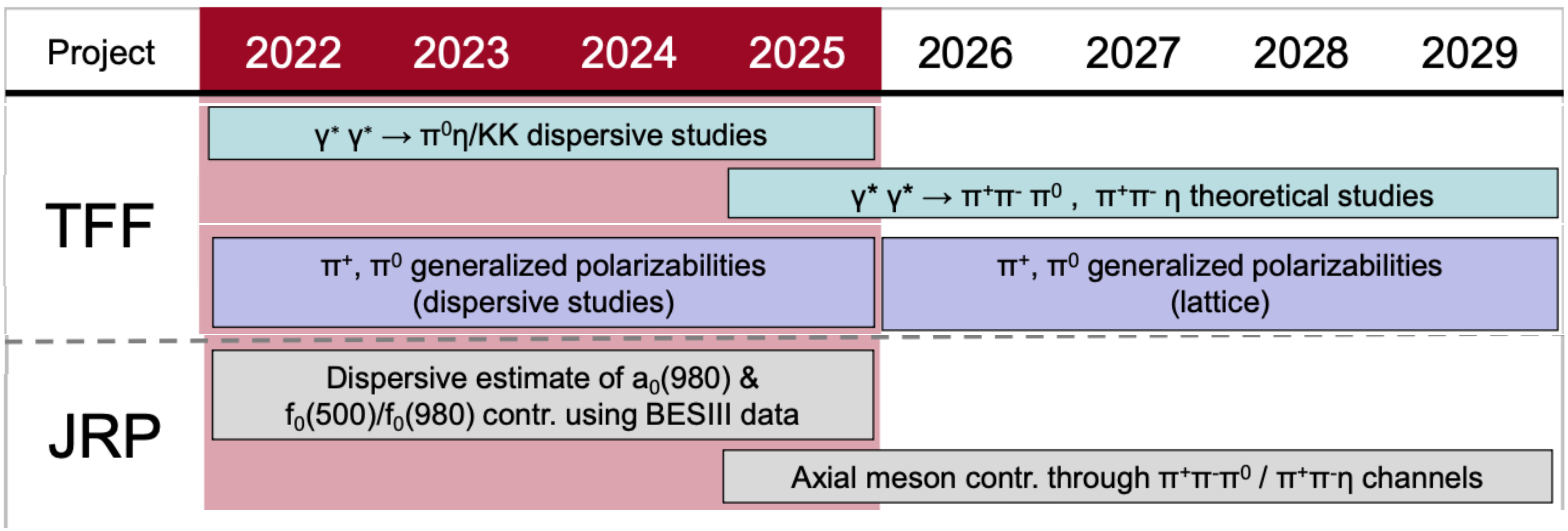


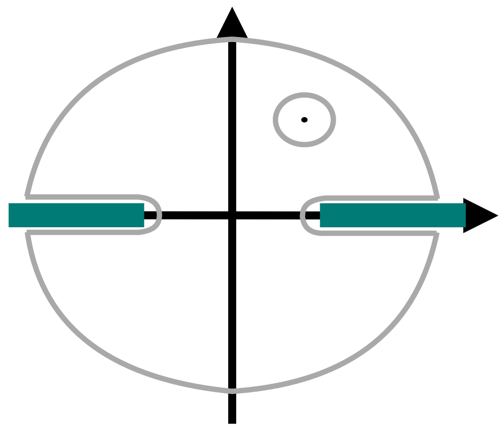
- **Why it matters?** precision dispersive HLbL piece of a_μ, hadron structure observables (form factors, polarisabilities), dispersive determination of $a_0(980)$ resonance parameters

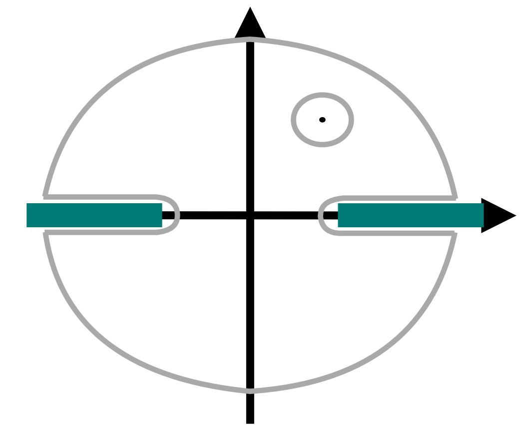
- **Bottleneck:** Results are ❗ limited by lack of BESIII single-virtual data for all three tasks

Framework

- Strategy: Apply dispersion relations to the **kinematically unconstrained** p.w. amplitudes

$$t_{ab}(s) = \int_L \frac{ds'}{\pi} \frac{\text{Im}\, t_{ab}(s')}{s' - s} + \sum_c \int_R \frac{ds'}{\pi} \frac{t_{ac}(s')\, \rho_c(s')\, t^*_{cb}(s')}{s' - s}$$





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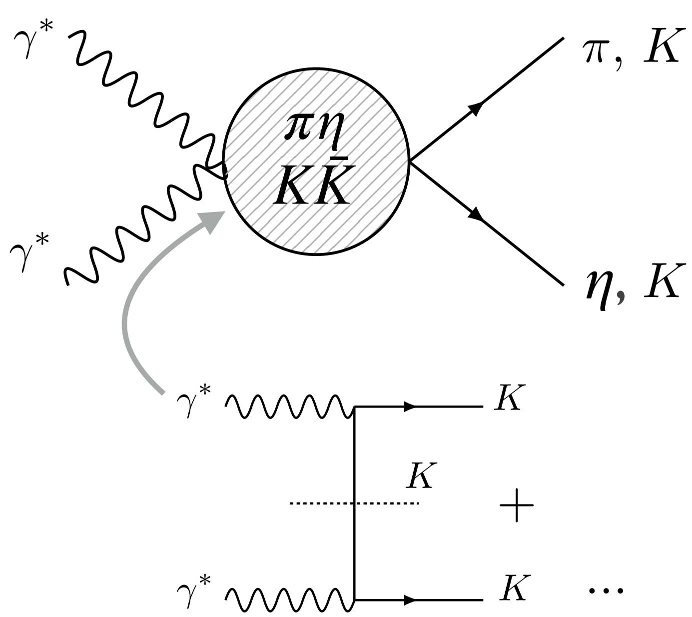
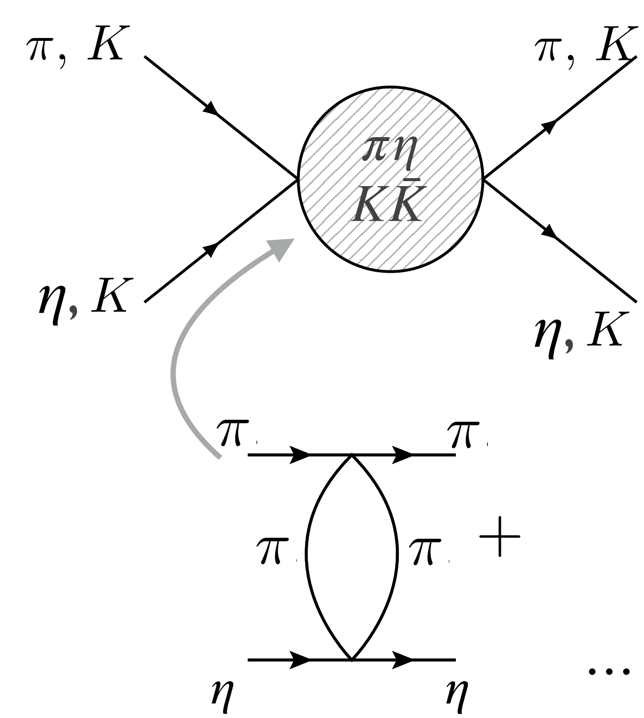
- Key features:
  1) **Left-hand cuts differ** for each channel
  2) The coupled-channel system $\gamma^*\gamma^*/\pi\eta/K\bar{K}$ can be reduced to to the off-diagonal $\gamma^*\gamma^* \to \pi\eta/K\bar{K}$, which requires the hadronic rescattering input





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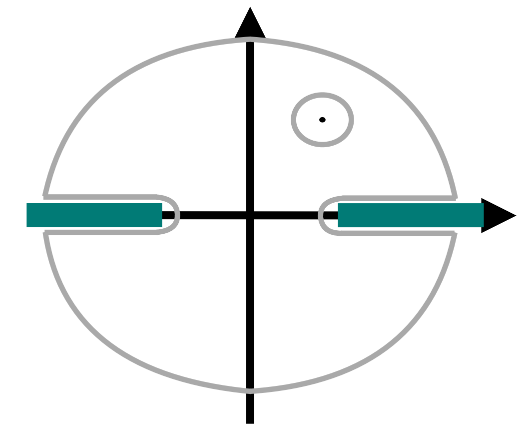

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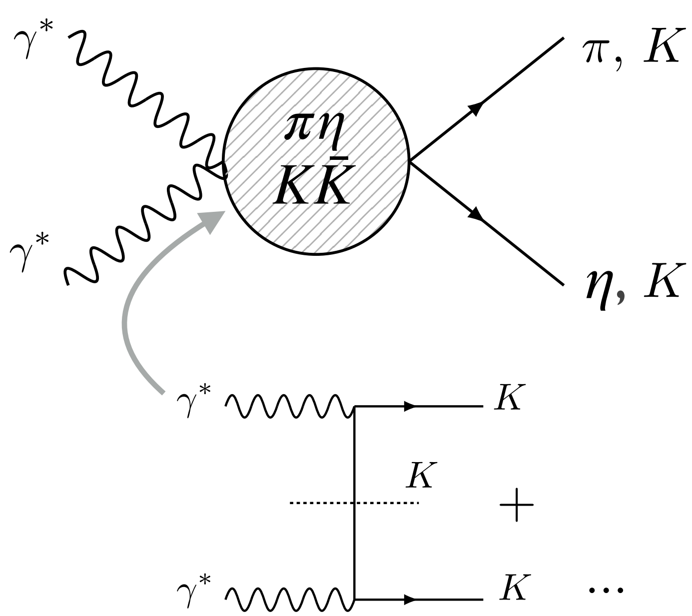
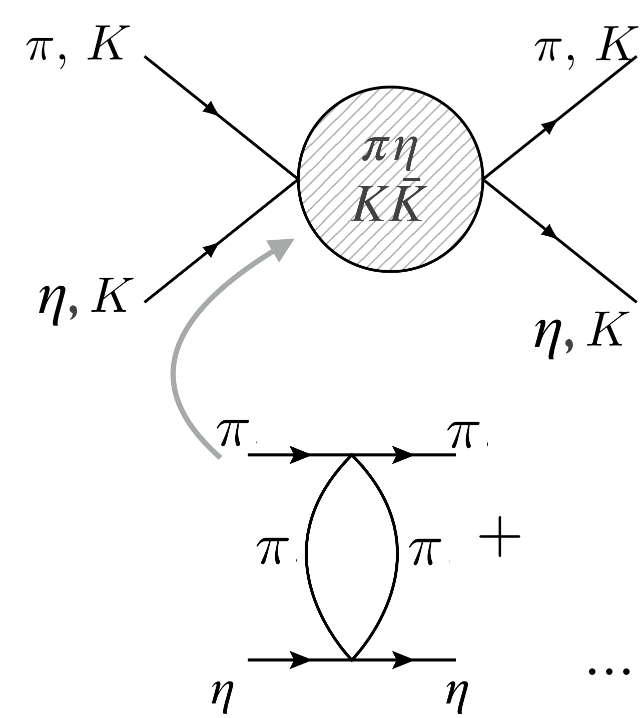
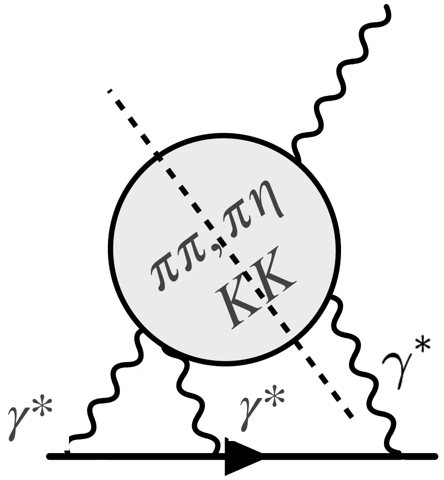
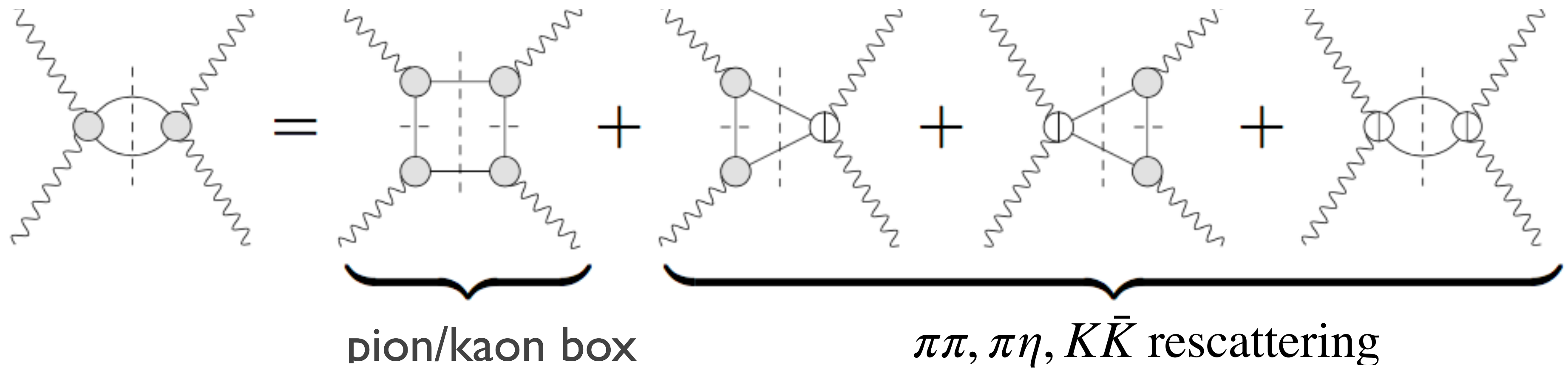




- Details of the analysis:
  1) $\pi\eta/K\bar{K}$ solved via **N/D method** with **conformal mapping** for left-hand cuts (subtracted)
  2) $\gamma^*\gamma^* \to \pi\eta/K\bar{K}$ solved using **modified Muskhelishvili-Omnès formalism** (subtracted and unsubtr.)

Challenges (TFF-2, JRP-1)





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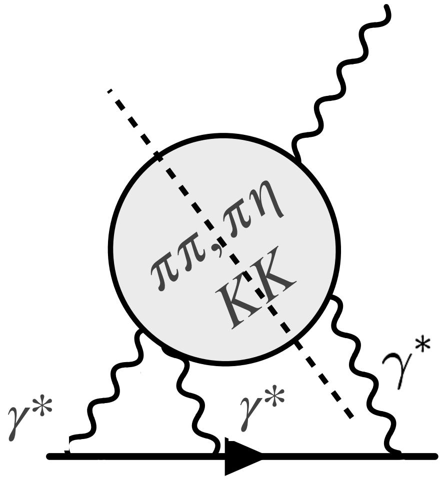
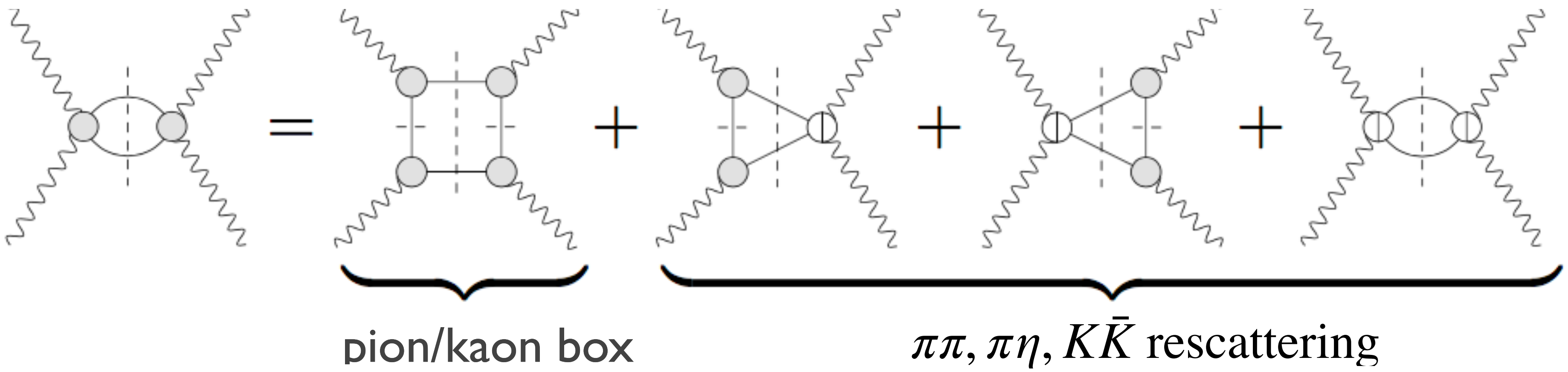




[Colangelo et al. (2014-2017), I.D. et al. (2019, 2021)]

$a_\mu[\text{S-wave}, I=0]_{\pi\pi} = -9.3(0.9) \times 10^{-11}$ $f_0(500)$

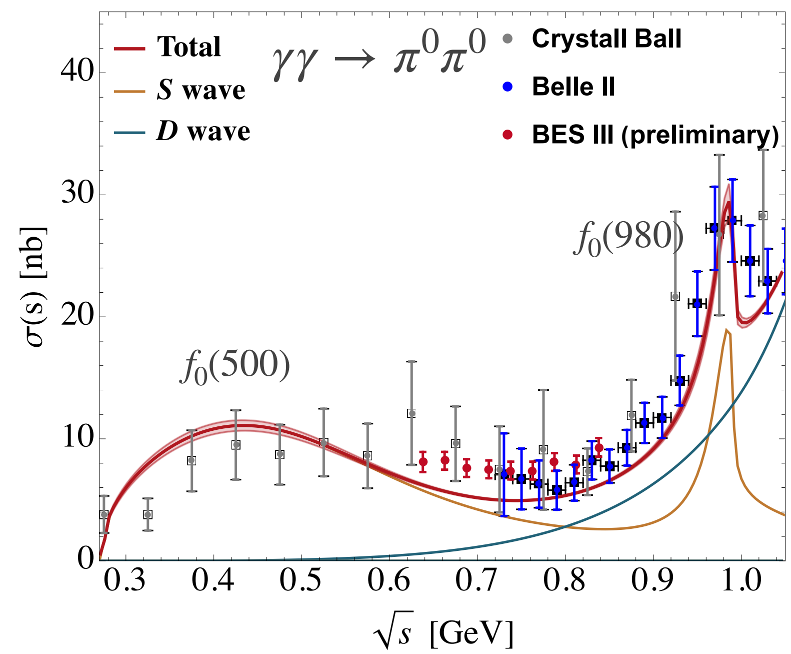
$a_\mu[\text{S-wave}, I=0]_{\pi\pi,K\bar{K}} = -9.8(1.0) \times 10^{-11}$ $f_0(500)/f_0(980)$

Unsubtracted dispersion relation for $\gamma^*\gamma^* \to \pi\pi/K\bar{K}$

Left-hand cuts: π/K pole with vector form factors $F_{\pi,K}(Q^2)$

Data used: $\pi\pi/K\bar{K}$ scattering data (Roy analyses)

$\gamma\gamma \to \pi^0\pi^0$ used to justify left-hand cut approximation



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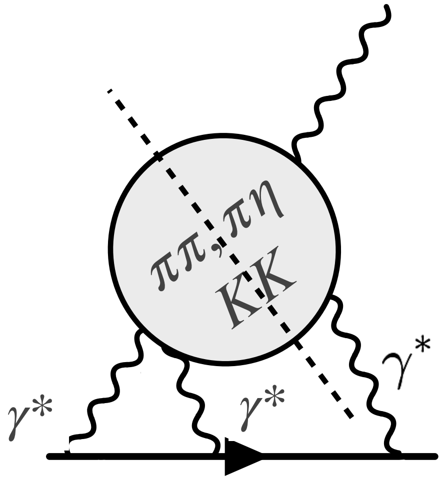

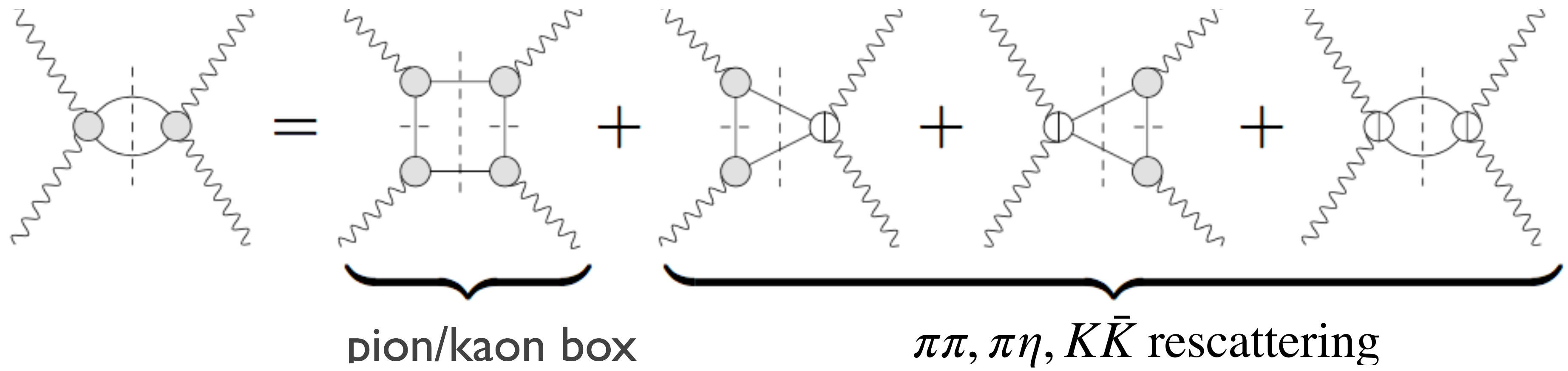

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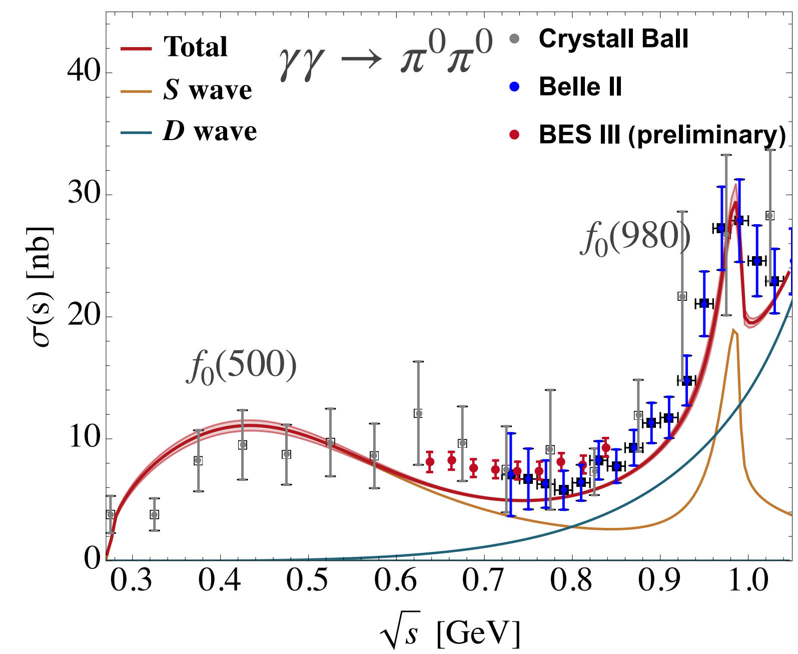


$a_\mu[\text{S-wave}, I=1]_{\pi\eta,K\bar{K}} = ?$ $a_0(980)$

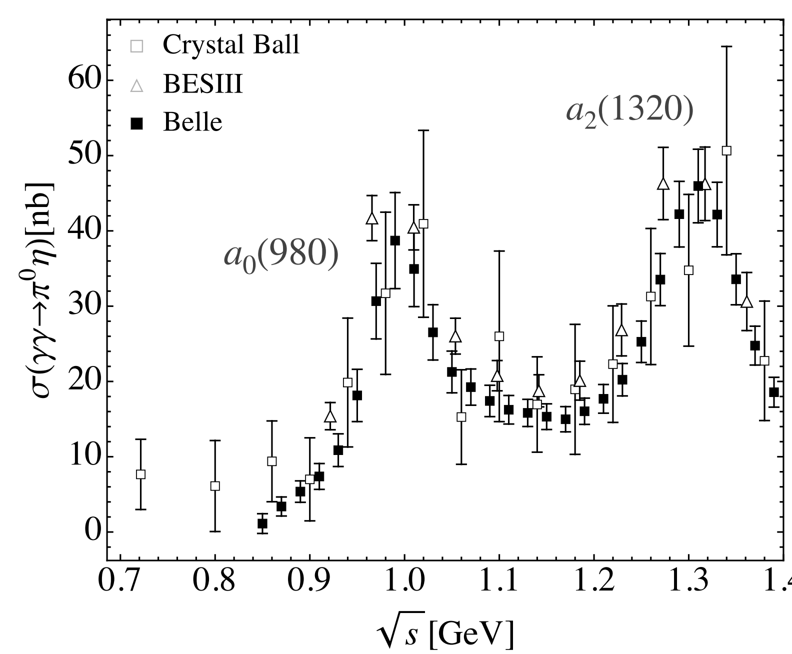
Unsubtracted dispersion relation for $\gamma^*\gamma^* \to \pi\eta/K\bar{K}$

Left-hand cuts: K pole with vector form factor $F_K(Q^2)$

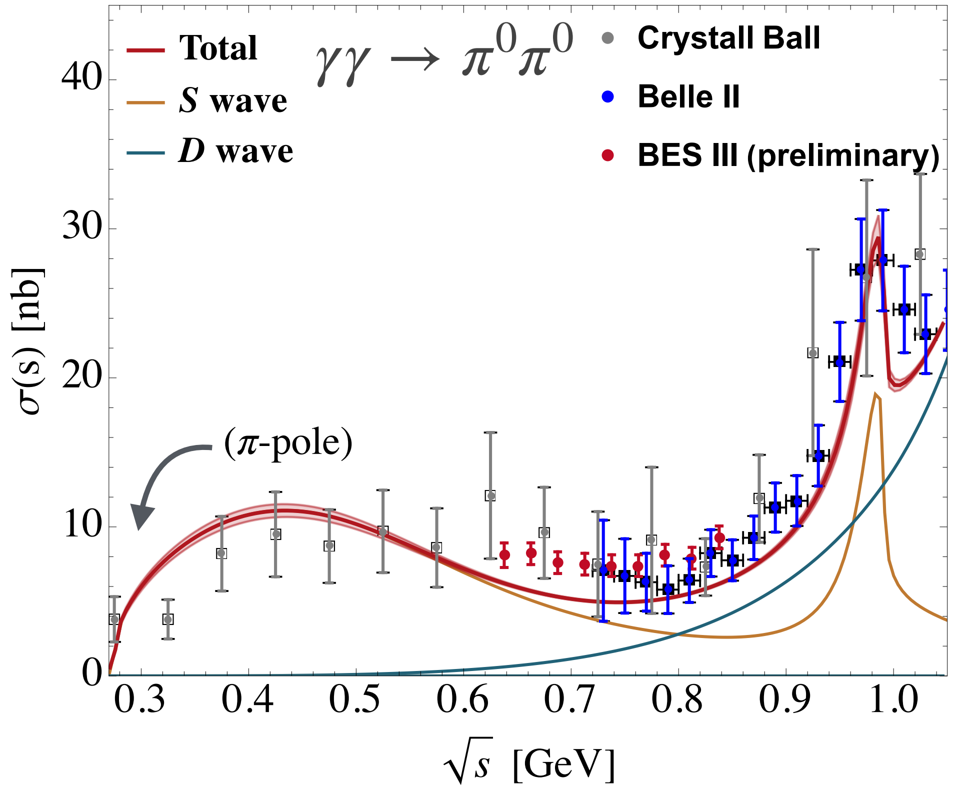
Challenge: ❗ no direct $\pi\eta/K\bar{K}$ scattering data

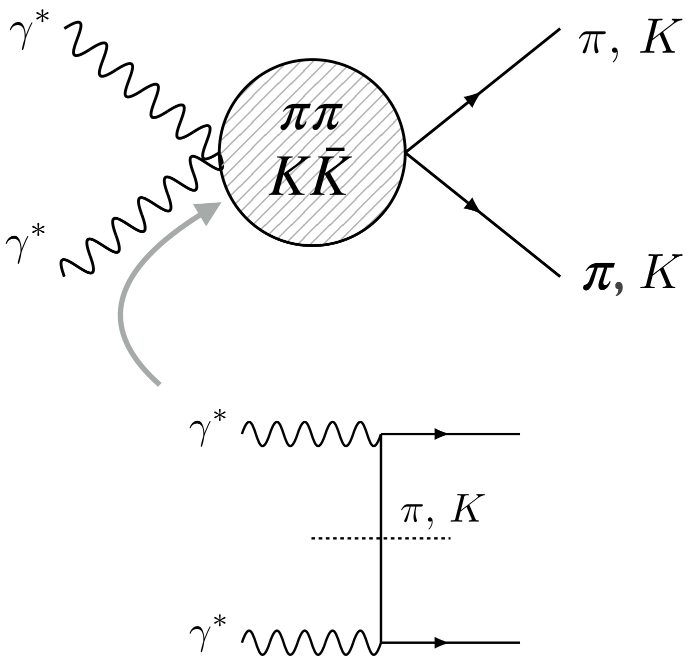
$\gamma\gamma \to \pi\eta, K\bar{K}$ data used to constraint $\pi\eta/K\bar{K}$ amplitude

Required check: assess importance of heavier left-hand cuts



Pion polarizabilities (TFF-4)

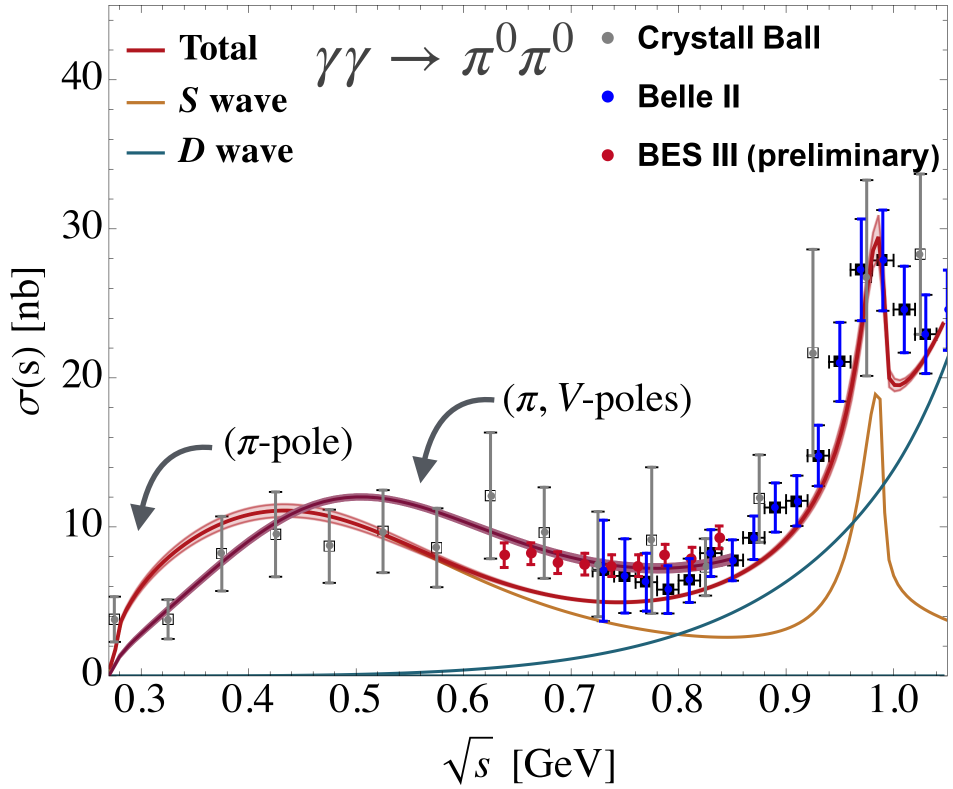


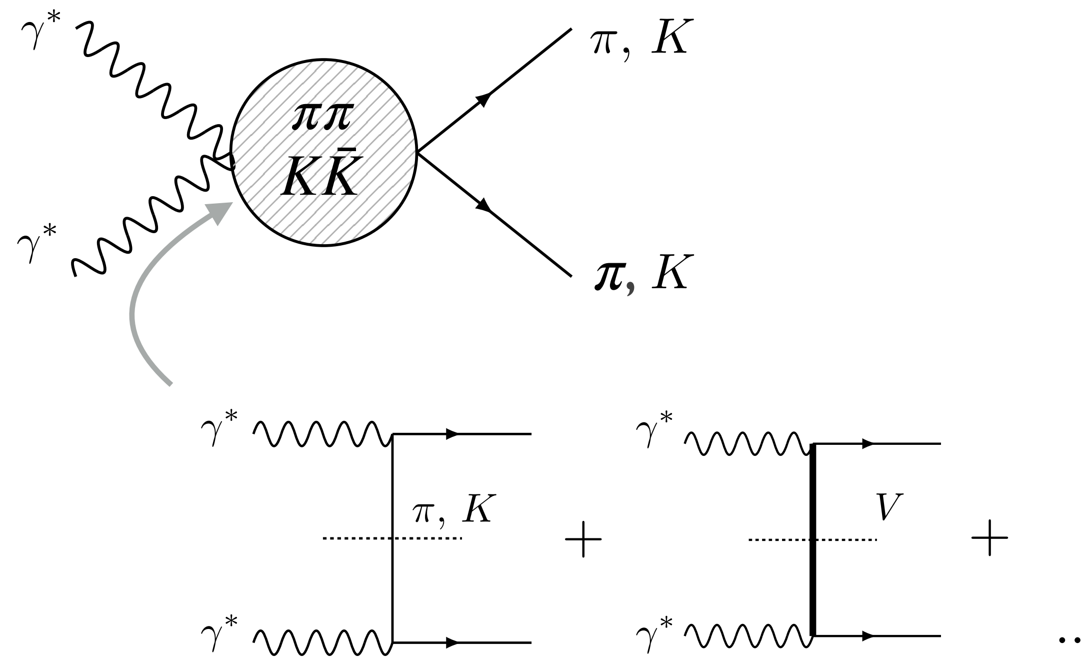


Unsubtracted dispersion relation for $\gamma\gamma \to \pi\pi/K\bar{K}$

- **Left-hand cuts**: π/K pole
- $\Gamma_{\gamma\gamma}(f_0(500), f_0(980))$ consistent with other analyses
- $(\alpha_1 - \beta_1)_{\pi^\pm}$ consistent with ChPT
- $(\alpha_1 - \beta_1)_{\pi^0} \sim 9 \times 10^{-4}\,\mathrm{fm}^3$ (no Adler zero $\gamma\gamma \to \pi^0\pi^0$) vs $(\alpha_1 - \beta_1)^{\chi PT}_{\pi^0} = -1.9(2) \times 10^{-4}\,\mathrm{fm}^3$

[Colangelo et al. (2017)]
[I.D. et al. (2019)]





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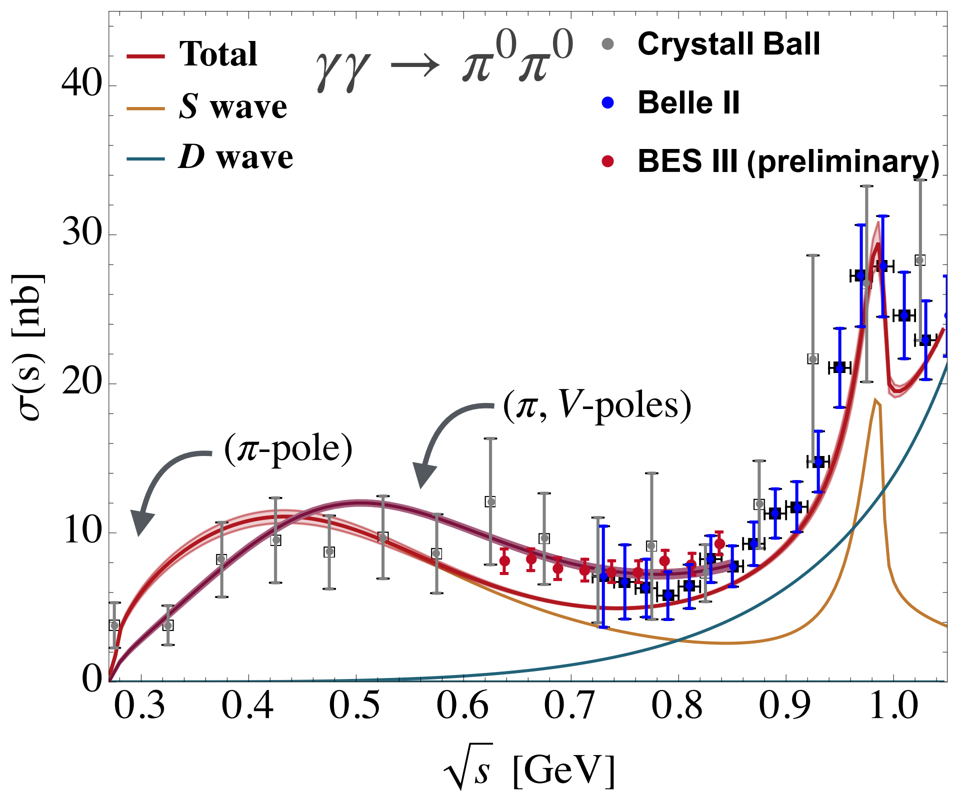
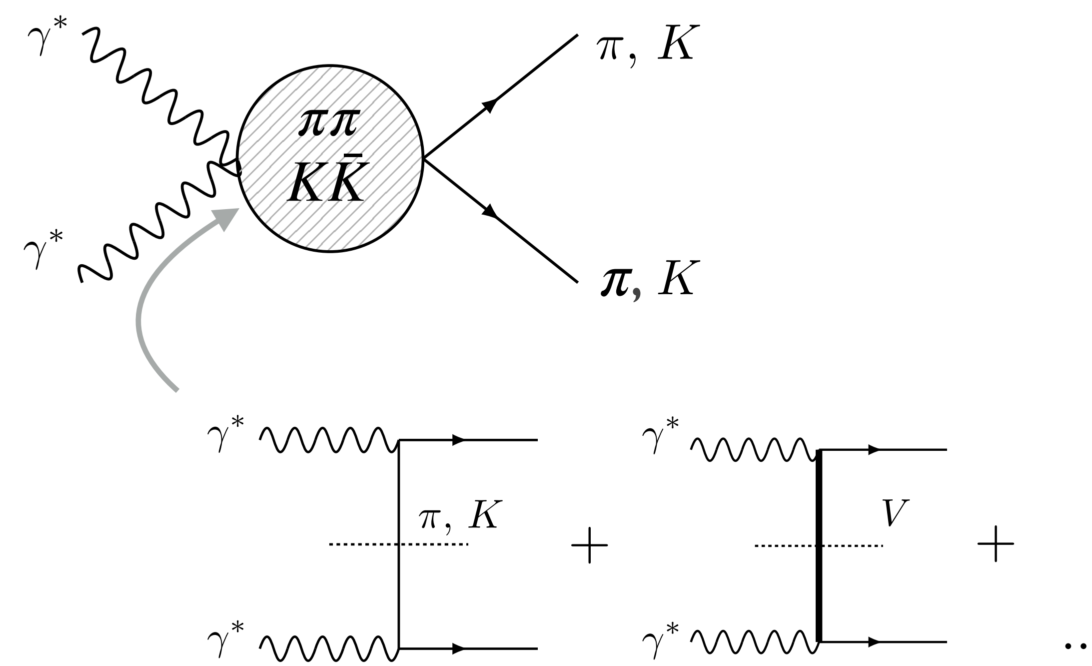
Subtracted dispersion relation for $\gamma\gamma \to \pi\pi/K\bar{K}$

- **Left-hand cuts**: π/K pole + V pole (ω exchange dominates)
- Accurate fit to $\gamma\gamma \to \pi^0\pi^0$ via subtraction constants
- Cure $(\alpha_1 - \beta_1)_{\pi^0}$ by inclusion of the Adler zero
- ❗ $d\sigma/d\cos\theta$ from BESIII is crucial for pion (possibly kaon) polarizabilities
- ❗ $\gamma\gamma^* \to \pi\pi$ data required to determine Q^2 dependence of $(\alpha_1 - \beta_1)_\pi$

[Garcia-Matrin et al. (2010)]
[Ermolina et al. EPJ Web Conf (2024)]

Pion polarizabilities (TFF-4)





		Dispersive			ChPT	Experiment
		Present work		Garcia-Martin et al.		
		π pole	π, V poles	π, V, A, T pole	NNLO	COMPASS
$(\alpha_1-\beta_1)_{\pi^\pm}$	$10^{-4}\mathrm{fm}^3$	5.1	$2.4\,(4)(3)^{+1.0}_{-0.0}$	4.7	5.7 (1.0)	$4.0\,(1.2)_{\rm stat}\,(1.4)_{\rm sys}$
$(\alpha_1-\beta_1)_{\pi^0}$	$10^{-4}\mathrm{fm}^3$	8.4	$-1.3\,(3)(0)^{+0.0}_{-0.3}$	−1.25 (17)	−1.9 (2)	–
$(\alpha_2-\beta_2)_{\pi^\pm}$	$10^{-4}\mathrm{fm}^5$	18.1	$16.5\,(4)(2)^{+2.1}_{-0.0}$	14.7 (2.1)	16.2/21.6	–
$(\alpha_2-\beta_2)_{\pi^0}$	$10^{-4}\mathrm{fm}^5$	24.8	$30.0\,(4)(3)^{+4.3}_{-0.0}$	32.1 (2.1)	37.6 (3.3)	–

Subtracted dispersion relation for $\gamma\gamma \to \pi\pi/K\bar{K}$

- **Left-hand cuts**: π/K pole + V pole (ω exchange dominates)
- Accurate fit to $\gamma\gamma \to \pi^0\pi^0$ via subtraction constants
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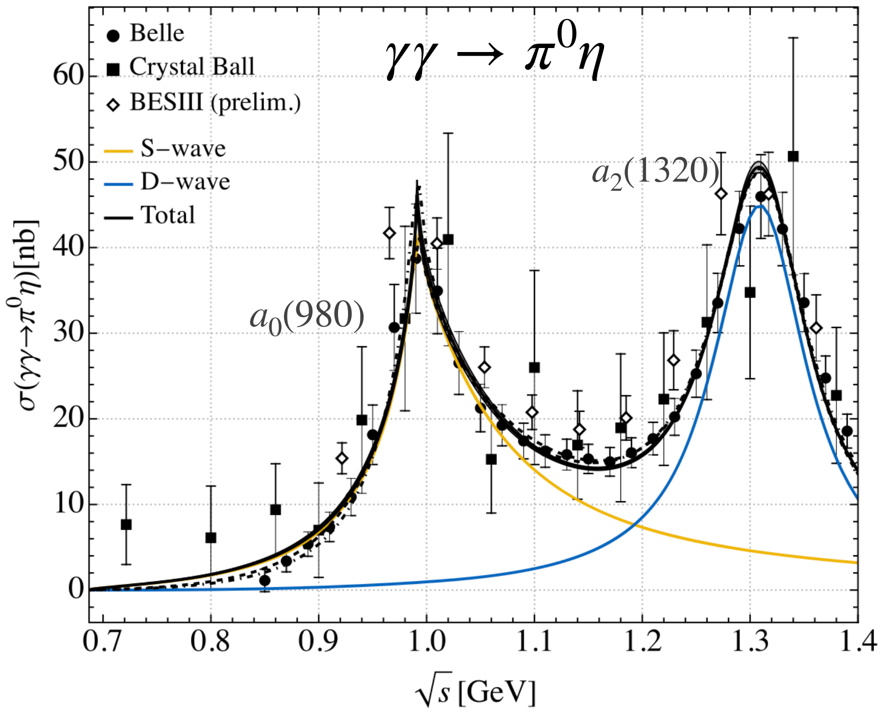
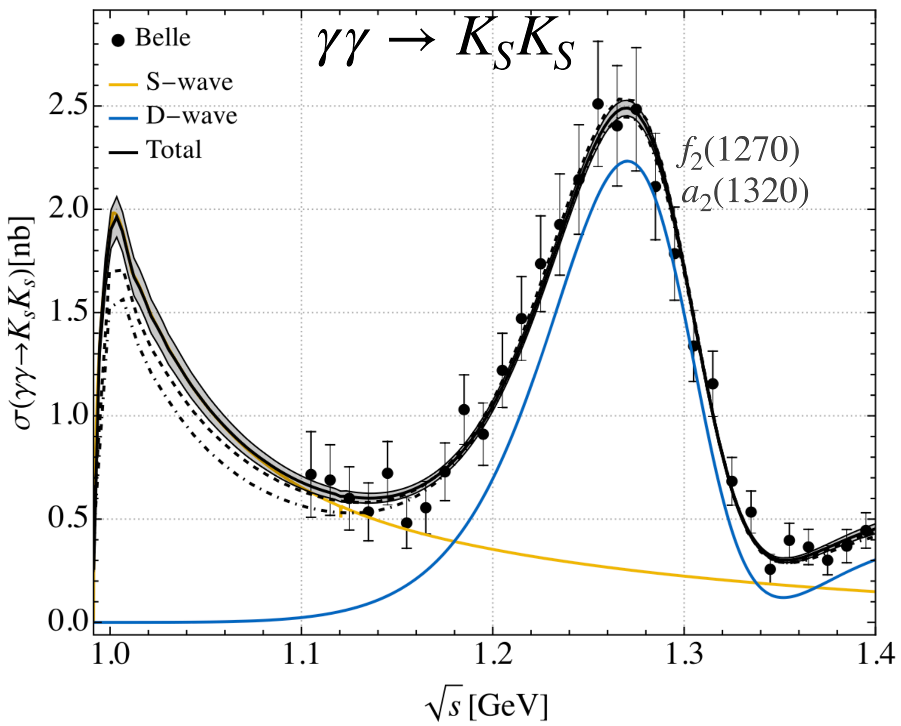
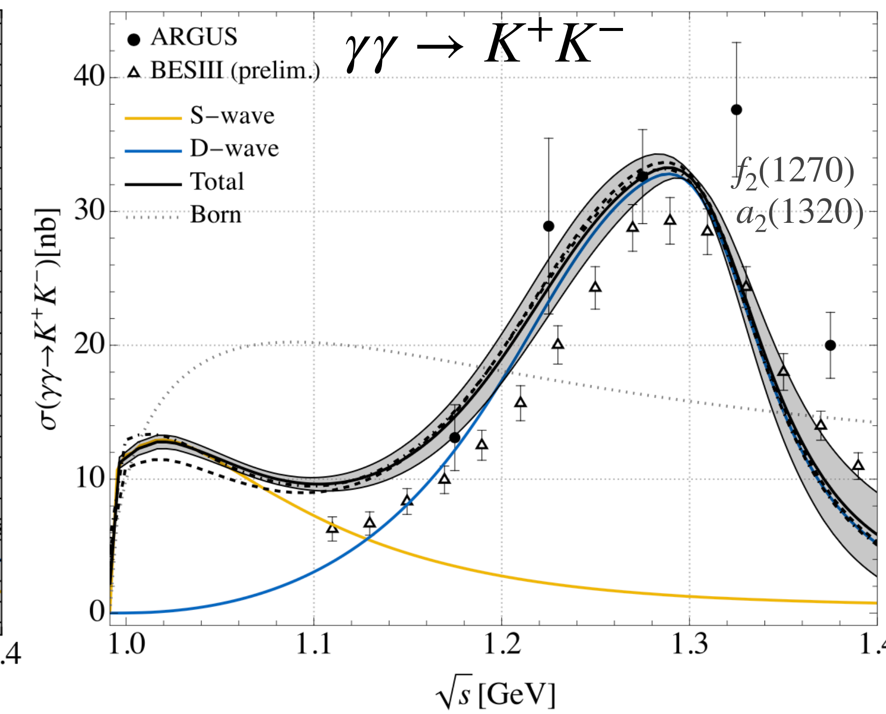
Results (TFF-2)







Chiral constraints on $\pi\eta/K\bar{K}$:

- Adler zero $\pi\eta \to K\bar{K}$
- $t_{\pi\eta\to\pi\eta}(s_{th}), t_{\pi\eta\to K\bar{K}}(s_{th})$

Unsubtracted dispersion relation (S-wave):

- **Left-hand cuts**: K pole
- no Adler zero $\gamma\gamma \to \pi^0\eta$
- Prediction for $\gamma^*\gamma^* \to \pi\eta, K\bar{K}_{I=1}$
- $a_\mu[\text{S-wave}, I=1]_{\pi\eta,K\bar{K}} = -0.44(5)\times 10^{-11}$

more details later

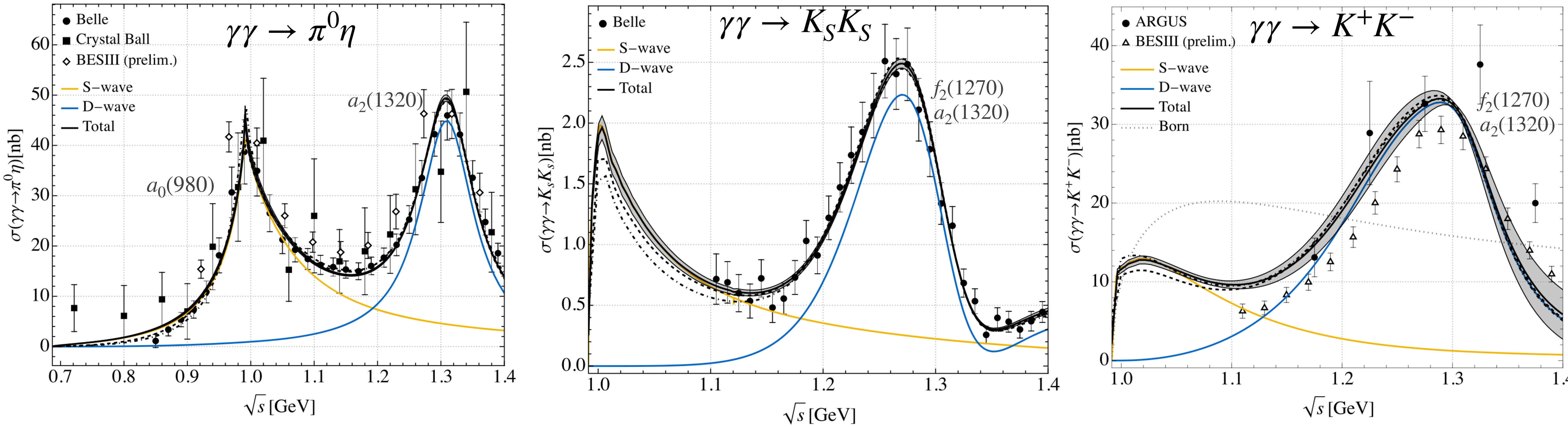
D-wave

- Breit-Wigner parametrization for $a_2(1320), f_2(1270)$

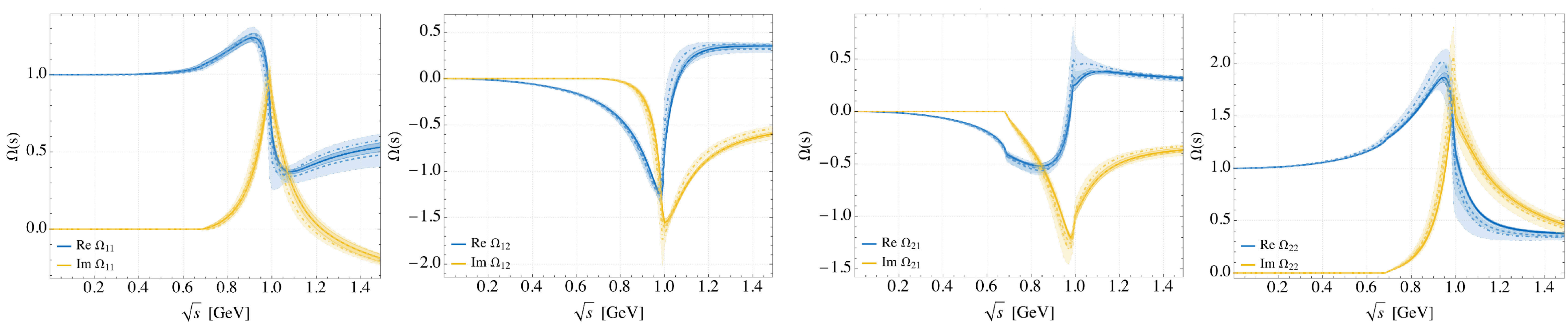
Subtracted dispersion relation (S-wave):

- **Left-hand cuts**: K pole + V pole
- Adler zero $\gamma\gamma \to \pi^0\eta$
- Require $\gamma^*\gamma^* \to \pi\eta$ data to determine Q^2 dependence of the subtraction constants

Results (TFF-2)



- **Unsubtracted** and **Subtracted** dispersion relations for $\gamma\gamma \to \pi\eta/K\bar{K}$ yield very similar $\pi\eta/K\bar{K}$ amplitudes



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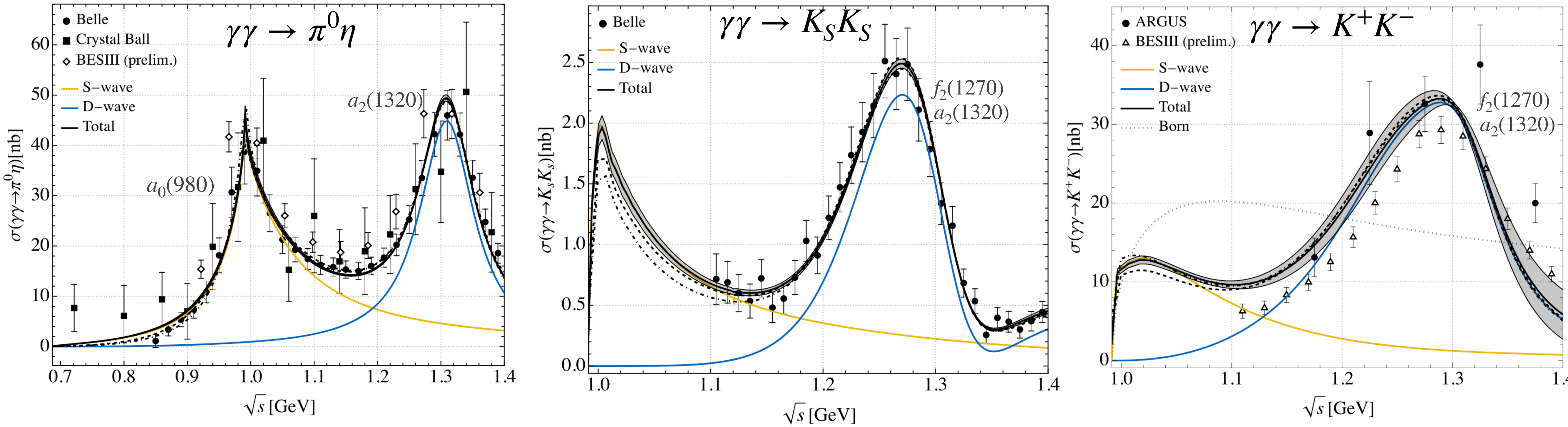

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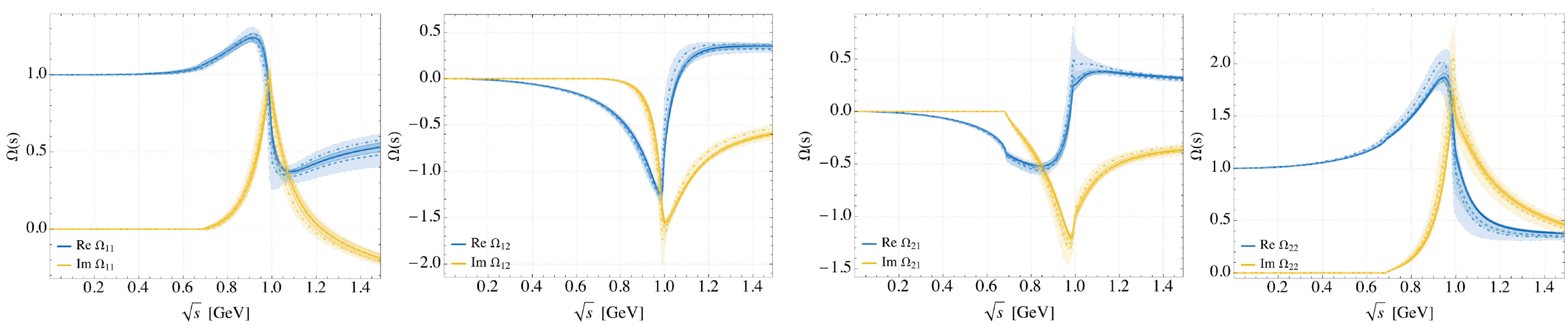
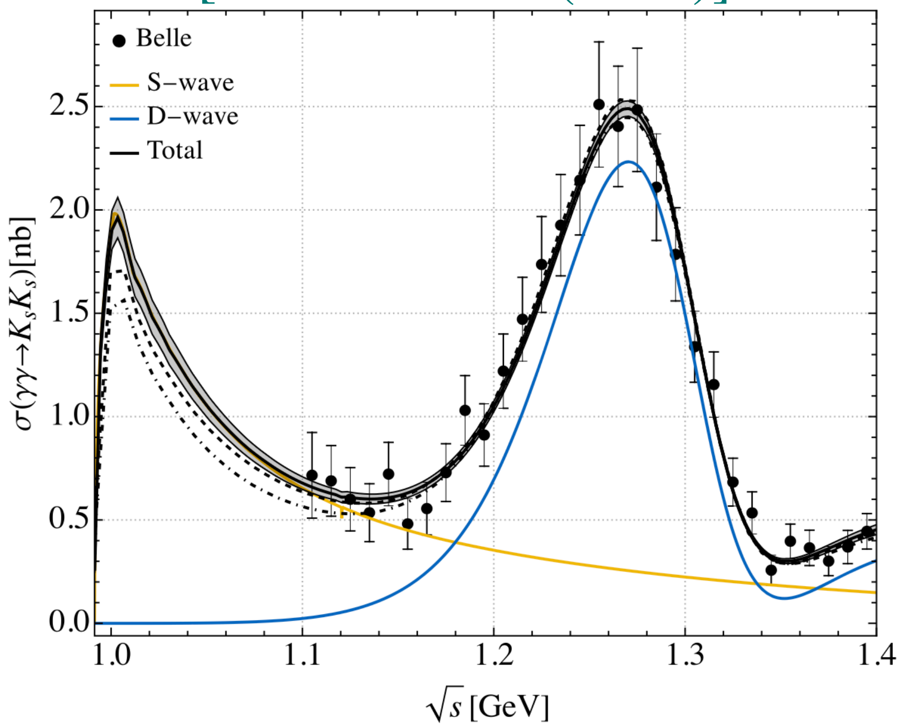


		Pole Position (MeV)	$\pi\eta$ (GeV)	$K\bar{K}$ (GeV)	$\gamma\gamma$ (MeV)
Deineka et al. 2024	RSII	$1047(18) - i\,72(17)$	3.8(3)	5.2(4)	7.3(5)
	RSIII	$930(25) - i\,80(10)$	2.9(1)	2.0(1)	8.9(3)
Lu et al. 2020	RSII	$1000^{+13}_{-1} - i\,37^{+13}_{-3}$	$2.2^{+0.6}_{-0.2}$	$4.0^{+0.3}_{-0.2}$	$5.0^{+0.9}_{-0.5}$

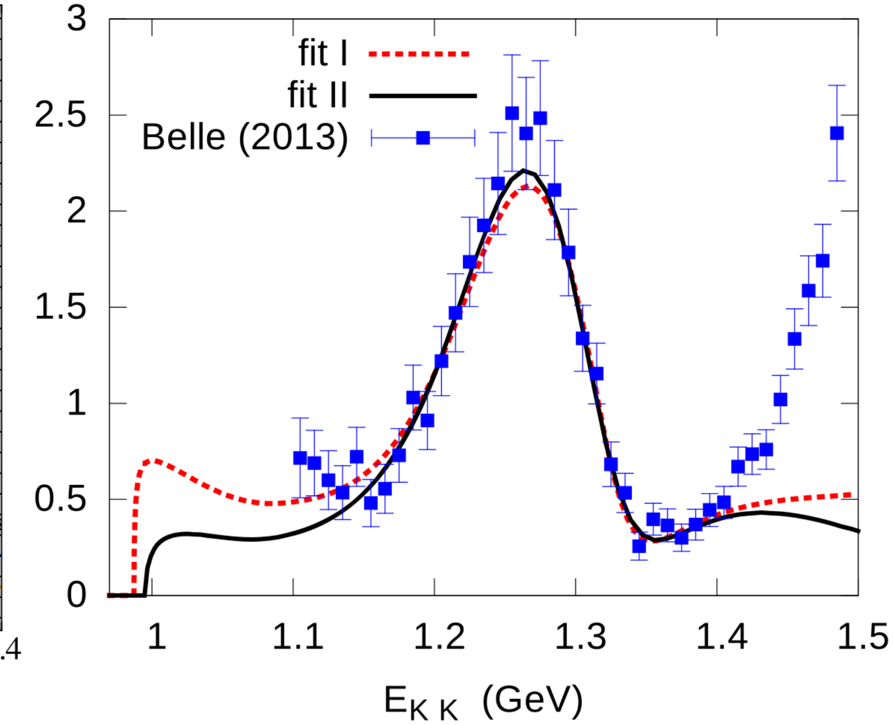
PDG (MeV):
(960...1030) − i(20...70)

Differences from other theoretical approaches

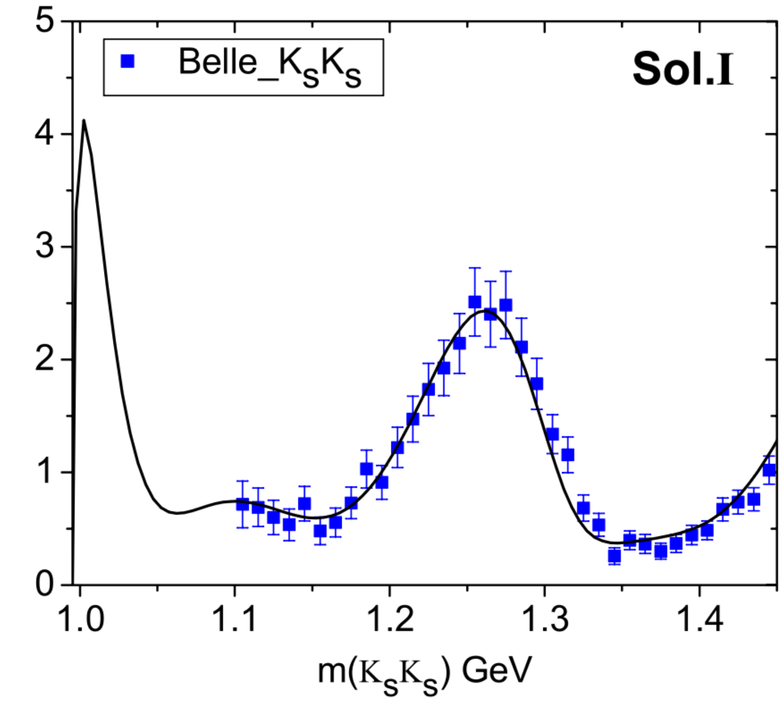
[Deineka et al. (2024)]



[Lu, Moussallam (2020)]



[Dai, Pennington (2014)]



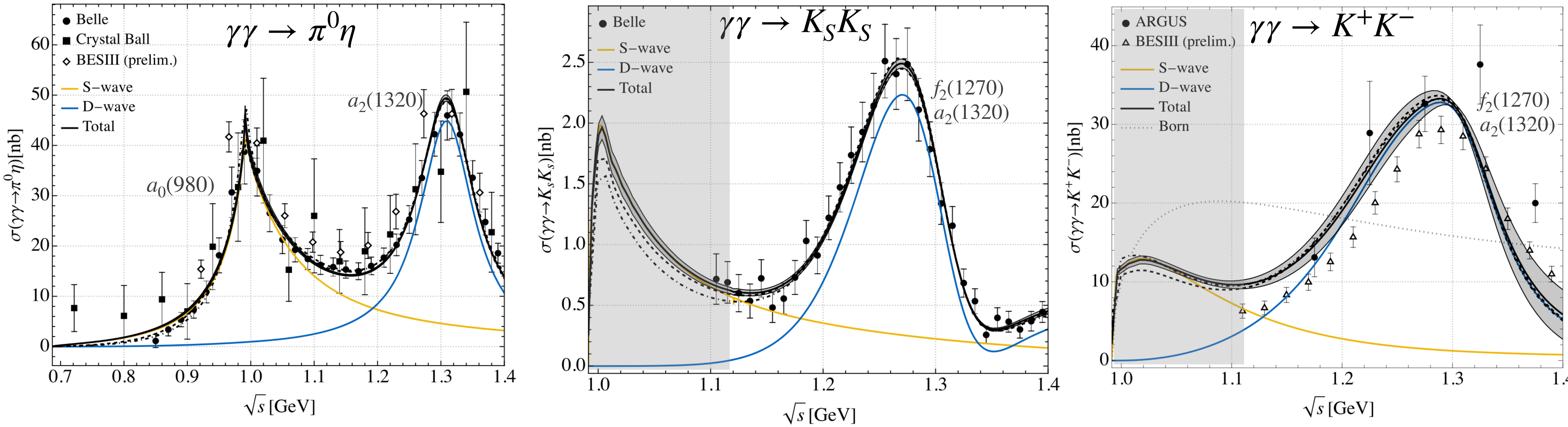
- [Oller et al. (1998)] Summed loops using unitarized tree-level ChPT amplitudes

- [Danilkin et al. (2012, 2017)] Dispersive (N/D) method with Omnès matrix fixed by ChPT
  No soft-photon constraint or Adler zero for $\gamma\gamma \to \pi^0\eta$

- [Dai, Pennington (2014)] Amplitude analysis of $\gamma\gamma \to \pi\pi/K\bar{K}$.
  Isovector channel parametrized phenomenologically by polynomials in s

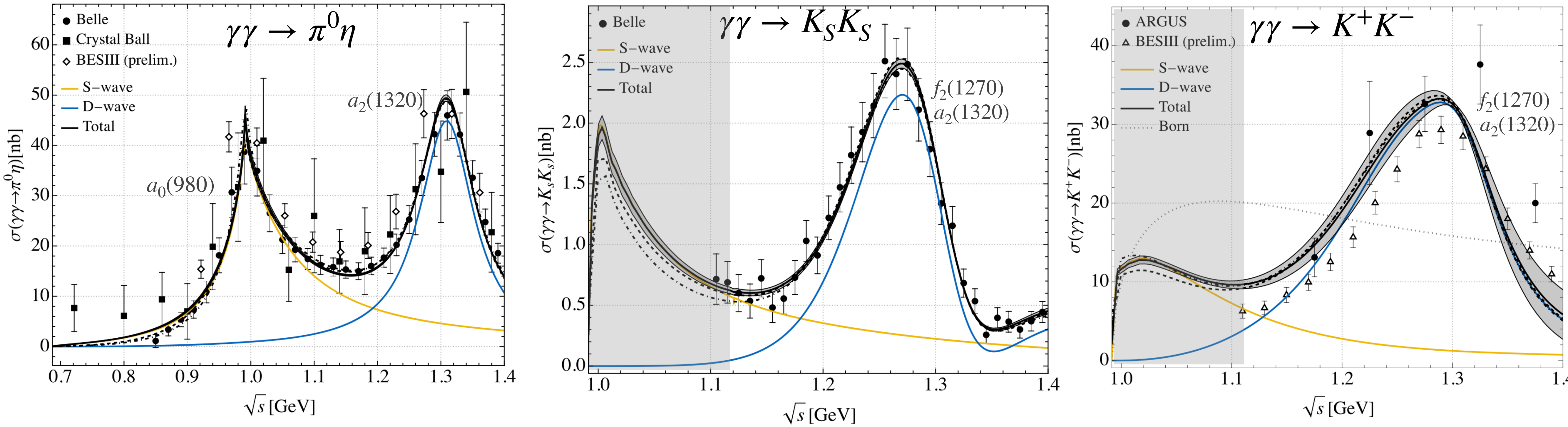
- [Lu, Moussallam (2020)] Direct Omnès solution, **heavily** based on ChPT (only partly on $\gamma\gamma$ data)
  $\Omega_{ab} \sim O(1/s)$; requires inclusion of a_0' resonance
  Only subtracted DR for $\gamma\gamma \to \pi^0\eta$ is possible
  No prediction for of $\gamma^*\gamma^* \to \pi^0\eta$

$$\Omega_{ab}(s) = \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{t^*_{ac}(s')\,\rho_c(s')\,\Omega_{cb}(s')}{s'-s}$$



- How to constraint more $\pi\eta/K\bar{K}$ amplitudes?
  ⏳ TFF-2 plan: “Include KLOE data on $\phi \to \pi^0\eta\gamma$ in our analysis” (talk by Bai-Long)

Results (TFF-2)



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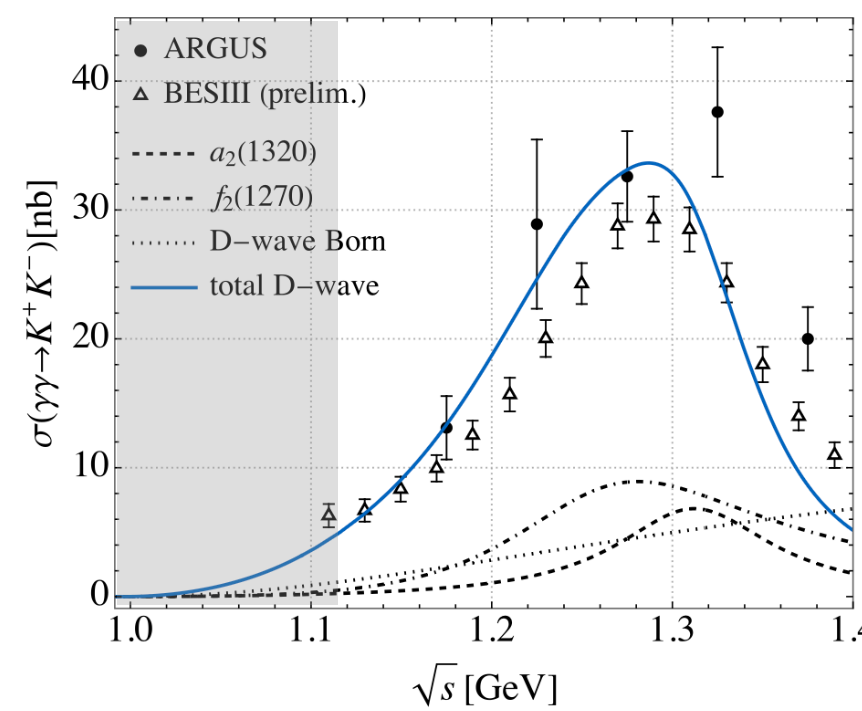
- Other strategies?
  ❗ $d\sigma/d\cos\theta$ data **near** $K\bar{K}$ threshold threshold is urgently needed (e.g. from BESIII)

  Theoretical requirement:
  Need dispersive $\pi\pi/K\bar{K}$ **D-wave** description of $f_2(1270)$

$$\Omega_{ab}(s) = \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{t^*_{ac}(s')\,\rho_c(s')\,\Omega_{cb}(s')}{s'-s}$$

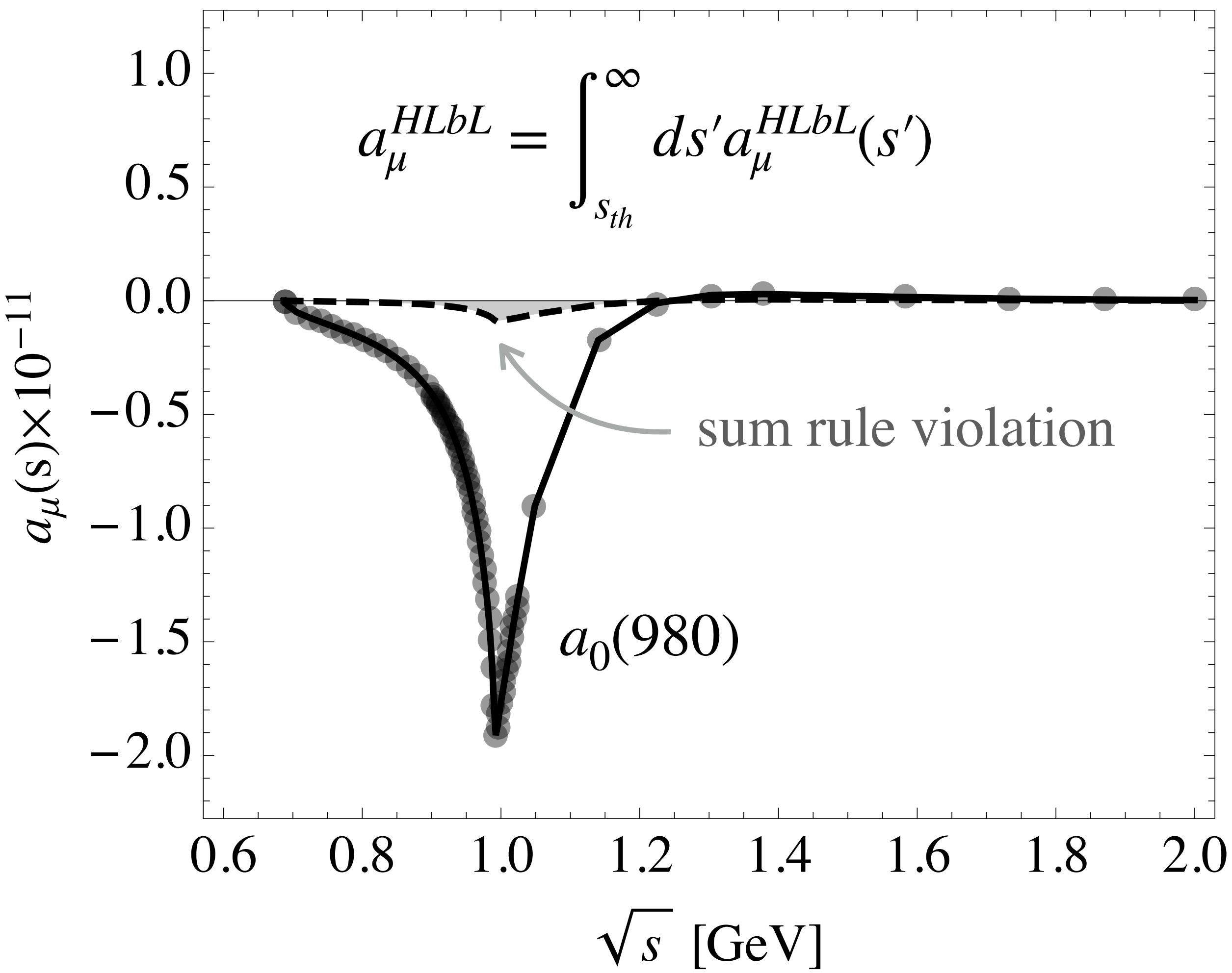


(possible project for FPII)

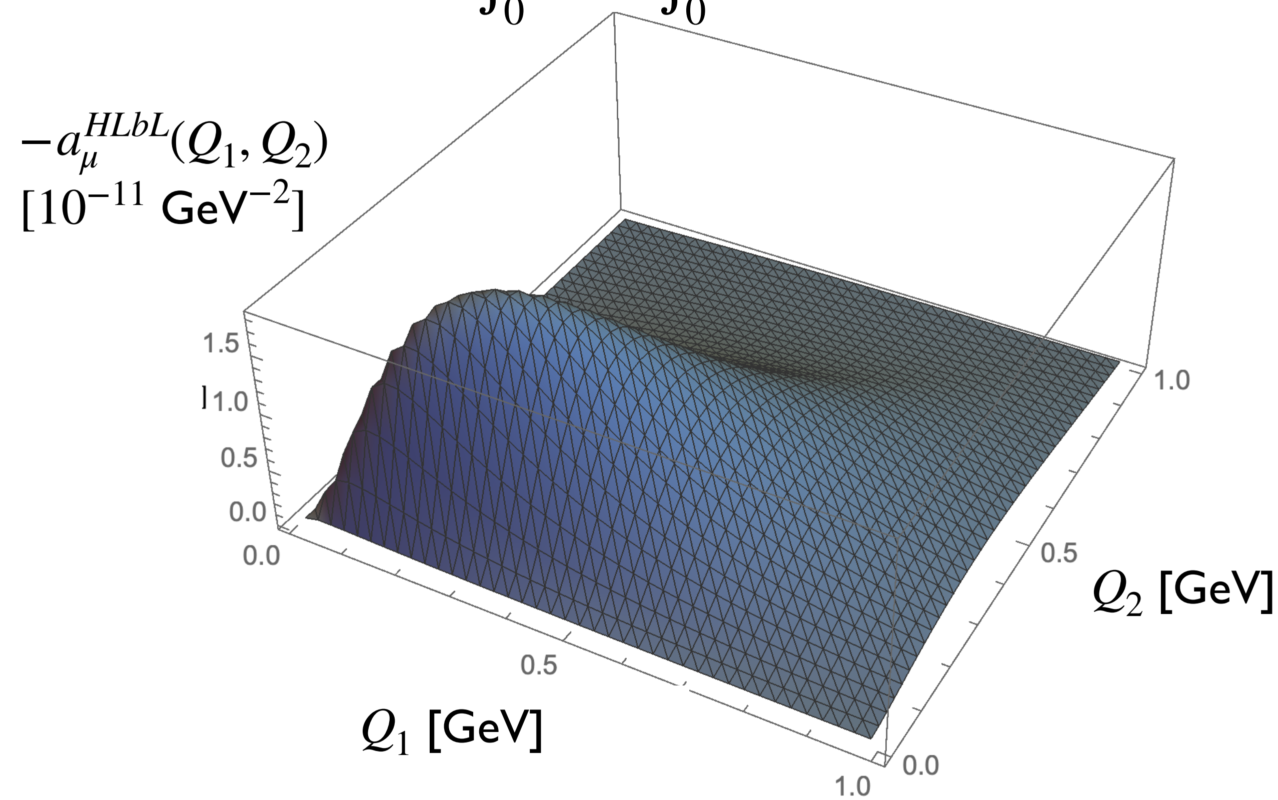
$a_0(980)$ contribution to HLbL piece of a_μ

$$a_\mu^{HLbL} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,Q_3)\bar{\Pi}_i(Q_1,Q_2,Q_3)$$

[Colangelo et al. (2014-2017)]



$$a_\mu^{HLbL} = \int_0^\infty dQ_1 \int_0^\infty dQ_2\, a_\mu^{HLbL}(Q_1,Q_2)$$



- Error budget: $\pi\eta/K\bar{K}$, TFF input, sum rule violation

$$a_\mu[\text{S-wave}, I=1]_{\pi\eta,K\bar{K}} = -\,0.44(3)(3)(2)\times 10^{-11}$$ [Deineka at al. (2024), white paper (2025)]

$$a_\mu[\text{NWA}]_{a_0(980)} = -\left([0.3,0.6]^{+0.2}_{-0.1}\right)\times 10^{-11}$$ [Danilkin et al. (2021), Schuler et al. (1998)]

Theory contribution to HadroTOPS: $\pi\pi$ channel

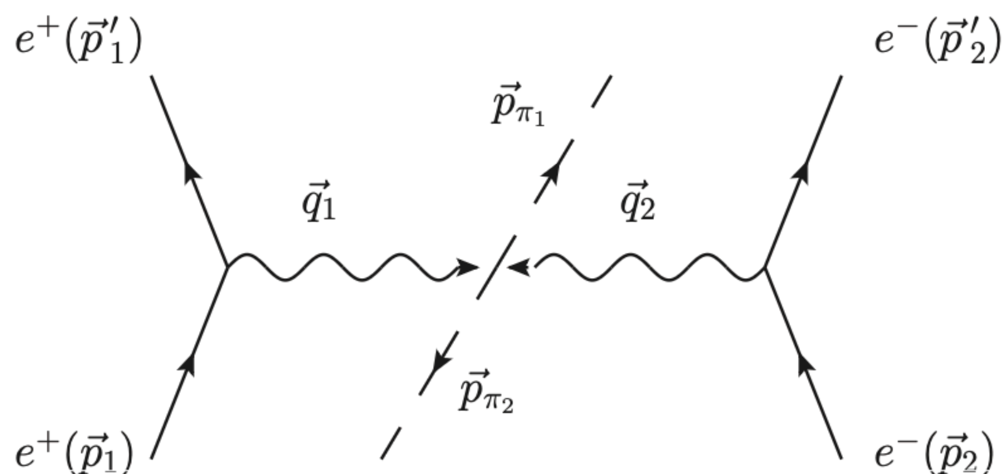
Cross section for the exclusive process $e^+e^- \to e^+e^-\pi\pi$ given by

$$d\sigma_{h_1h_2} = \frac{1}{F}\, d\text{Lips} \sum_{h_1',h_2'} |\mathcal{M}|^2 = \frac{1}{F}\, d\text{Lips}\, \frac{e^4}{Q_1^4 Q_2^4} \times \begin{cases} L_{1,\mu\mu'}\, L_{2,\nu\nu'}\, H^{\mu\nu}\, (H^{\mu'\nu'})^* \\ \sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'} \rho_{h_1}^{\lambda_1\lambda_1'}\, \rho_{h_2}^{\lambda_2\lambda_2'}\, M_{\lambda_1\lambda_2}\, M^*_{\lambda_1'\lambda_2'} \end{cases}$$

Lorentz-covariant form (used in **Ekhara**)

equivalent form (used in **HadroTOPS**)

where $M_{\lambda_1\lambda_2} \equiv \epsilon^\mu(q_1,\lambda_1)\, \epsilon^\nu(q_2,\lambda_2)\, H_{\mu\nu}$ are the helicity amplitudes for $\gamma^*\gamma^* \to \pi\pi$



Theory contribution to HadroTOPS: $\pi\pi$ channel

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Lorentz-covariant form (used in **Ekhara**)

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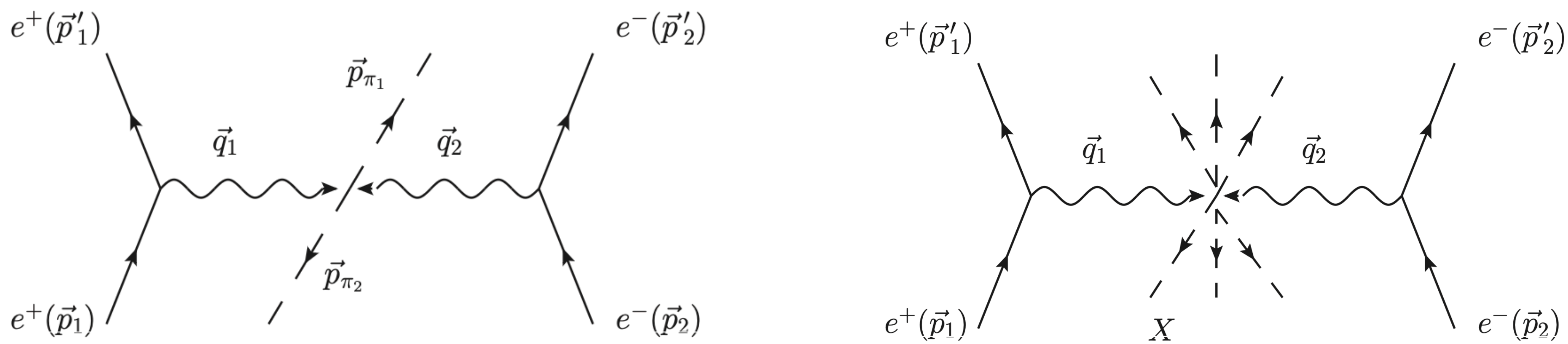
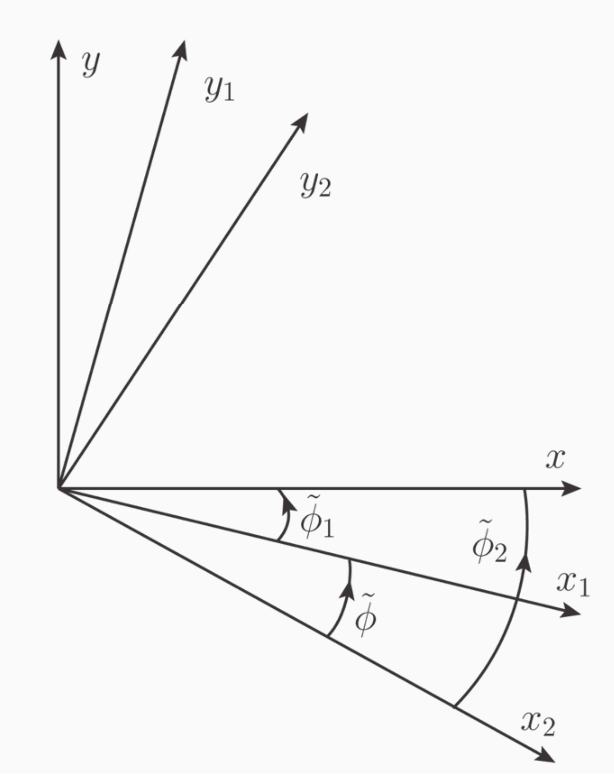


The inclusive $e^+e^- \to e^+e^-X$ cross-section can be written compactly in terms of 8 independent response functions $\text{Im}\, M_{++,++}, \ldots, \text{Im}\, M_{0+,-0}$ [Bonneau et al. (1975), Budnev et al. (1975)]

$$\text{Im}\, M_{\lambda_1'\lambda_2',\lambda_1\lambda_2} = \frac{1}{2} \sum_X \int d\Gamma_X\, (2\pi)^4\, \delta^4(q_1+q_2-p_X)\, M_{\lambda_1\lambda_2}\, M^*_{\lambda_1'\lambda_2'} = \sum_{X=\pi,\ldots} + \sum_{X=\pi\pi,\ldots} + \ldots$$

Theory contribution to HadroTOPS: $\pi\pi$ channel

For the exclusive process $e^+e^- \to e^+e^-\pi\pi$ the formula becomes much longer ($15^{\text{(unpol)}} + 10 = 25$ differential hadronic response functions) with additional differential dependence on azimuthal angles



$$
\begin{aligned}
d\sigma^{(unpol)} &= \frac{\alpha^2}{8\pi^4 Q_1^2 Q_2^2} \frac{\sqrt{X}}{s(1-4m^2/s)^{1/2}} \frac{d^3\vec{p}_1{}'}{E_1'} \frac{d^3\vec{p}_2{}'}{E_2'} \frac{d\Omega_\pi}{2\pi} \frac{4}{(1-\varepsilon_1)(1-\varepsilon_2)} \\
&\times \Bigg\{ \frac{1}{2}\left(\frac{d\sigma_0}{d\cos\theta_\pi} + \frac{d\sigma_2}{d\cos\theta_\pi}\right) + \left[\varepsilon_1 + \frac{2m^2}{Q_1^2}(1-\varepsilon_1)\right]\left[\varepsilon_2 + \frac{2m^2}{Q_2^2}(1-\varepsilon_2)\right]\frac{d\sigma_{LL}}{d\cos\theta_\pi} \\
&+ \left[\varepsilon_2 + \frac{2m^2}{Q_2^2}(1-\varepsilon_2)\right]\left(1+\varepsilon_1\cos\left(2\tilde{\phi}_1\right)\right)\frac{d\sigma_{TL}}{d\cos\theta_\pi} + \left[\varepsilon_1 + \frac{2m^2}{Q_1^2}(1-\varepsilon_1)\right]\left(1+\varepsilon_2\cos\left(2\tilde{\phi}_2\right)\right)\frac{d\sigma_{LT}}{d\cos\theta_\pi} \\
&+ \frac{1}{2}\varepsilon_1\varepsilon_2\left[\cos 2(\tilde{\phi}_2-\tilde{\phi}_1)\frac{d\sigma_0}{d\cos\theta_\pi} + \cos 2(\tilde{\phi}_1+\tilde{\phi}_2)\frac{d\sigma_2}{d\cos\theta_\pi}\right] - \left[\varepsilon_1\cos\left(2\tilde{\phi}_1\right) + \varepsilon_2\cos\left(2\tilde{\phi}_2\right)\right]\frac{d\tau_{T2}}{d\cos\theta_\pi} \\
&+ \left[\varepsilon_1(1+\varepsilon_1) + \frac{4m^2}{Q_1^2}\varepsilon_1(1-\varepsilon_1)\right]^{1/2}\left[\varepsilon_2(1+\varepsilon_2) + \frac{4m^2}{Q_2^2}\varepsilon_2(1-\varepsilon_2)\right]^{1/2} \\
&\times \left[\cos\left(\tilde{\phi}_2-\tilde{\phi}_1\right)\left(\frac{d\tau_0}{d\cos\theta_\pi} + \frac{d\tau_1}{d\cos\theta_\pi}\right) + \cos\left(\tilde{\phi}_1+\tilde{\phi}_2\right)\left(\frac{d\tau_1}{d\cos\theta_\pi} - \frac{d\tau_{L2}}{d\cos\theta_\pi}\right)\right] \\
&+ \left[\varepsilon_1(1+\varepsilon_1) + \frac{4m^2}{Q_1^2}\varepsilon_1(1-\varepsilon_1)\right]^{1/2}\left[\cos\tilde{\phi}_1\left(\frac{d\tau_{-12}}{d\cos\theta_\pi} - \frac{d\tau_{-1T}}{d\cos\theta_\pi}\right) - 2\left[\varepsilon_2 + \frac{2m^2}{Q_2^2}(1-\varepsilon_2)\right]\cos\tilde{\phi}_1\frac{d\tau_{1L}}{d\cos\theta_\pi}\right. \\
&\qquad\qquad \left. +\varepsilon_2\cos\left(\tilde{\phi}_1+2\tilde{\phi}_2\right)\frac{d\tau_{-12}}{d\cos\theta_\pi} - \varepsilon_2\cos\left(2\tilde{\phi}_2-\tilde{\phi}_1\right)\frac{d\tau_{-1T}}{d\cos\theta_\pi}\right] \\
&+ \left[\varepsilon_2(1+\varepsilon_2) + \frac{4m^2}{Q_2^2}\varepsilon_2(1-\varepsilon_2)\right]^{1/2}\left[\cos\tilde{\phi}_2\left(\frac{d\tau_{12}}{d\cos\theta_\pi} - \frac{d\tau_{1T}}{d\cos\theta_\pi}\right) - 2\left[\varepsilon_1 + \frac{2m^2}{Q_1^2}(1-\varepsilon_1)\right]\cos\tilde{\phi}_2\frac{d\tau_{-1L}}{d\cos\theta_\pi}\right. \\
&\qquad\qquad \left. +\varepsilon_1\cos\left(2\tilde{\phi}_1+\tilde{\phi}_2\right)\frac{d\tau_{12}}{d\cos\theta_\pi} - \varepsilon_1\cos\left(2\tilde{\phi}_1-\tilde{\phi}_2\right)\frac{d\tau_{1T}}{d\cos\theta_\pi}\right]\Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
d\sigma^{(unpol)}|_{Q_2^2\to 0} &= \frac{\alpha^2}{8\pi^4 Q_1^2 Q_2^2} \frac{\sqrt{X}}{s(1-4m^2/s)^{1/2}} \frac{d^3\vec{p}_1{}'}{E_1'} \frac{d^3\vec{p}_2{}'}{E_2'} \frac{d\Omega_\pi}{2\pi} \frac{4}{(1-\varepsilon_1)(1-\varepsilon_2)} \\
&\times \Bigg\{ \frac{1}{2}\left(\frac{d\sigma_0}{d\cos\theta_\pi} + \frac{d\sigma_2}{d\cos\theta_\pi}\right) + \left[\varepsilon_1 + \frac{2m^2}{Q_1^2}(1-\varepsilon_1)\right]\frac{d\sigma_{LT}}{d\cos\theta_\pi} \\
&-\varepsilon_1\cos\left(2\tilde{\phi}_1\right)\frac{d\tau_{T2}}{d\cos\theta_\pi} + \left[\varepsilon_1(1+\varepsilon_1) + \frac{4m^2}{Q_1^2}\varepsilon_1(1-\varepsilon_1)\right]^{1/2}\cos\tilde{\phi}_1\left(\frac{d\tau_{-12}}{d\cos\theta_\pi} - \frac{d\tau_{-1T}}{d\cos\theta_\pi}\right)
\end{aligned}
$$

Summary and Outlook

PHYSICAL REVIEW D **111**, 034009 (2025)

Dispersive estimate of the $a_0(980)$ contribution to $(g-2)_\mu$

Oleksandra Deineka, Igor Danilkin, and Marc Vanderhaeghen
Institut für Kernphysik and PRISMA+ Cluster of Excellence, Johannes Gutenberg Universität, D-55099 Mainz, Germany

(Received 18 October 2024; accepted 18 January 2025; published 10 February 2025)

- First dispersive estimate of $a_0(980)$ to $(g-2)_\mu$ (result included in white paper)
- Improves PDG parameters of $a_0(980)$
- Hadronic Omnès output applicable to different processes with $\pi\eta/K\bar{K}$ final state

EPJ Web of Conferences **303**, 01007 (2024) https://doi.org/10.1051/epjconf/202430301007
MENU 2023

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- ⏳ Waiting for **BESIII** data: $\gamma\gamma^* \to \pi^+\pi^-, \pi^0\pi^0$ or $\gamma\gamma \to \pi^0\pi^0$

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Publications planned before completion of FP1

- Dispersive analysis of $\phi \to \pi^0\eta\gamma, \pi^0\pi^0\gamma$ (talk by Bai-Long)
- Joint paper with experimental colleagues on HadroTOPS
- Pion generalized polarizabilities (depends on data)
- Validation/update of $f_0(500)/f_0(980)/a_0(980)$ to $(g-2)_\mu$ (depends on $\gamma\gamma^* \to \pi^+\pi^0, \pi^0\pi^0, \pi^0\eta$ data)

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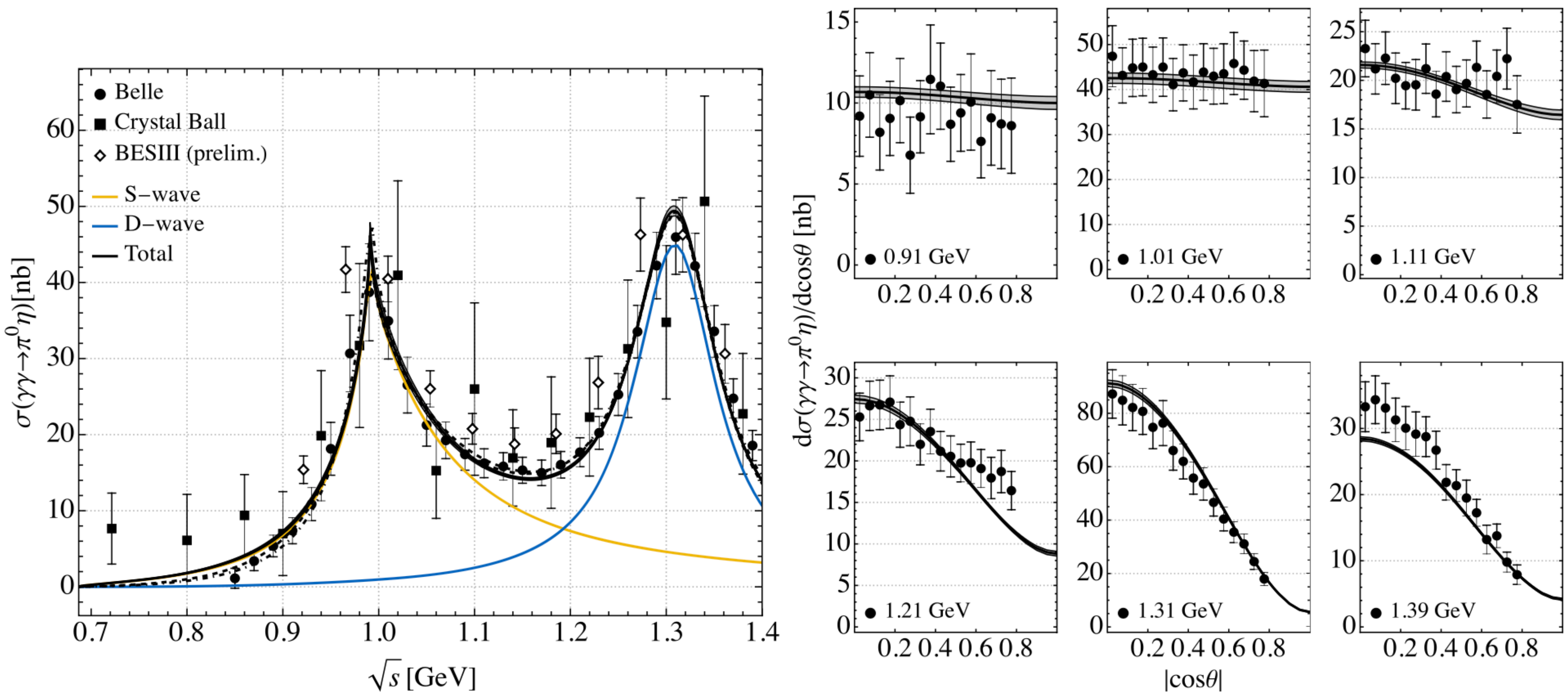
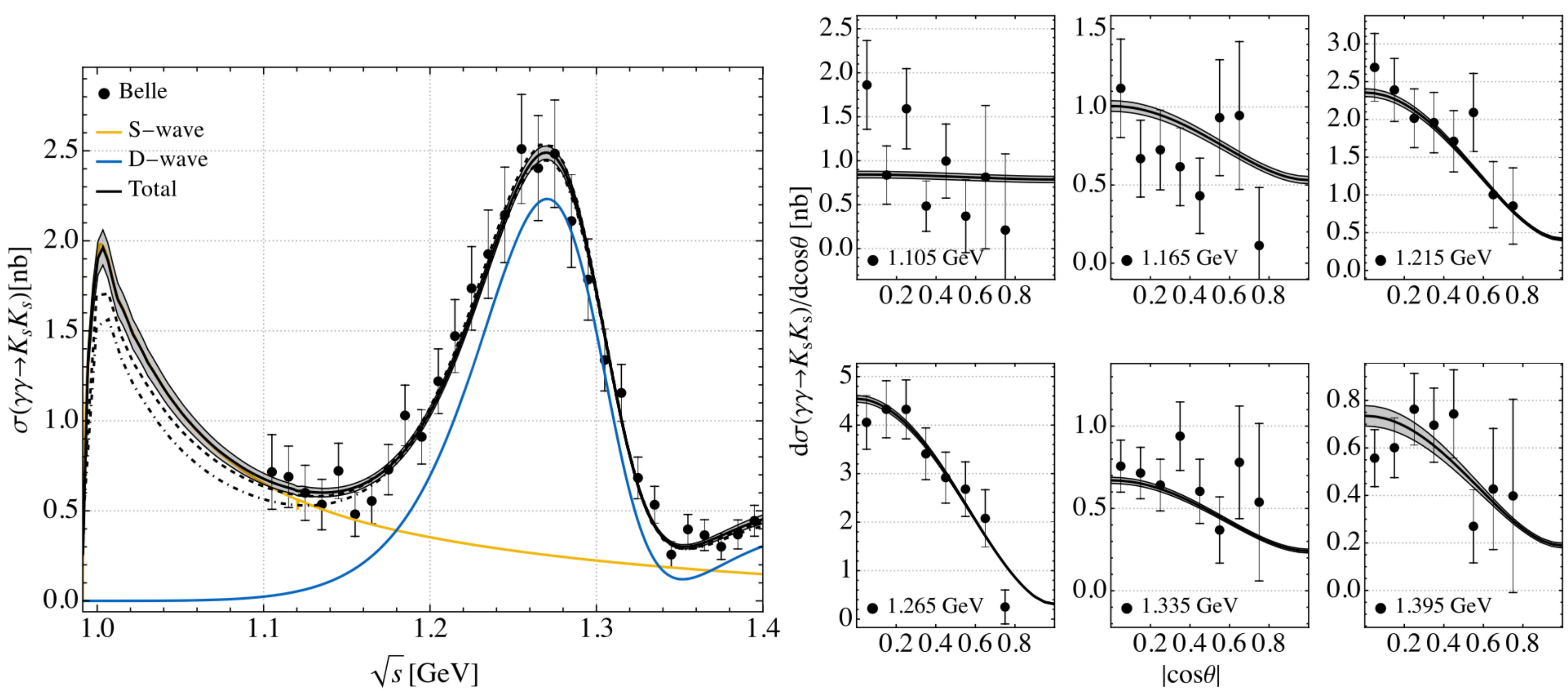
Possible projects for FP2

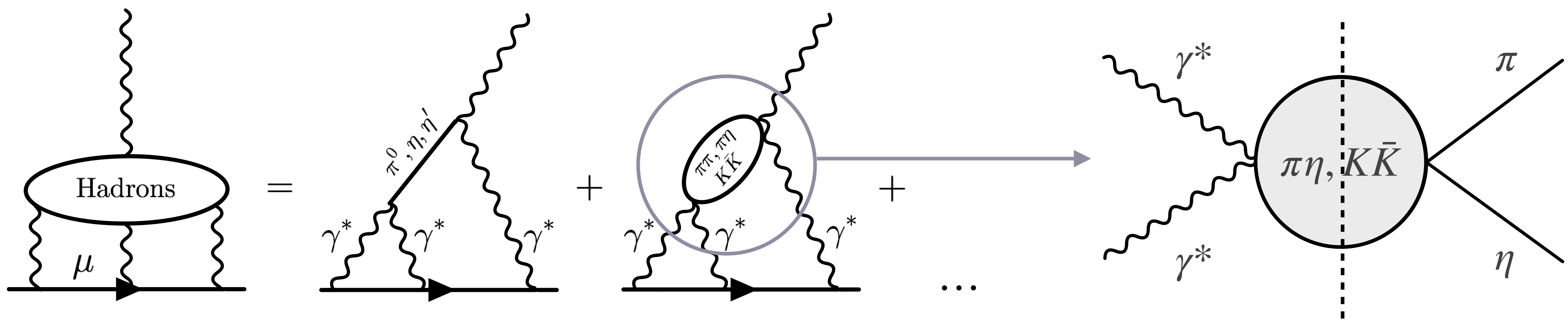
- $J/\psi \to \pi^0\pi^0\gamma$; probing gravitational form factors of proton/pion (talk by Bai-Long)
- Coupled-channel dispersive analysis of $f_2(1270)$ needed for $\gamma\gamma \to K^+K^-$ and extraction of kaon polarizabilities

EXTRA


Belle
Crystal Ball
BESIII (prelim.)
S−wave
D−wave
Total
σ(γγ→π⁰η)[nb]
√s [GeV]
dσ(γγ→π⁰η)/dcosθ [nb]
0.91 GeV
1.01 GeV
1.11 GeV
1.21 GeV
1.31 GeV
1.39 GeV
|cosθ|


Belle
S−wave
D−wave
Total
σ(γγ→KsKs)[nb]
√s [GeV]
dσ(γγ→KsKs)/dcosθ [nb]
1.105 GeV
1.165 GeV
1.215 GeV
1.265 GeV
1.335 GeV
1.395 GeV
|cosθ|

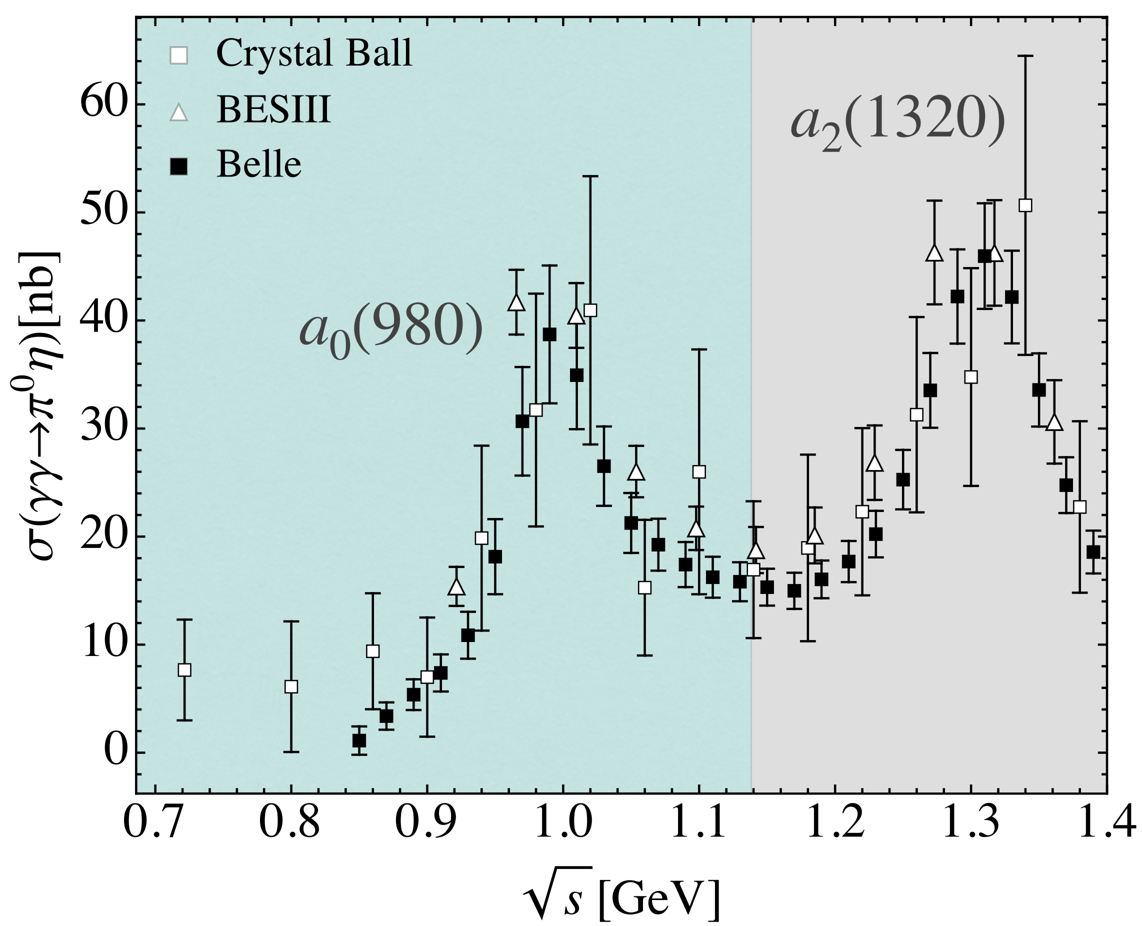
Motivation

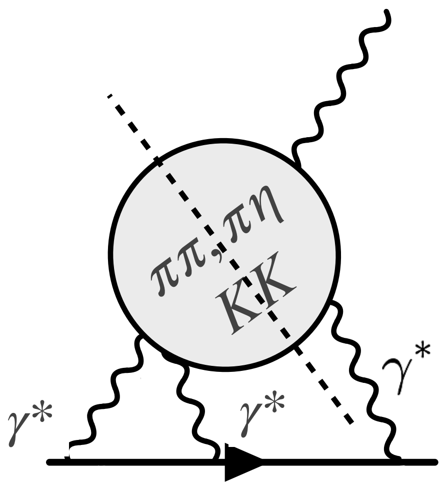


Ingredients for HLbL: $\gamma^*\gamma^* \to \pi\pi, \pi\eta, K\bar{K}\ldots$
for spacelike γ^*: $q = -Q^2 < 0$

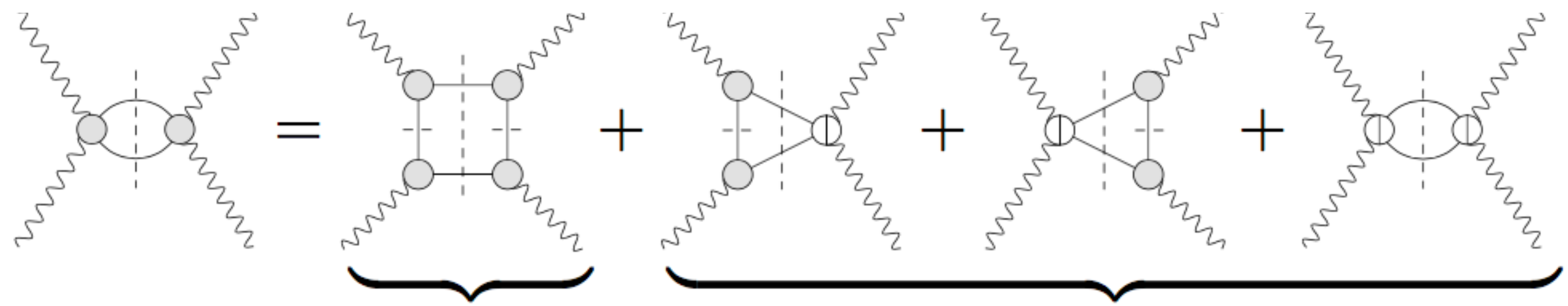
Contribution	Our estimate
π^0, η, η'-poles	93.8(4.0)
π, K-loops/boxes	−16.4(2)
S-wave $\pi\pi$ rescattering	−8(1)
subtotal	69.4(4.1)
scalars tensors	} − 1(3)
axial vectors	6(6)
u, d, s-loops / short-distance	15(10)
c-loop	3(1)
total	92(19)

[White paper (2020)]





$$a_\mu^{HLbL} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,Q_3)\bar{\Pi}_i(Q_1,Q_2,Q_3)$$



$a_\mu[\text{box}]_{\pi\pi,K\bar{K}} = -16.4(0.2) \times 10^{-11}$ [Colangelo et al. (2014-2017)]

Input: pion (kaon) vector form factors $F_{\pi,K}(Q^2)$

$a_\mu[\text{S-wave}, I=2]_{\pi\pi} = +1.1(0.1) \times 10^{-11}$ [Colangelo, Hoferichter, Procura, Stoffer (2017)]

$a_\mu[\text{S-wave}, I=0]_{\pi\pi} = -9.3(0.9) \times 10^{-11}$ (σ)

$a_\mu[\text{S-wave}, I=0]_{\pi\pi,K\bar{K}} = -9.8(1.0) \times 10^{-11}$ (σ, f_0) [I.D, Hoferichter, Stoffer (2021)]

Unsubtracted dispersion relation for $\gamma^*\gamma^* \to \pi\pi/K\bar{K}$

Left-hand cuts: π/K pole with vector form factors $F_{\pi,K}(Q^2)$

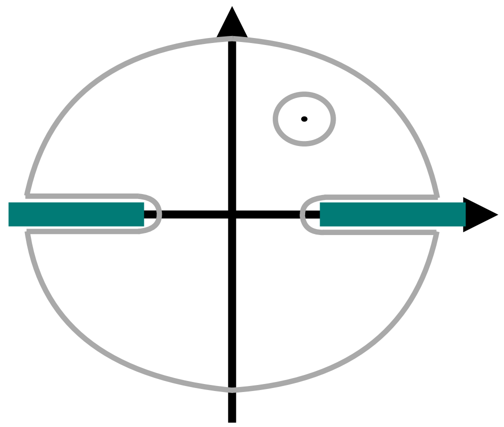
Data used: $\pi\pi/K\bar{K}$ scattering data (Roy analyses)

$\gamma\gamma \to \pi^0\pi^0$ used to **justify** left-hand cut approximation

Coupled-channel Omnès matrix

- Once-subtracted p.w. dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(0) + \frac{s}{\pi}\int_L \frac{ds'}{s'}\frac{\mathrm{Im}\, t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s}{\pi}\sum_c \int_R \frac{ds'}{s'}\frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$



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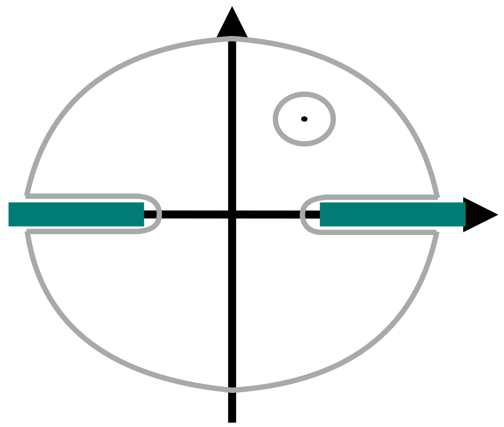


can be solved using the N/D method with input from $U_{ab}(s)$ above threshold

[Chew, Mandelstam (1960)]
[Luming (1964)]
[Johnson, Warnock (1981)]

$$t_{ab}(s) = \sum_c D^{-1}_{ac}\, N_{cb}(s)$$

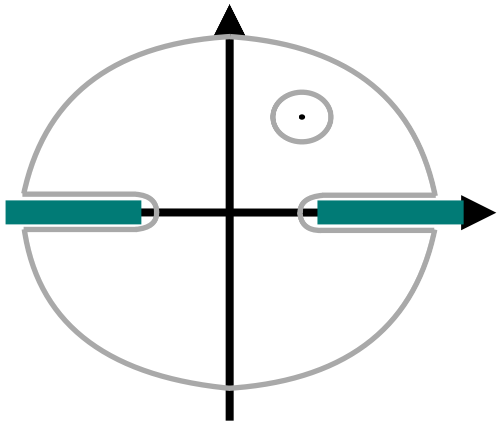
$$N_{ab}(s) = U_{ab}(s) + \frac{s}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{N_{ac}(s')\,\rho_c(s')\,(U_{cb}(s') - U_{cb}(s))}{s'-s}$$ (**left-hand cuts**)

$$D_{ab}(s) = \delta_{ab} - \frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{N_{ab}(s')\,\rho_b(s')}{s'-s}$$ (**right-hand cuts**)

Coupled-channel Omnès matrix

- Once-subtracted p.w. dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(0) + \frac{s}{\pi}\int_L \frac{ds'}{s'}\frac{\text{Im}\, t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s}{\pi}\sum_c \int_R \frac{ds'}{s'}\frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$



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$$D_{ab}(s) = \delta_{ab} - \frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'}\frac{N_{ab}(s')\,\rho_b(s')}{s'-s}$$ (**right-hand cuts**)

- The Omnès function fulfils the unitarity relation on the right-hand cut and is analytic everywhere else. For the case of **no bound states** or **no CDD poles**:

$$\Omega_{ab}(s) = D^{-1}_{ab}(s)$$

We used this method to obtain data driven $\pi\pi/K\bar{K}$ Omnès matrix for $f_0(500)$, $f_0(980)$
[I.D, Deineka, Vanderhaeghen (2020)]

Fitting parameters: left-hand cuts

- In general scattering problem, little is known about left-hand cuts, except their analytical structure in the complex plane. We approximate $U_{ab}(s)$ as an expansion in a **conformal mapping variable** $\xi(s)$

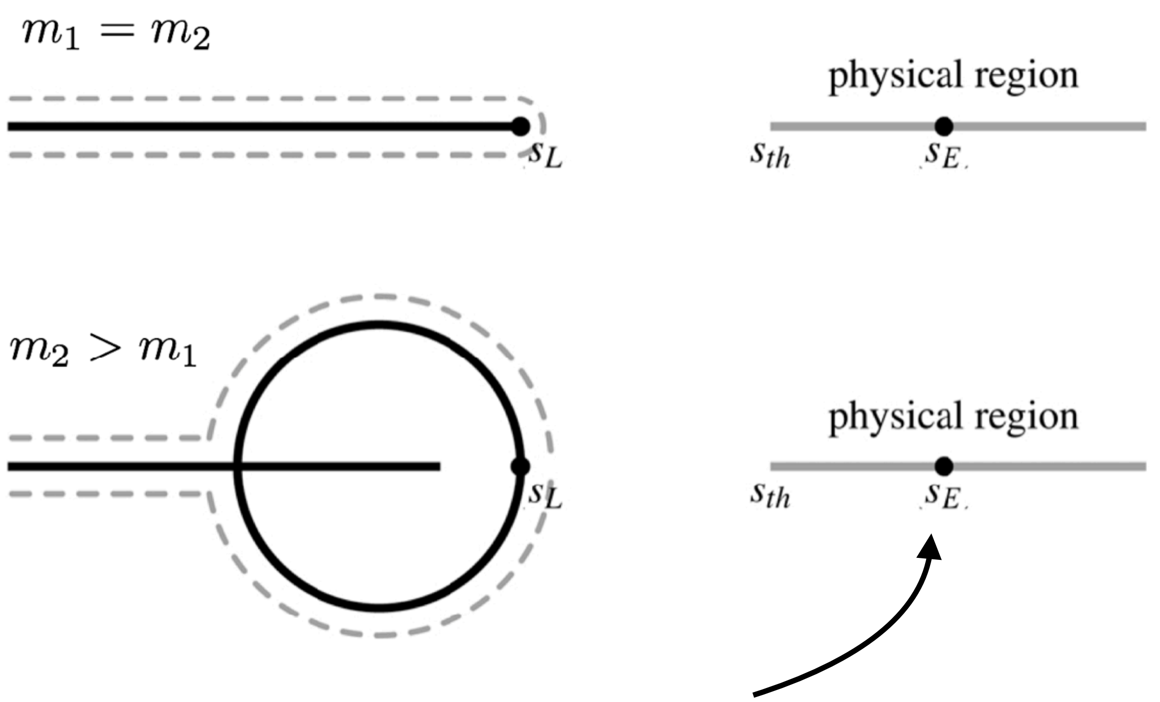
[Gasparyan, Lutz (2010)]

(asymptotically bounded unknown function)

$$U_{ab}(s) = t_{ab}(0) + \frac{s}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\text{Im}\; t_{ab}(s')}{s'-s}$$

$$\simeq \sum_{n=0}^{\infty} C_{ab,n}\,(\xi_{ab}(s))^n$$

unknown coefficients fitted to data or/and ChPT



$$\xi(s_E) = 0$$
$$\xi(s_L) = -1$$

$$\sqrt{s_E} = \frac{1}{2}\left(\sqrt{s_{th}} + \sqrt{s_{max}}\right)$$

source of the systematic uncertainties