Quantum metrology

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What will I talk about

QUANTUM METROLOGY

- Entanglement-based theory
- Squeezing-based theory

Measurements

Want to estimate a parameter φ written onto a probe by a transformation U_φ

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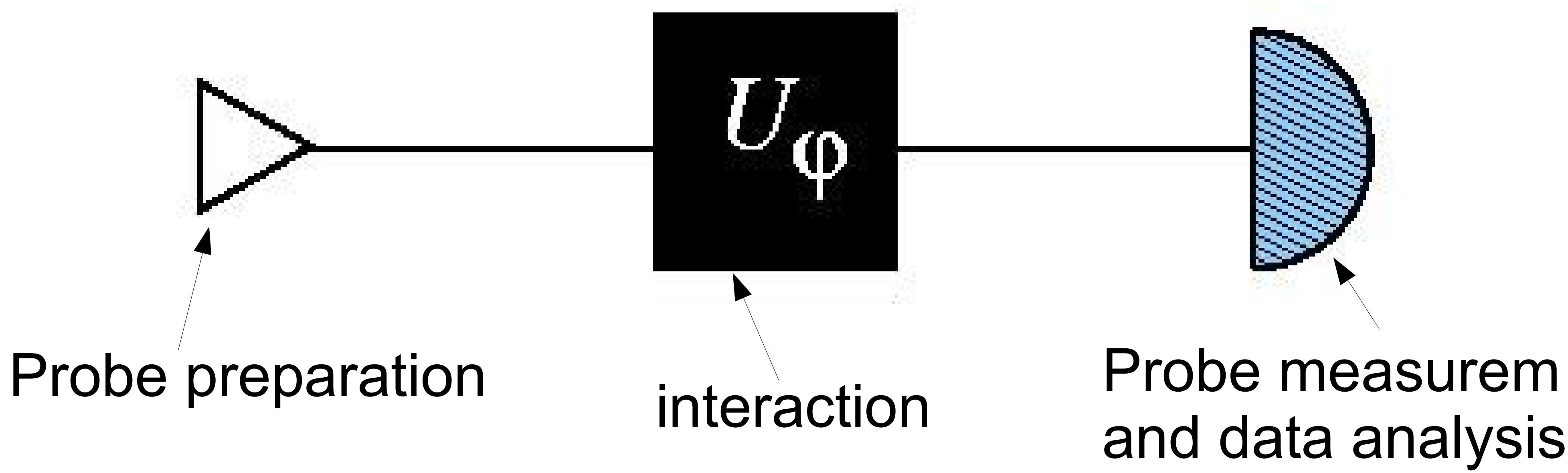

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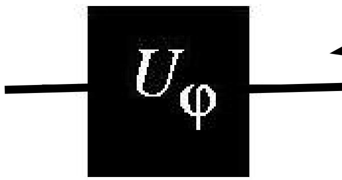

black box that implements a transformation $U_{\color{red}\varphi} = e^{i{\color{red}\varphi}H}$

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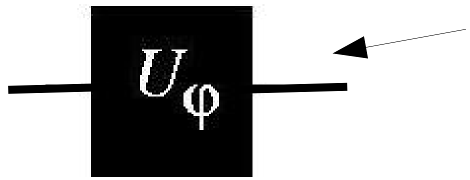

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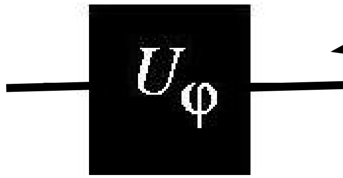

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GOAL: use it νN times and get the best estimate of φ

RESULT: quantum strategies: $\Delta\varphi \sim \frac{1}{\sqrt{\nu}N}$ Heisenberg bound

number of times the N-experiment is repeated

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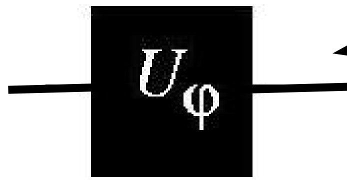

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$\sqrt{N}$ gain of quantum metrology

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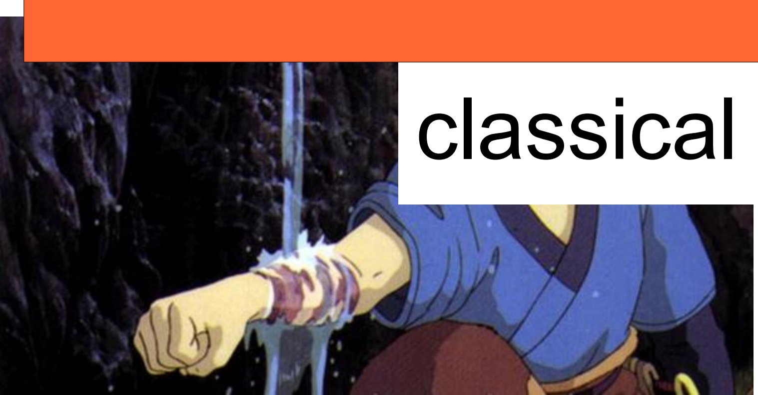

classical strategies:

Parallel strategies

PRL 96,010401

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PRL 96,010401

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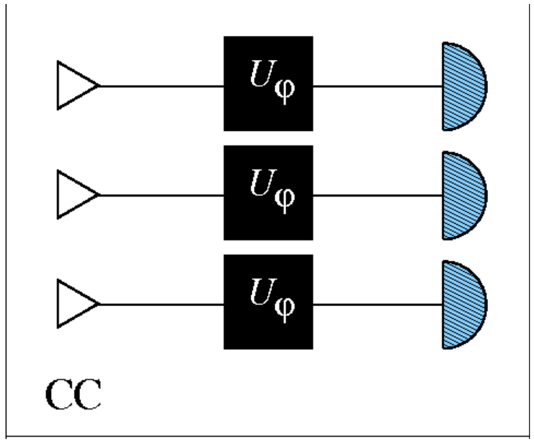

PRL 96,010401

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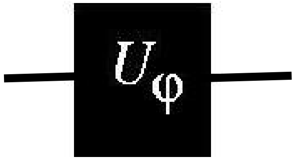
use  in parallel:

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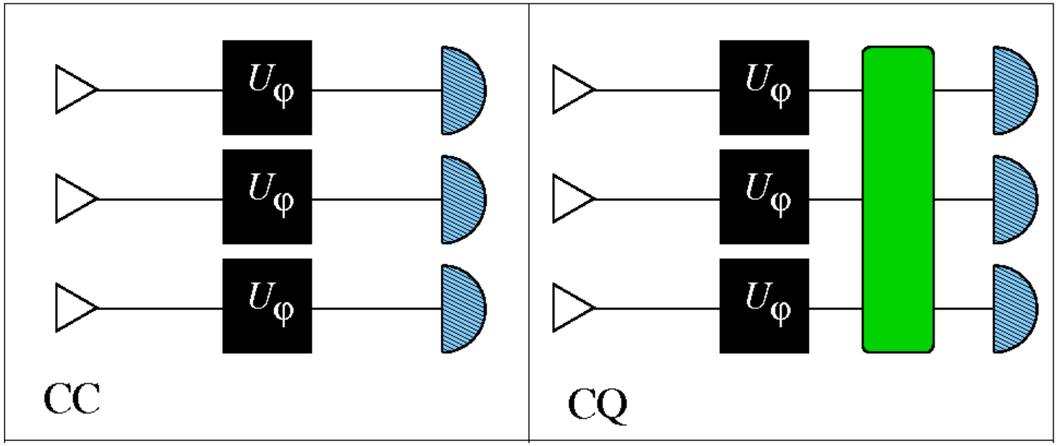

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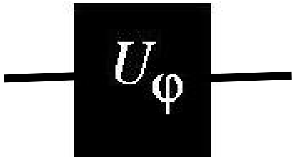
use 

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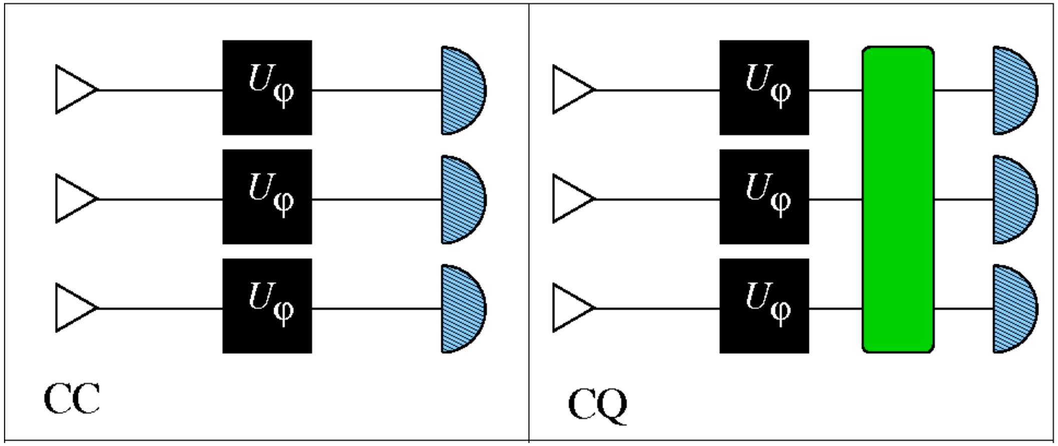

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(shot noise)

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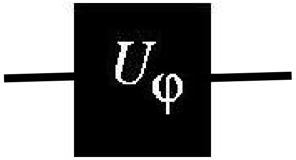
use 

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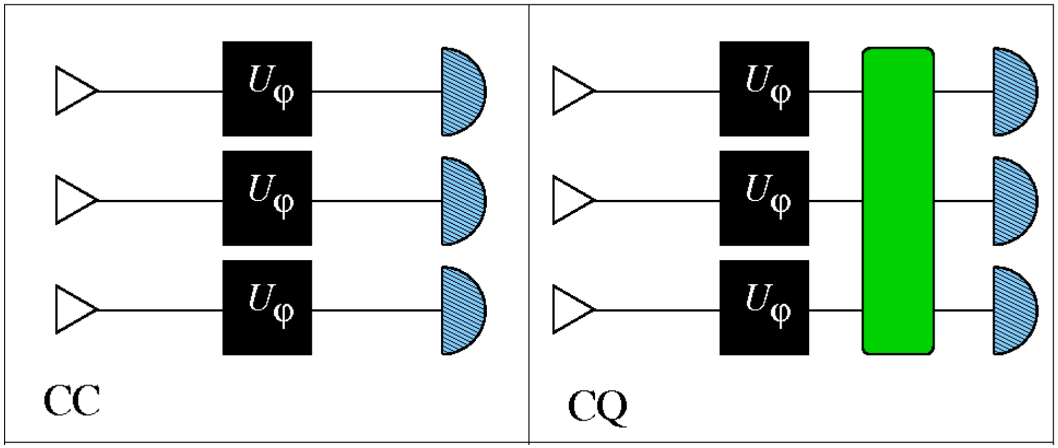

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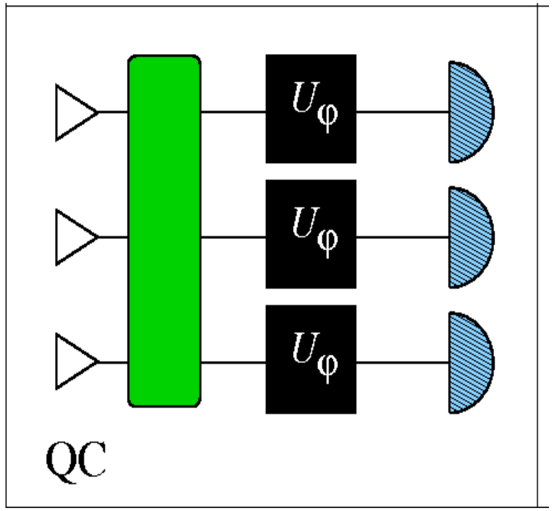

the *N* transformations act on an **entangled state**

PRL 96,010401

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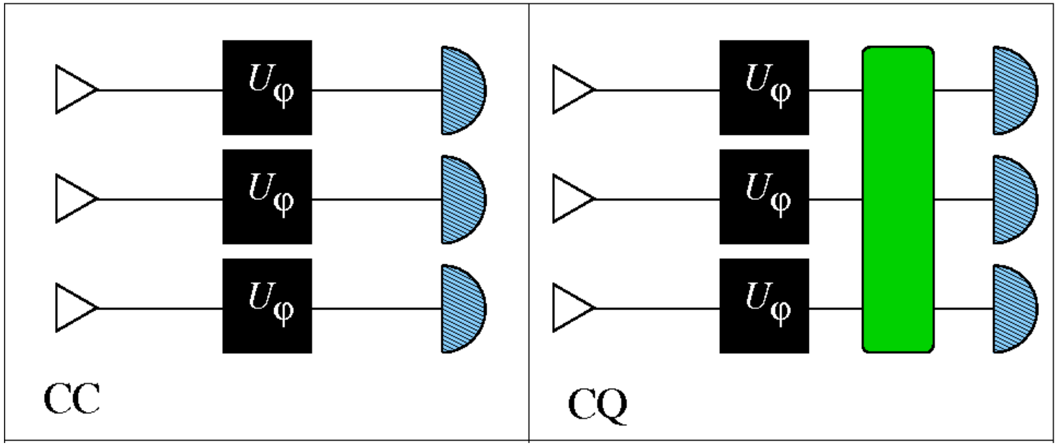

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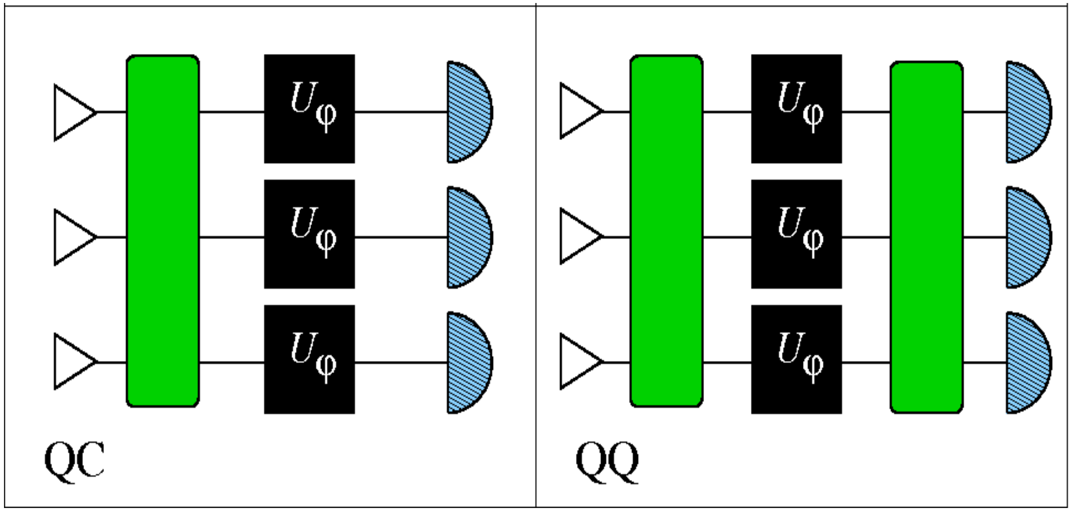

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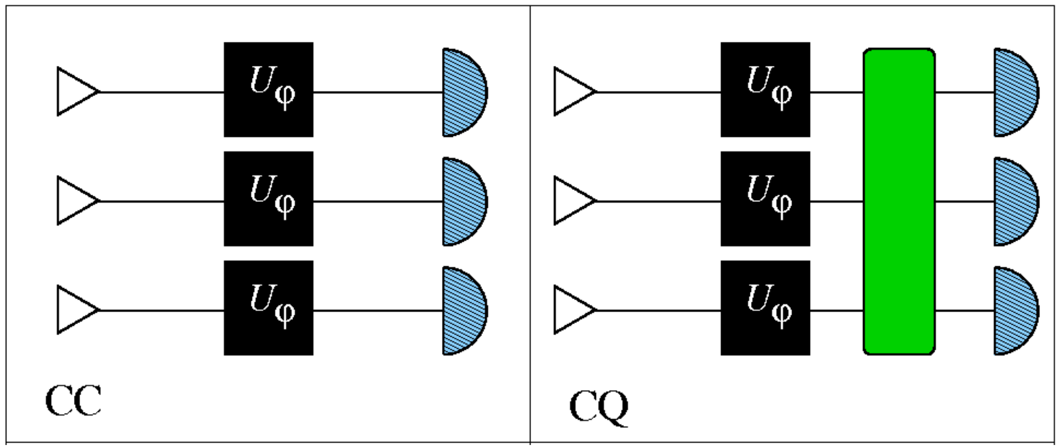

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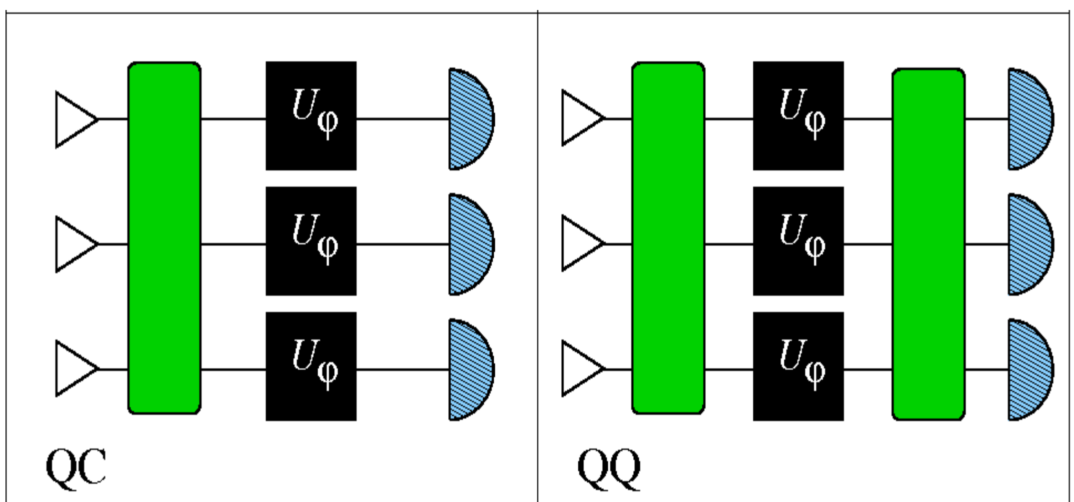

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(Heisenberg bound)

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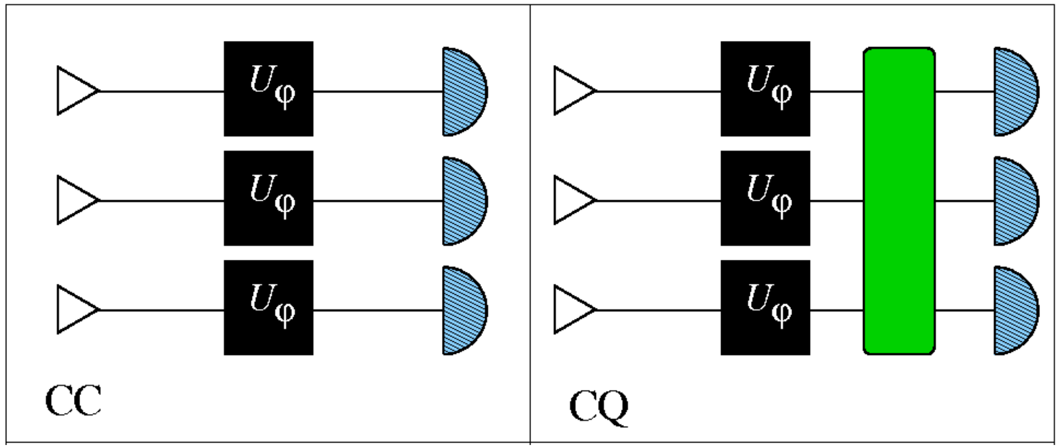

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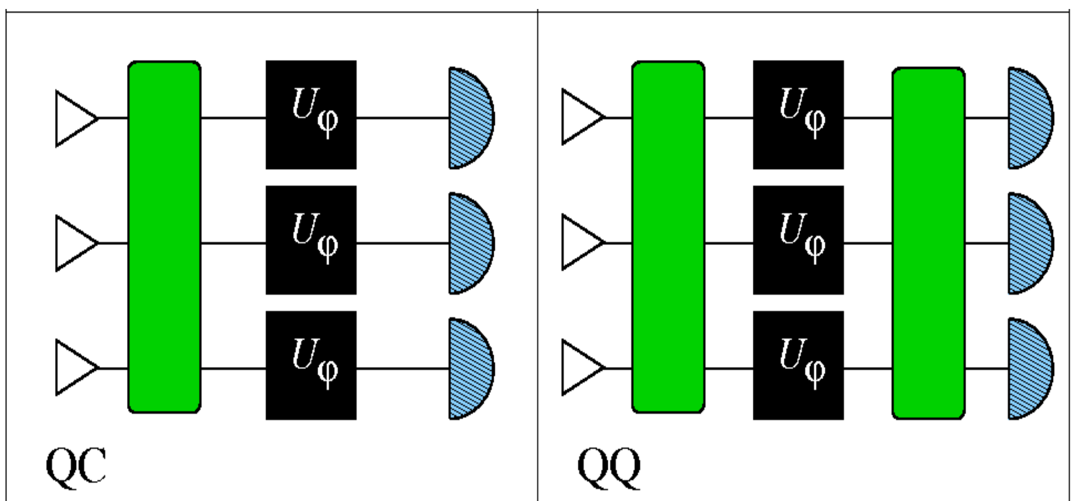

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Note: entanglement at the measurement stage is useless!

u

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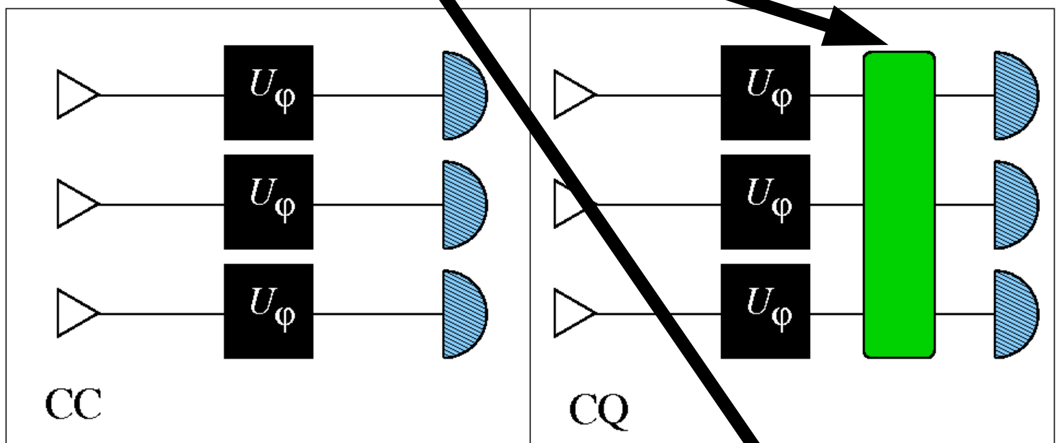

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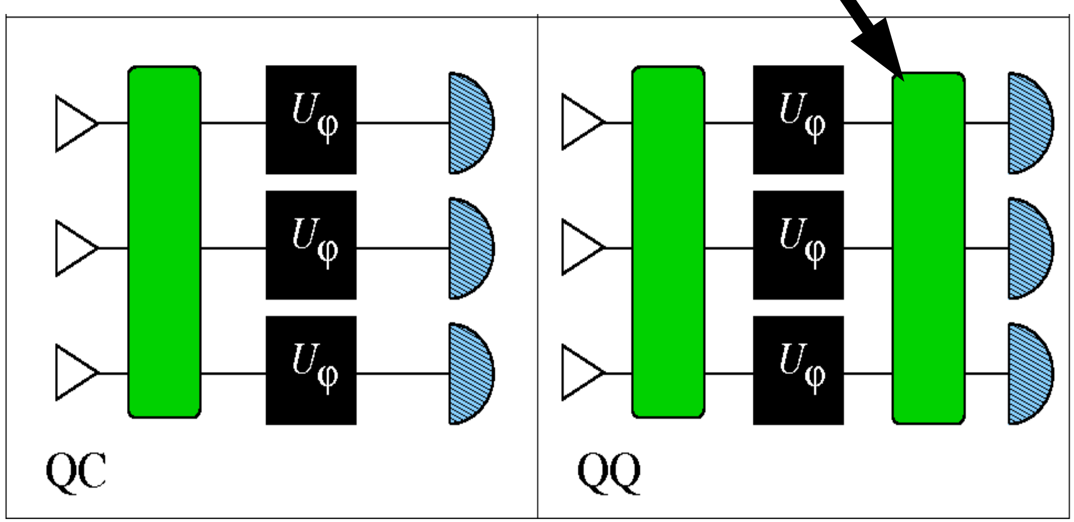

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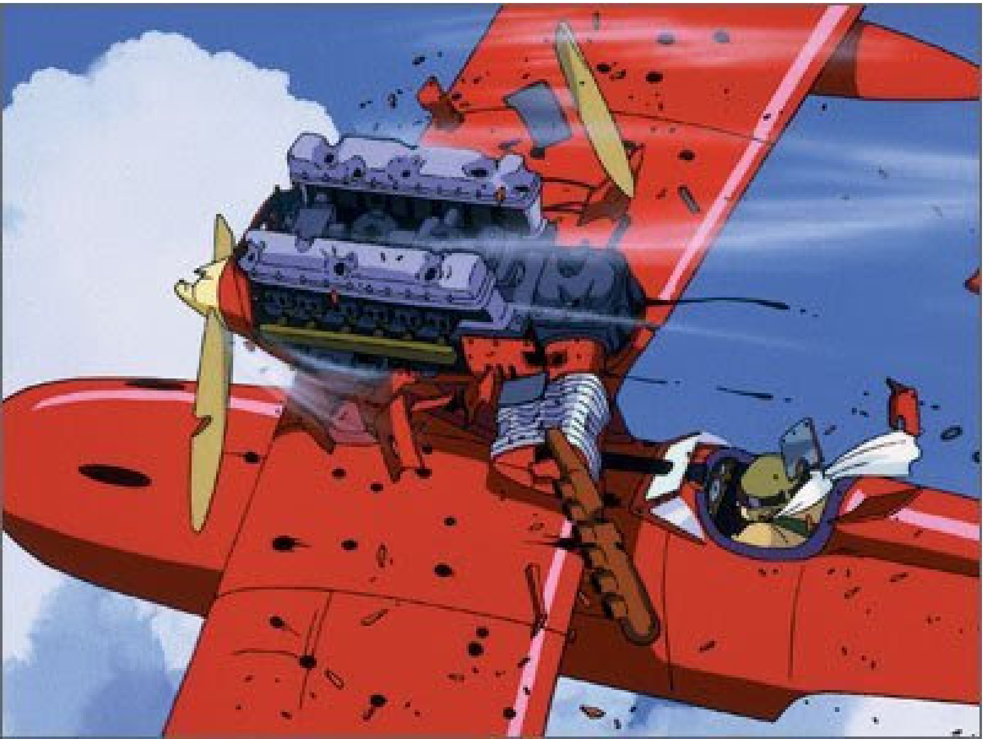
Sequential (multiround) strategies



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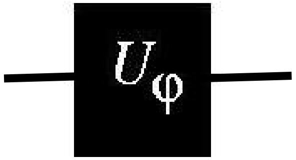
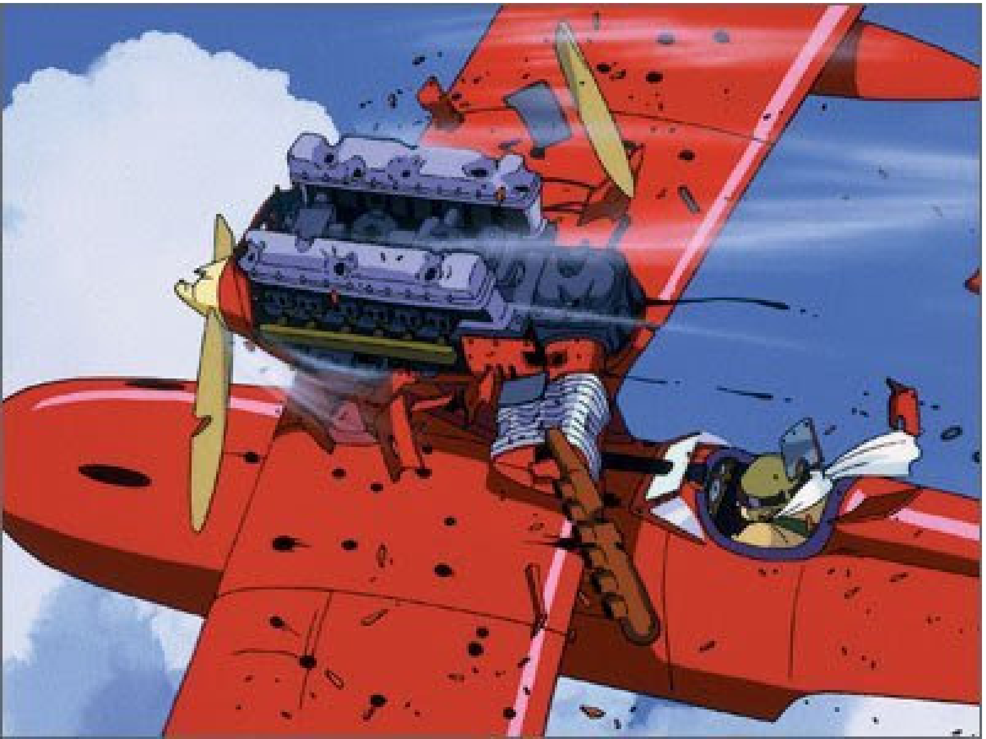
use  in series and start from $|+\rangle$ or $|-\rangle$ states:



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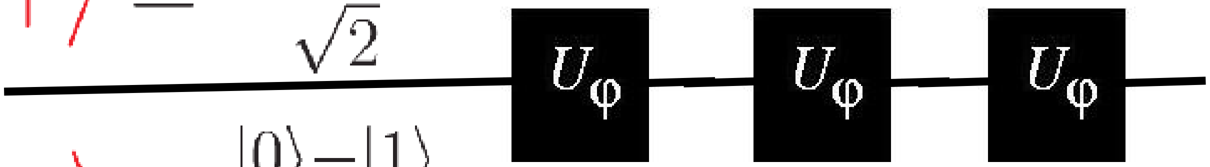
Sequential (multiround) strategies

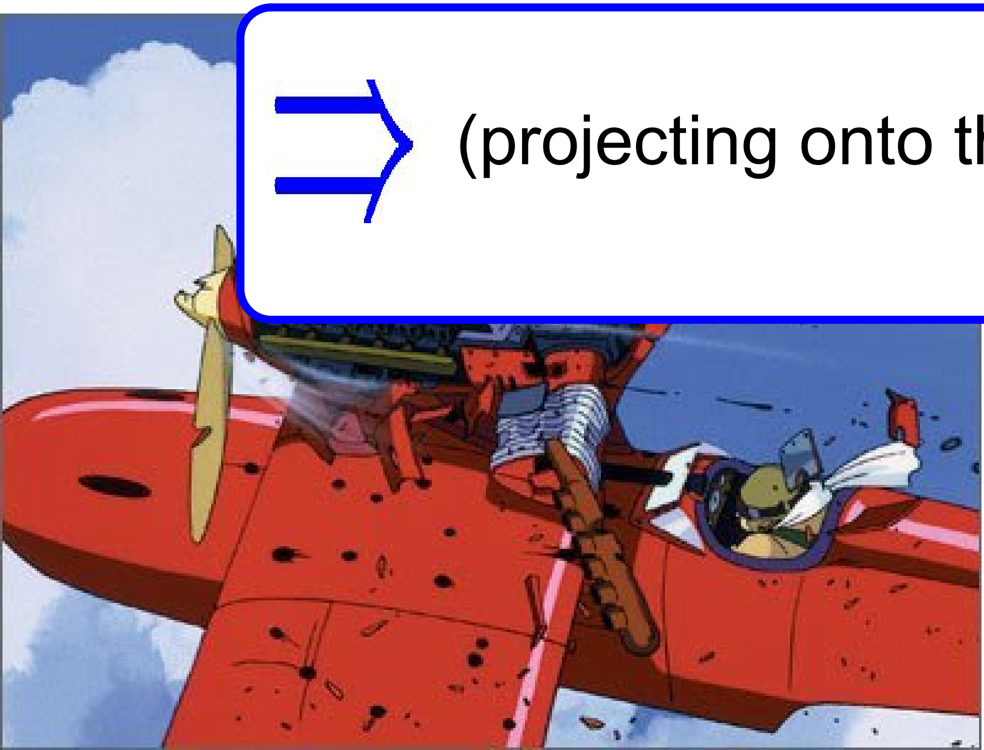
use U_φ in series and start from $|+\rangle$ or $|-\rangle$ states:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$



$$\frac{|0\rangle \pm e^{iN\varphi}|1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$\Rightarrow$ (projecting onto the initial state) $\quad \Delta\varphi \propto \frac{1}{N}$

"Heisenberg"-like scaling

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How to evaluate the goodness of the estimation? $\longrightarrow \Delta\varphi$ (RMS or standard deviation)

lower bound: generalized uncertainties

From the **Cramer-Rao bound**, it is possible to show that

[Ann. Phys. **247**, 135 (1996)]

if φ comes from a unitary (i.e. $\varrho(\varphi) = U_\varphi \varrho_0 U_\varphi^\dagger$) ***generated*** by h

i.e. $\frac{d\varrho}{d\varphi} = -i[h,\varrho]$

$$\Rightarrow \boxed{\Delta^2\varphi\;\Delta^2 h \geqslant \frac{1}{4\nu}}$$

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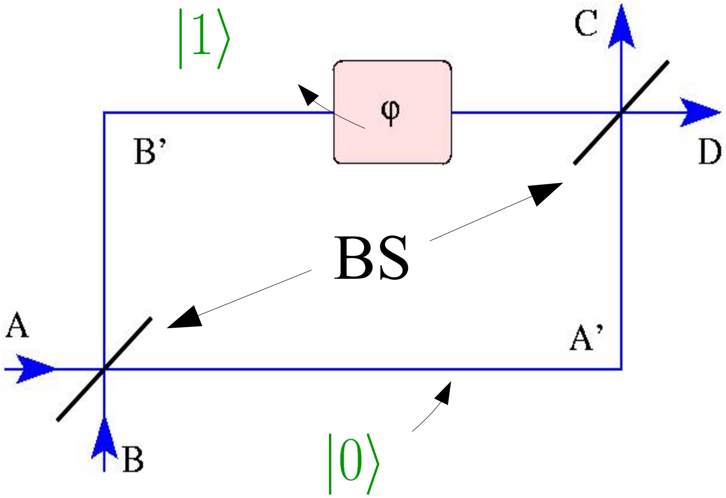
If it's **not** unitary, then the formula is more complicated, but same idea!

Prototypical example:
measure the phase in an interferometer by inserting N Photons in A and measuring the photons at each output port C and D



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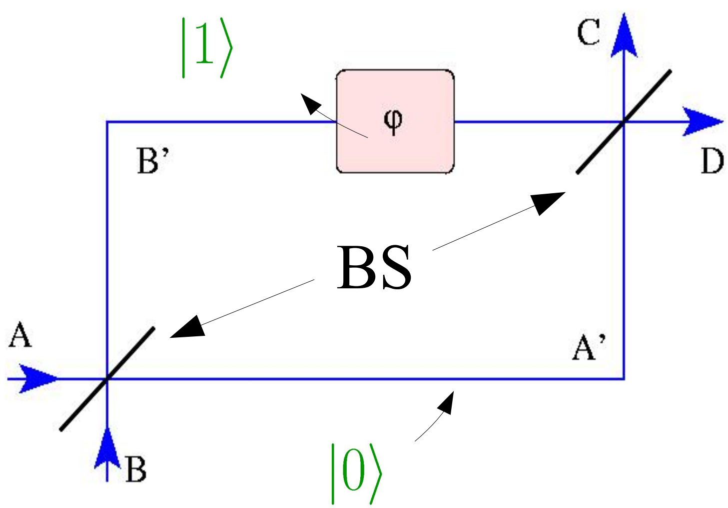
$$P_\varphi = |\langle \Psi_{in} | \Psi_{out} \rangle|^2 = \frac{1}{4} |(\langle 0| + \langle 1|)(|0\rangle + e^{i\varphi}|1\rangle)|^2 = \cos^2(\varphi/2)$$



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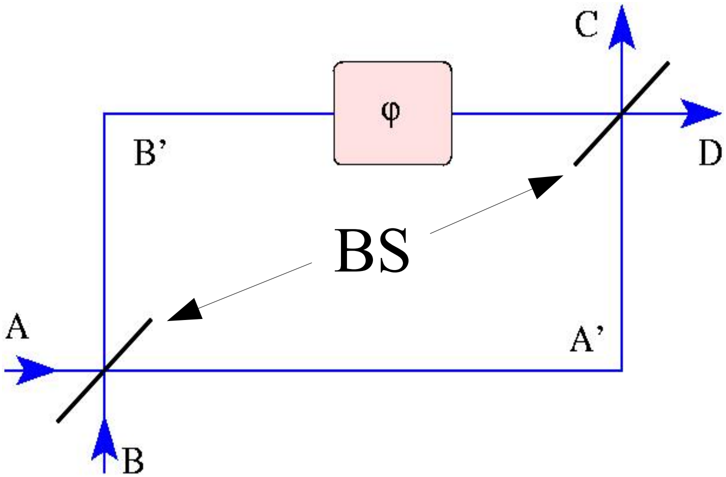

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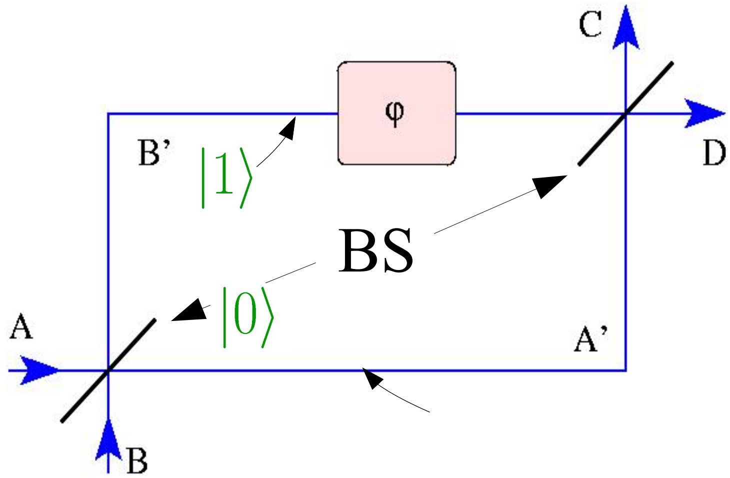

State in the interferometer:

$$\left[(|0\rangle + |1\rangle)/\sqrt{2}\right]^{\otimes N}$$

each of the *N* photons is treated independently in the interferometer

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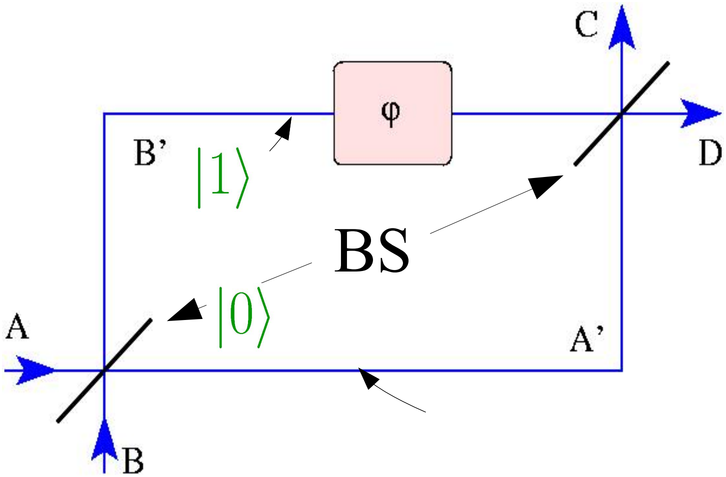

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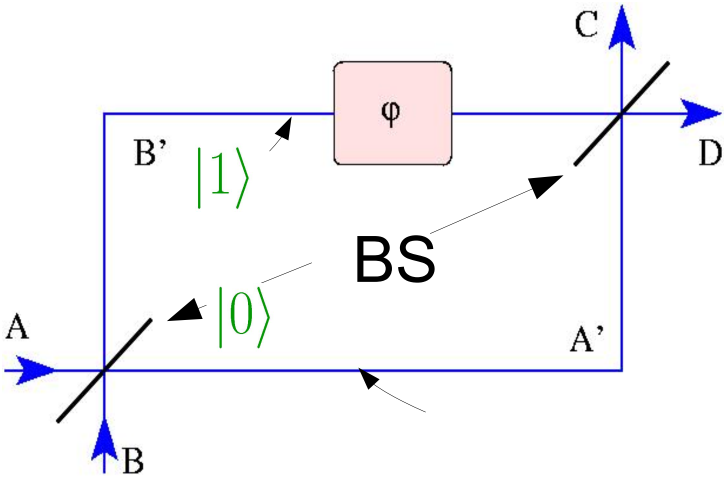

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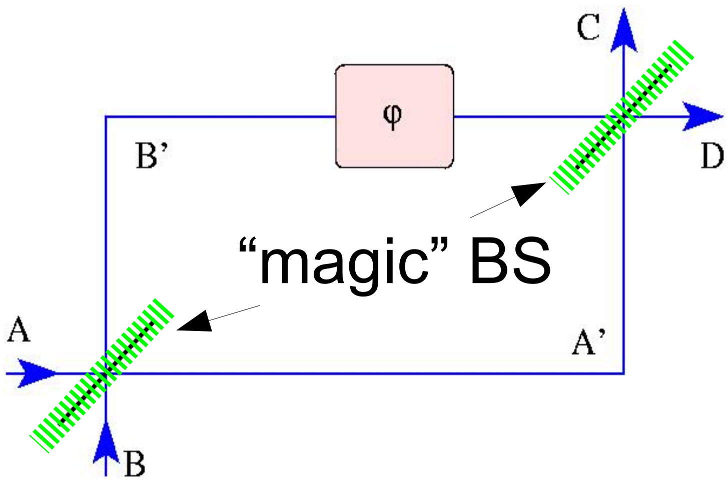

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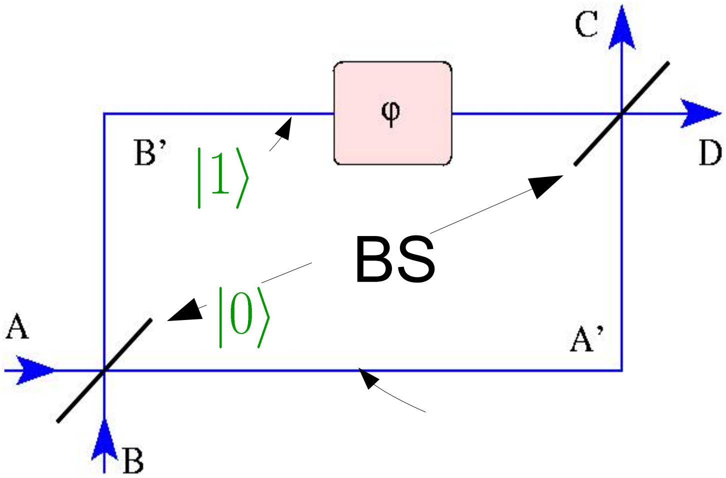

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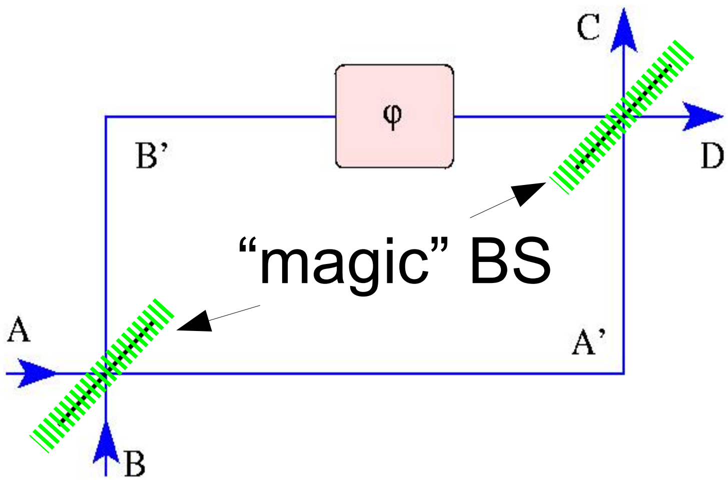

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the *N* photons are "collectively" employed (i.e. they are entangled) in the interferometer

$$\Rightarrow \Delta\varphi = 1/N$$

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HINT:

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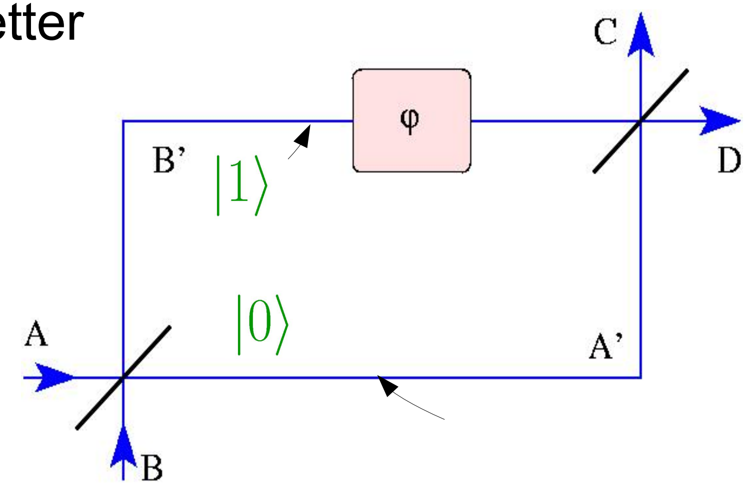

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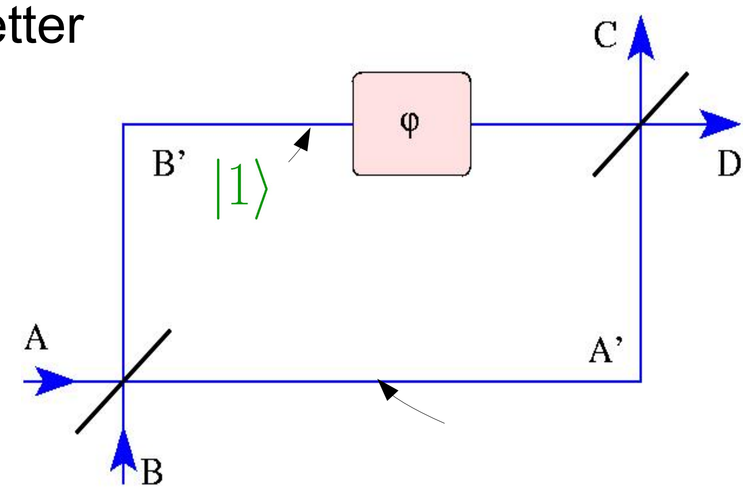
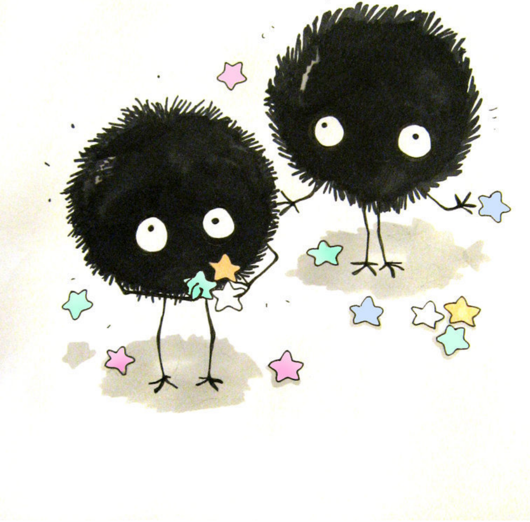


ENTANGLED STATE! Importance of entanglement

Entanglement?!?



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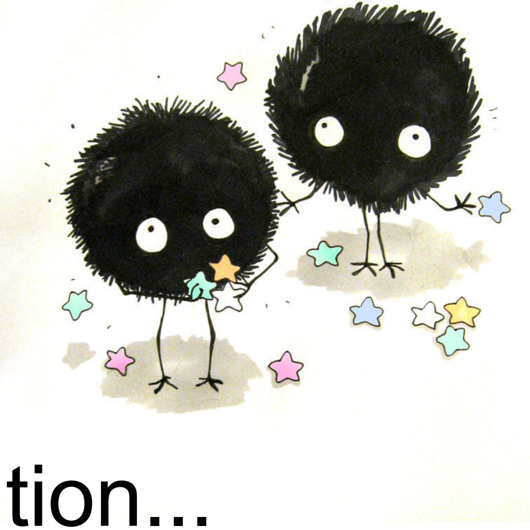

What is entanglement anyway?

An “intuitive” easy-to-understand definition...

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that cannot exist at the same time!

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2) let the probes interact with the system;

3) measure on a dual basis.

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2) let the probes interact with the system;

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Result: a $\sqrt{N}$ precision enhancement.

PRL 96, 010401 (2006) PHYSICAL REVIEW LETTERS week ending 13 JANUARY 2006

Quantum Metrology

Vittorio Giovannetti,[1] Seth Lloyd,[2] and Lorenzo Maccone[3]

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[2]*MIT, Research Laboratory of Electronics and Department of Mechanical Engineering, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139, USA*
[3]*QUIT–Quantum Information Theory Group, Dipartimento di Fisica "A. Volta" Università di Pavia, via Bassi 6,*

NATURE|Vol 440|30 March 2006 NEWS & VIEWS

QUANTUM METROLOGY

Size isn't everything

be optimal because it achieves the bound, known as the Cramér–Rao lower bound[2], that expresses the best accuracy that can be

REVIEW ARTICLES | FOCUS

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nature photonics

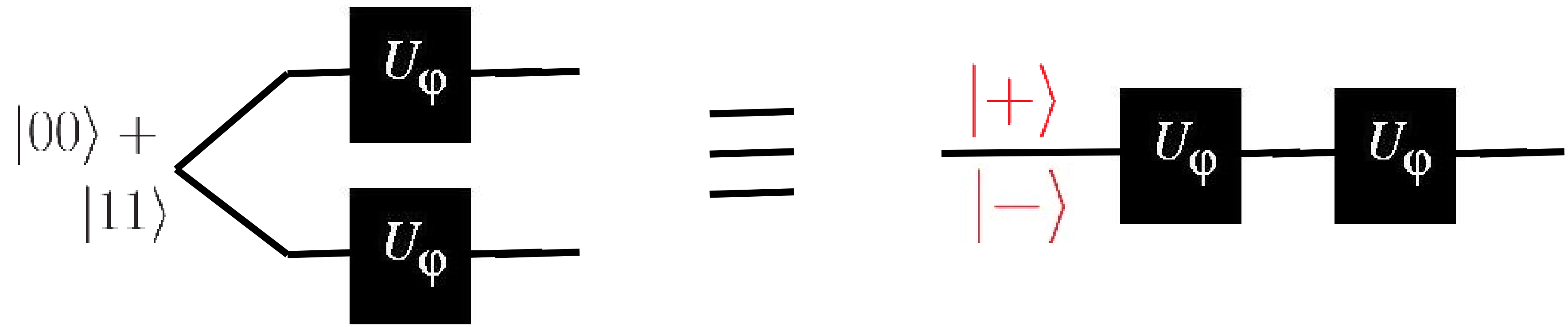
Advances in quantum metrology

Vittorio Giovannetti[1]*, Seth Lloyd[2] and Lorenzo Maccone[3]

The statistical error in any estimation can be reduced by repeating the measurement and averaging the results. The central limit theorem implies that the reduction is proportional to the square root of the number of repetitions. Quantum metrology is the use of quantum techniques such as entanglement to yield higher statistical precision than purely classical approaches. In this Review, we analyse some of the most promising recent developments of this research field and point out some of the new experiments. We then look at one of the major new trends of the field: analyses of the effects of noise and experimental imperfections.

So... Why entanglement?

BECAUSE



i.e. entanglement turns a **parallel** strategy **into** a **sequential** one.

Infinite dimensional systems

Up to now $\longrightarrow$ finite dimensional systems

(for N00N state interferometry, we were in the N-photon subspace of the radiation Hilbert space)

What happens in infinite-dimensions?

Obvious: infinite resources give infinite precision!!!

Infinite dimensional systems

Obvious: infinite resources give infinite precision!!!

... but infinite resources are irrealistic: we need to introduce some constraints (e.g. on the average energy)

what happens then?

What happens then?

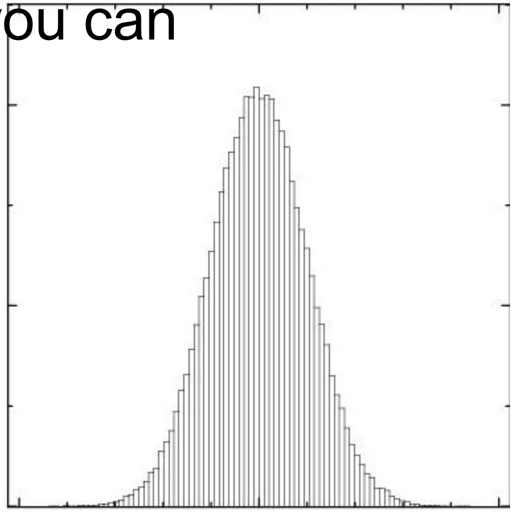
a generalization
of Heisenberg-type
uncertainty relations

Work in collaboration with
Vittorio Giovannetti and Seth Lloyd

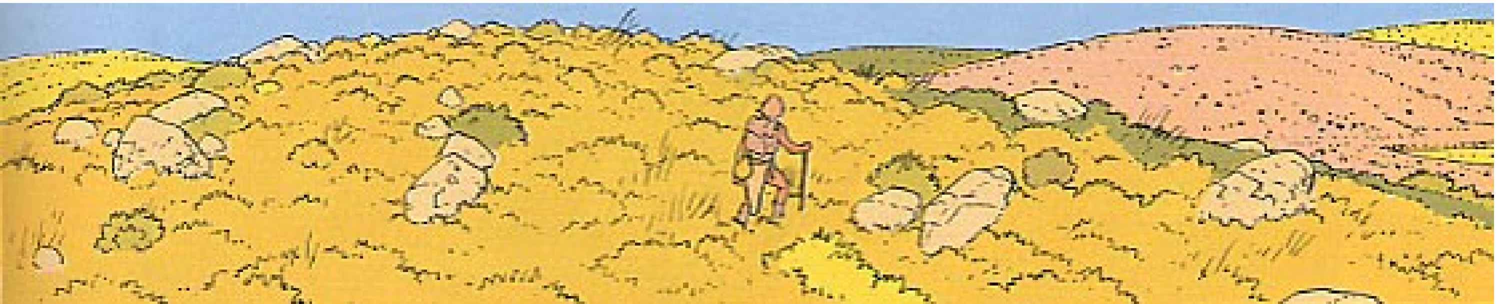
PRL **108**, 260405 (2012)
PRL **108**, 210404 (2012)

Heisenberg uncertainty relation

"If you have a probe system with spread Δp in momentum, you can discover its position with uncertainty Δx"



$$\Delta X \geqslant \frac{\hbar}{2\Delta p} \equiv \frac{1}{\Delta H}$$



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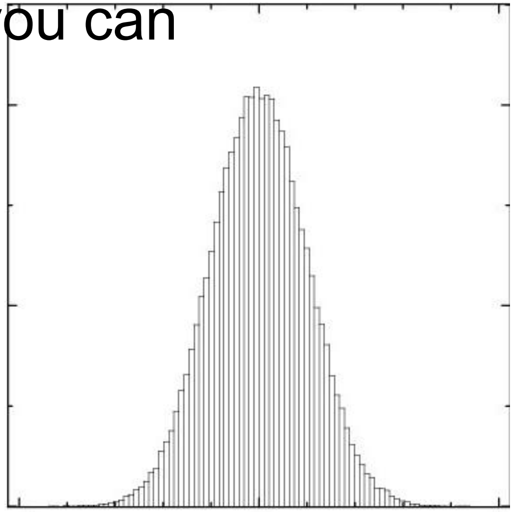

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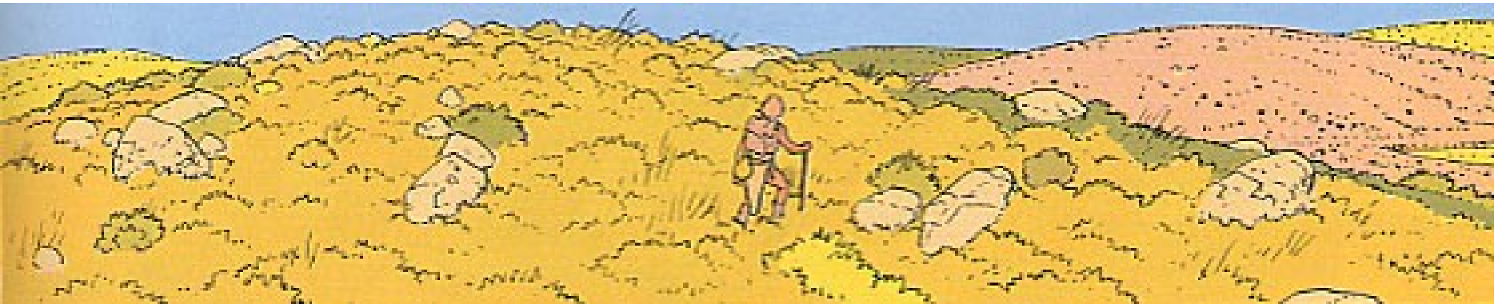
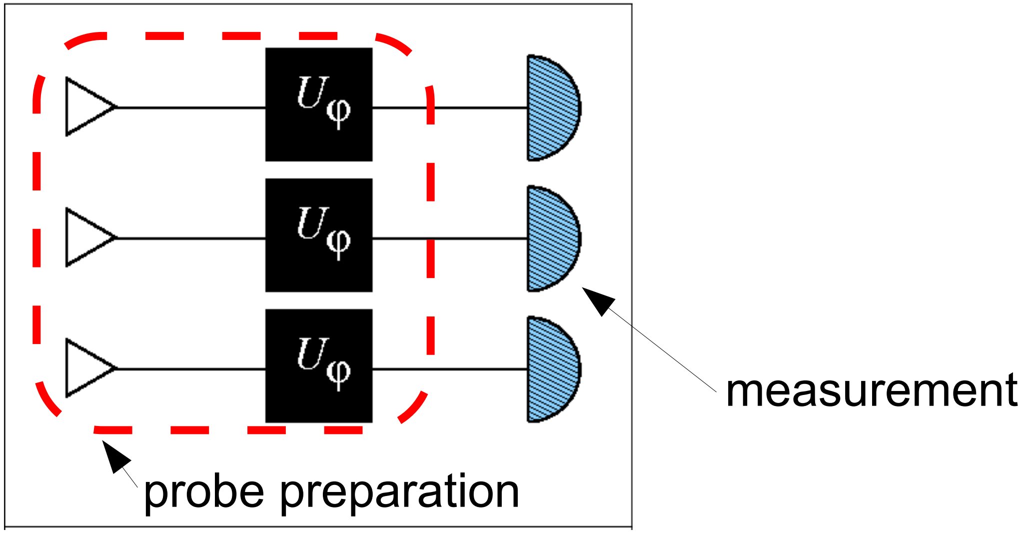
H is the generator of translations of X:

$$\rho_x = e^{-ixH} \rho_0 e^{ixH}$$



To increase precision, prepare and repeat the measurement ν times,



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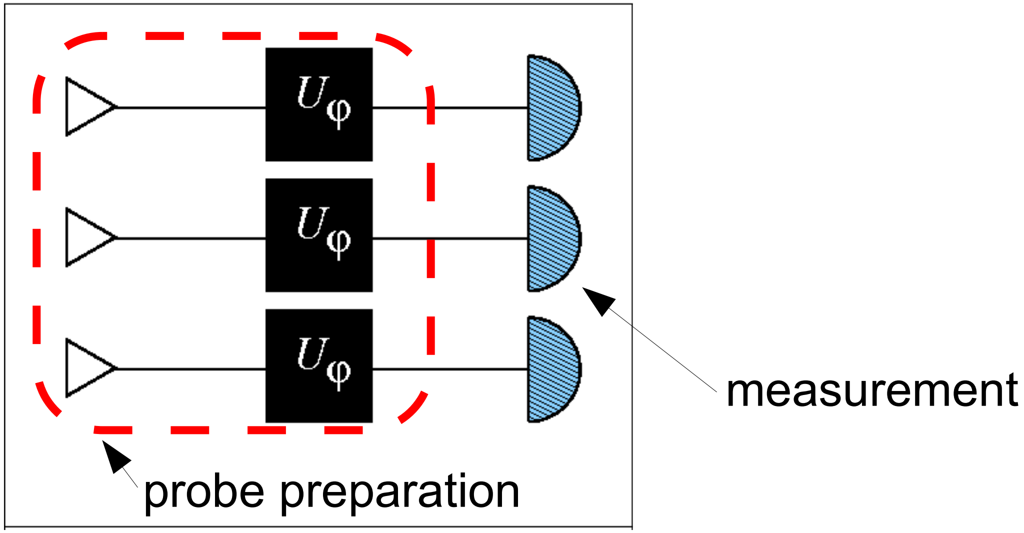

$\Longrightarrow$ the uncertainty for the **AVERAGE position** reduces

Heisenberg uncertainty

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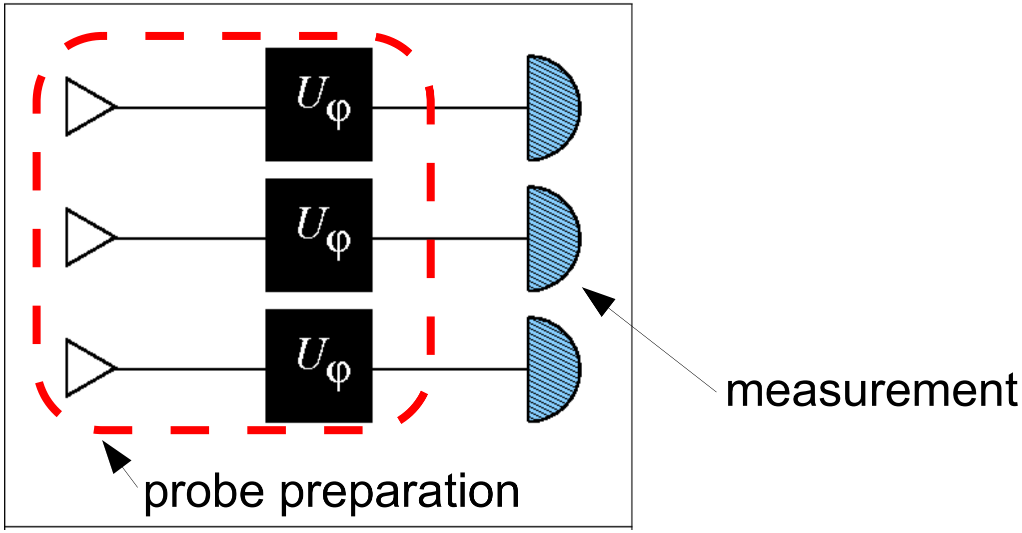

$\Longrightarrow$ the uncertainty for the **AVERAGE position** reduces

$$\boxed{\Delta X \geqslant \frac{\hbar}{\sqrt{\nu}2\Delta p} \equiv \frac{1}{\sqrt{\nu}\Delta H}}$$

New bound

Heisenberg / Cramer-Rao

$$\boxed{\Delta X \geqslant \frac{1}{\sqrt{\nu}\Delta H}}$$

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$$\mathcal{H} \equiv \langle H \rangle - E_0$$

κ constant O(1)

ground state (minimum eigenvalue of *H*)

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$$\mathcal{H} \equiv \langle H \rangle - E_0$$

κ constant O(1)

ground state (minimum eigenvalue of *H*)

precision bounded by the expectation value $\langle H \rangle$ (**first** moment) of the generator *H*

For interferometry:

$$H = a^\dagger a \ , \ \ x = \phi \ , \ \ \nu = 1$$

$$\boxed{\Delta\phi \geqslant \frac{\kappa}{N}}$$

$$N = \langle a^\dagger a \rangle \ , \ \ E_0 = 0$$

The Heisenberg bound for interferometry

PRL 108, 260405 (2012) PHYSICAL REVIEW LETTERS week ending 29 JUNE 2012

Quantum Measurement Bounds beyond the Uncertainty Relations

Vittorio Giovannetti,[1] Seth Lloyd,[2] and Lorenzo Maccone[3]

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[2]*Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[3]*Dipartimento Fisica Alessandro Volta, INFN Sezione Pavia, Università di Pavia, via Bassi 6, I-27100 Pavia, Italy*

PRL 108, 210404 (2012) PHYSICAL REVIEW LETTERS week ending 25 MAY 2012

Sub-Heisenberg Estimation Strategies Are Ineffective

Vittorio Giovannetti[1] and Lorenzo Maccone[2]

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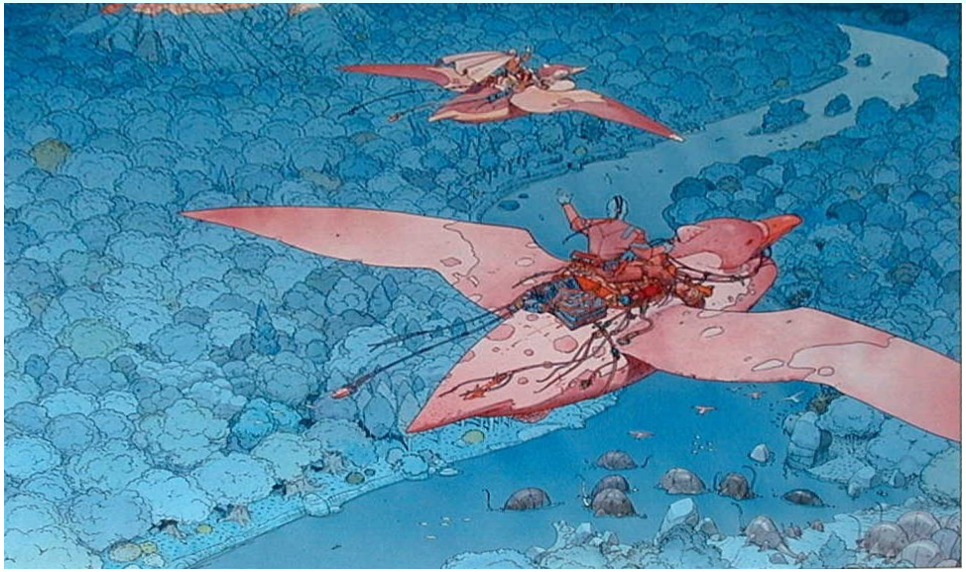
[2]*Dipartimento di Fisica "A. Volta," University of Pavia, Via Bassi 6, I-27100 Pavia, Italy*



Squeezing metrology

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Usual quantum metrology uses **entanglement** among different probes to get a better measurement precision

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Usual quantum metrology uses **entanglement** among different probes to get a better measurement precision

Here we study the effect of **quantum squeezing**

Main results

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- Heisenberg bound for squeezing

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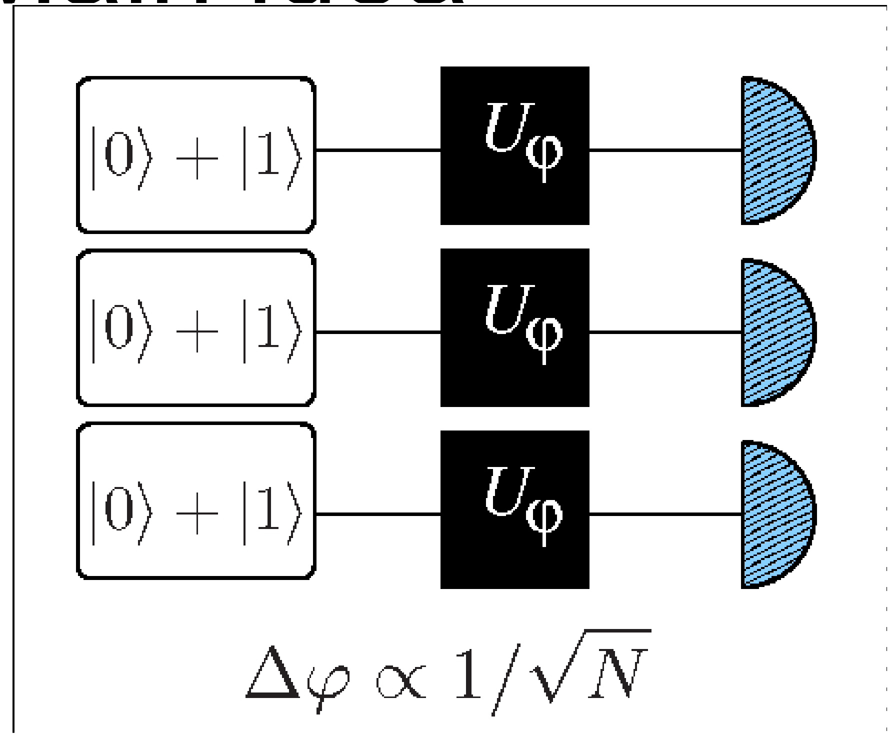
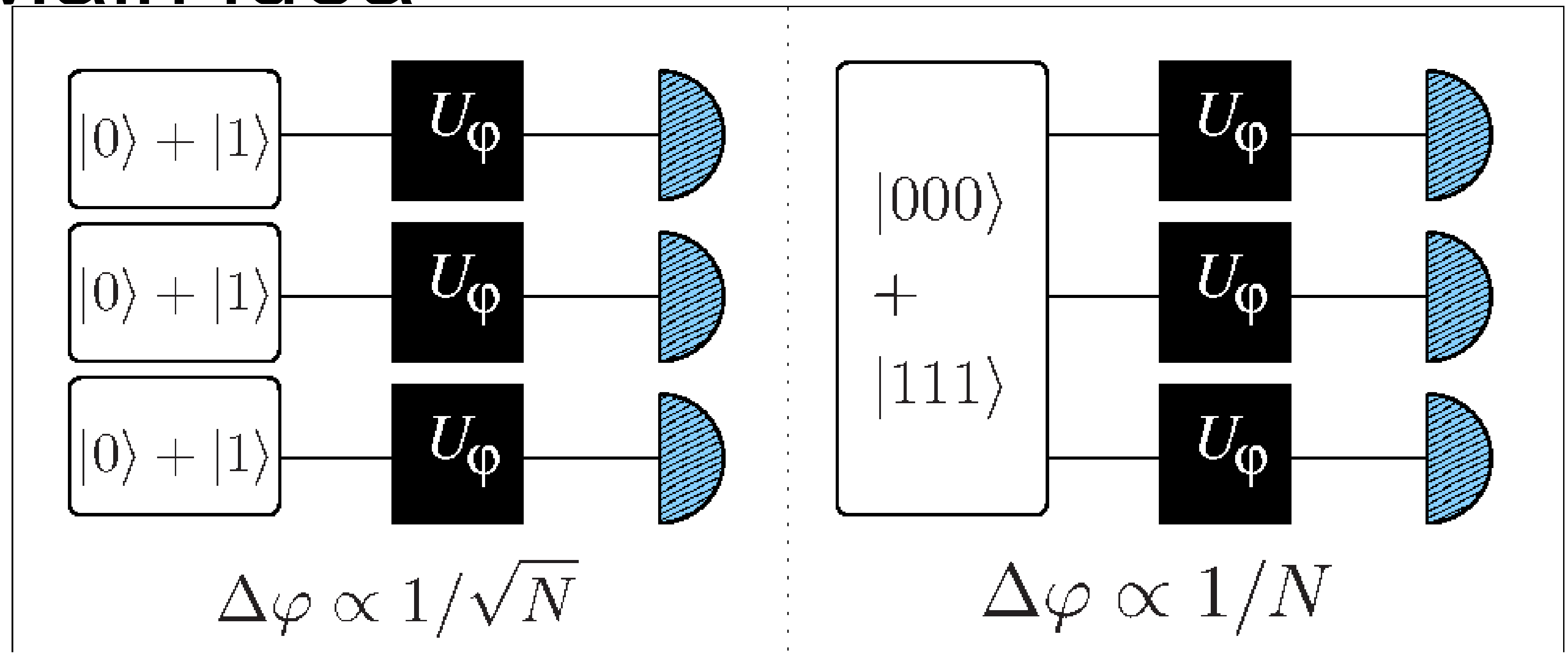
- Quadratic enhancement in precision in terms of the number of non-squeezed probes one could create with the squeezed probe's energy.

Main idea



$|0\rangle + |1\rangle$
U_φ
$|0\rangle + |1\rangle$
U_φ
$|0\rangle + |1\rangle$
U_φ
$\Delta\varphi \propto 1/\sqrt{N}$

Main idea


$|0\rangle + |1\rangle$
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$|0\rangle + |1\rangle$
U_φ
U_φ
U_φ
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$|000\rangle + |111\rangle$
U_φ
U_φ
U_φ
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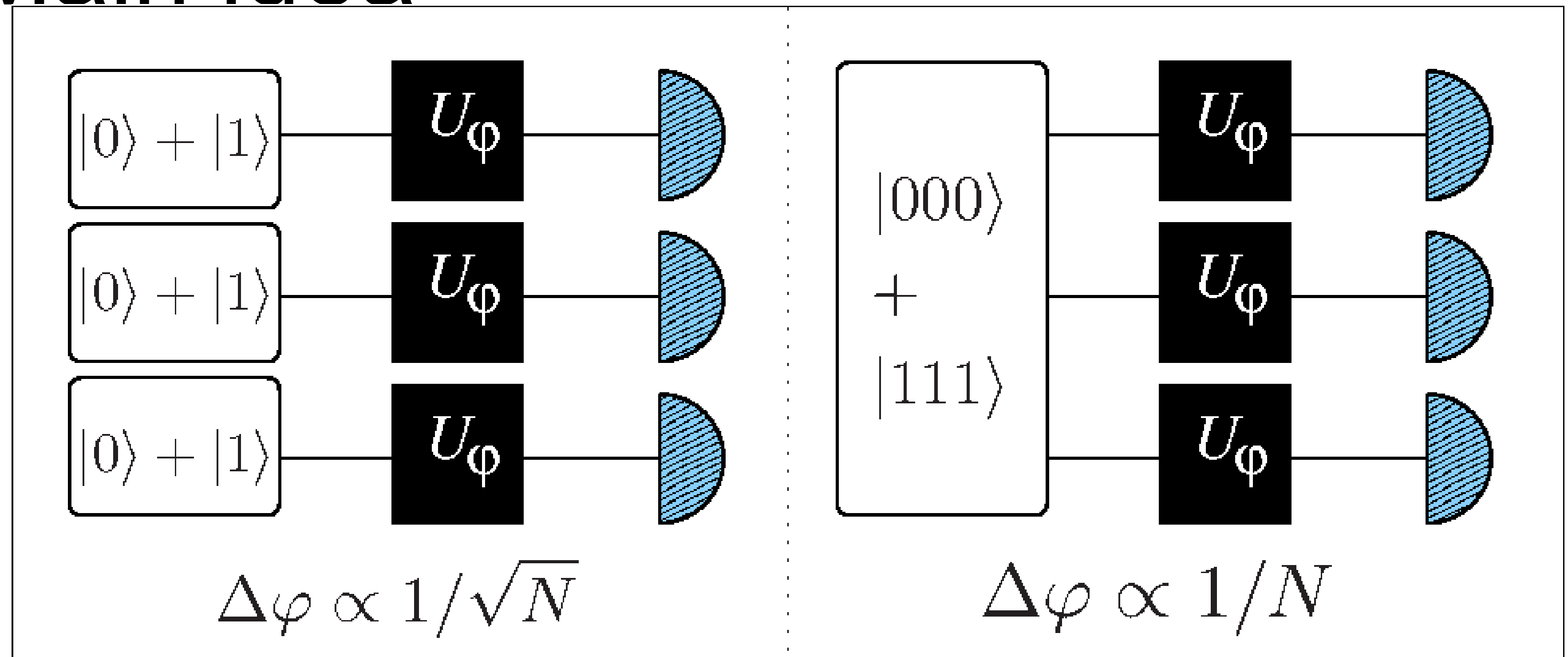
Main idea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U_φ
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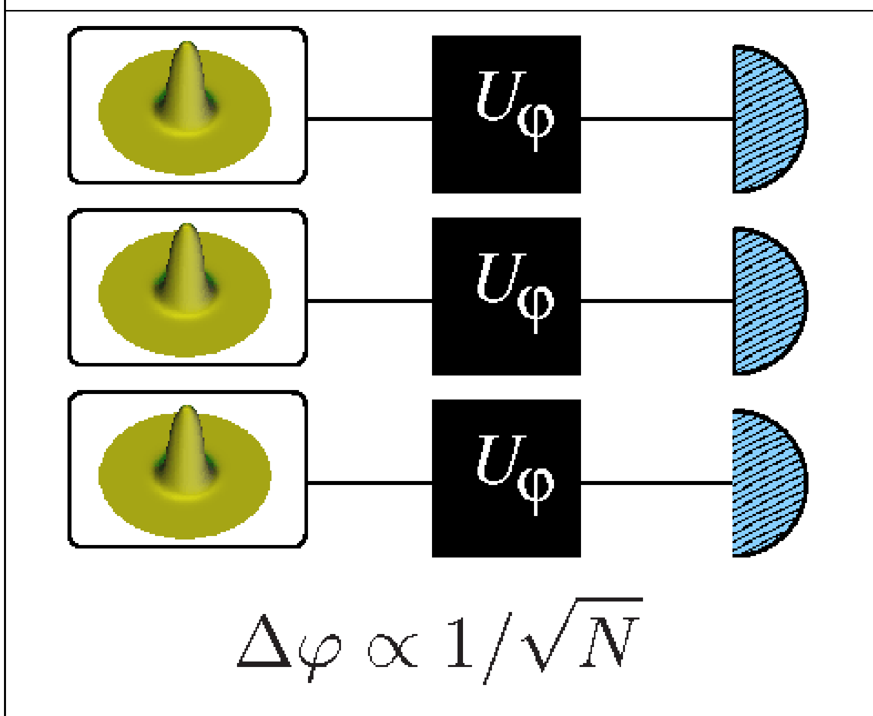
U_φ
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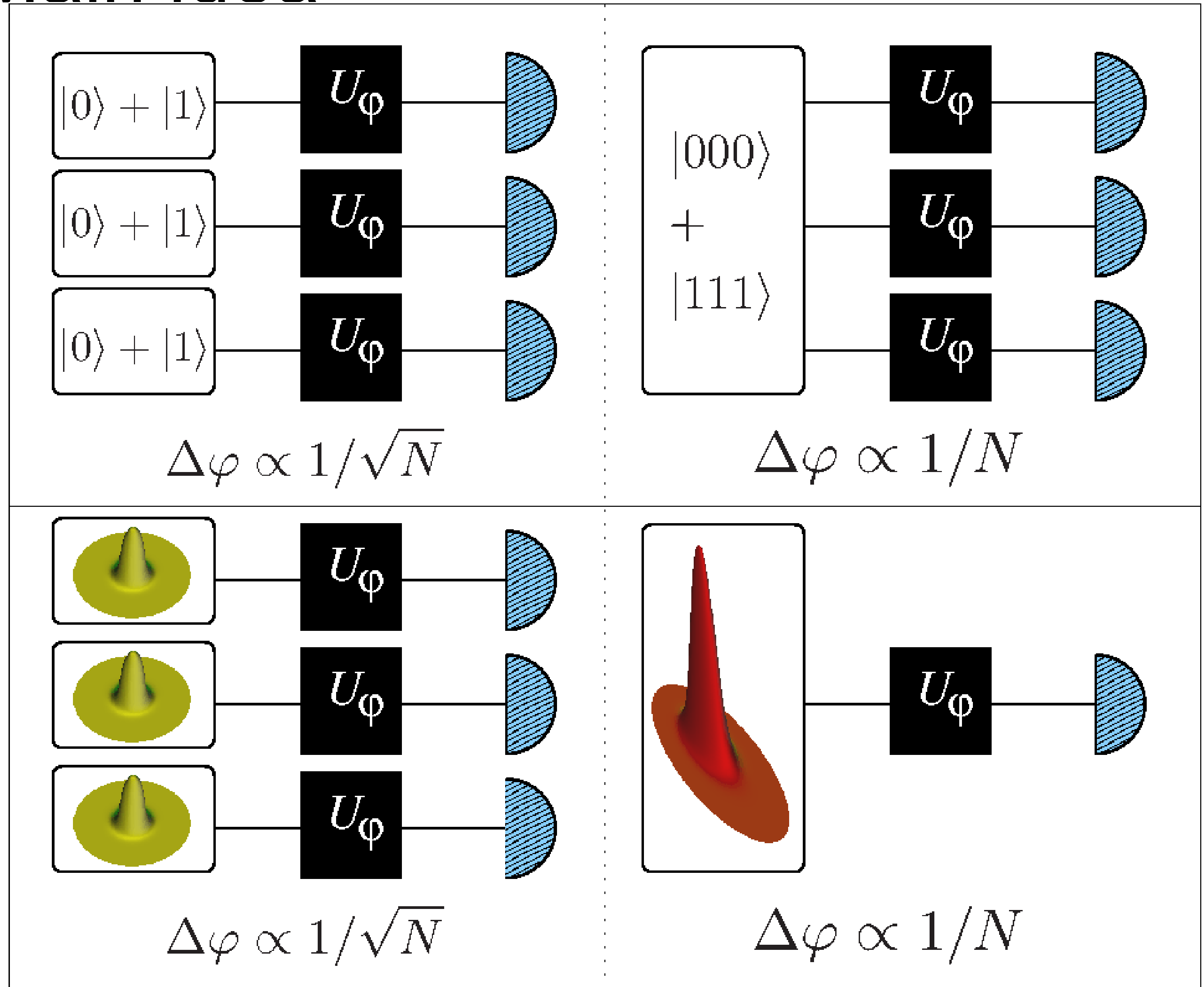
Main idea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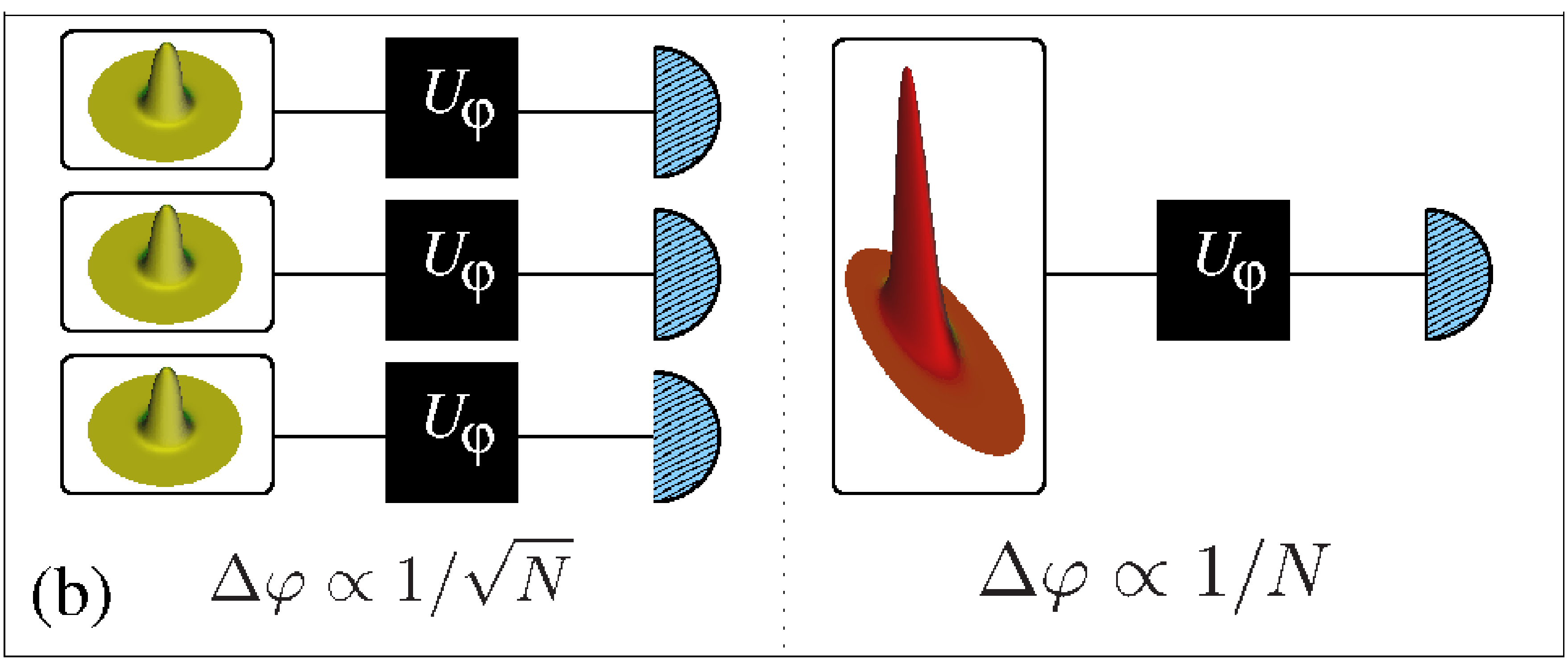
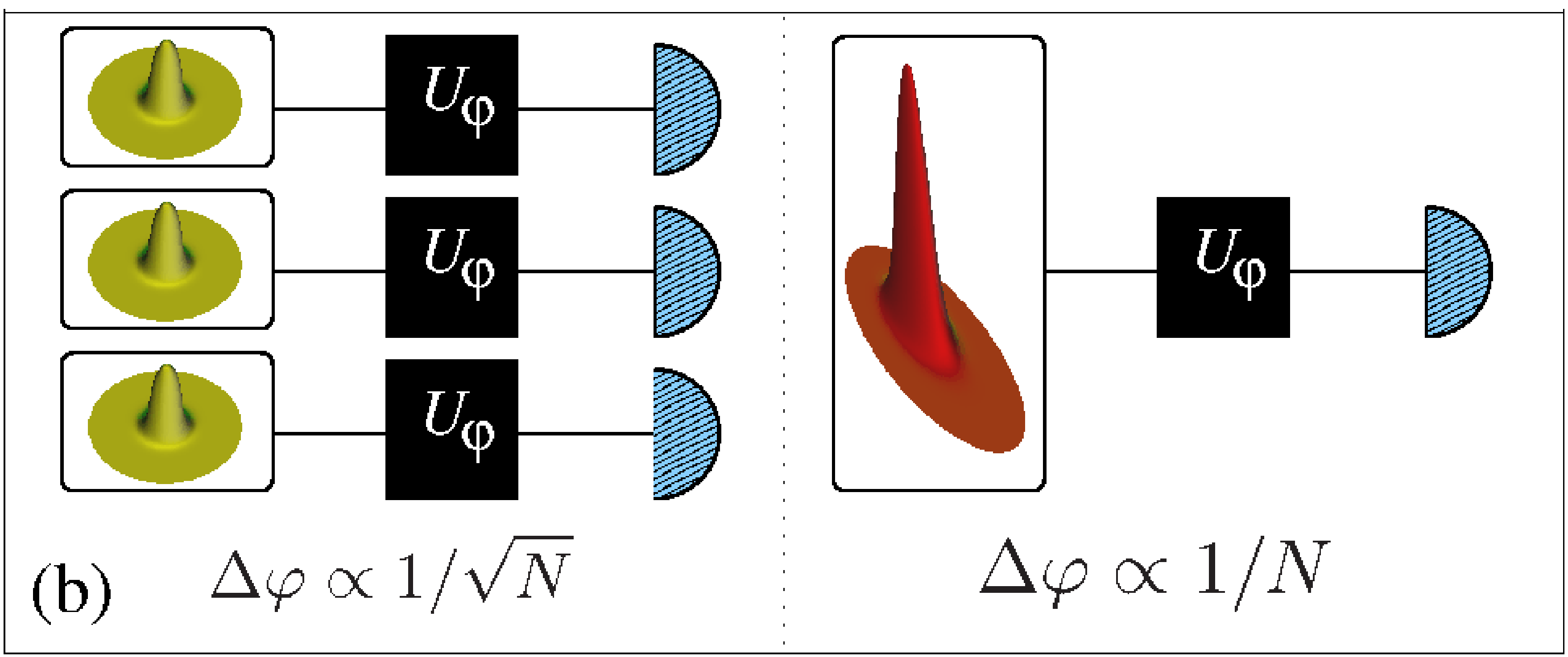
Main idea



U_φ
U_φ
U_φ
U_φ
(b)
$\Delta\varphi \propto 1/\sqrt{N}$
$\Delta\varphi \propto 1/N$

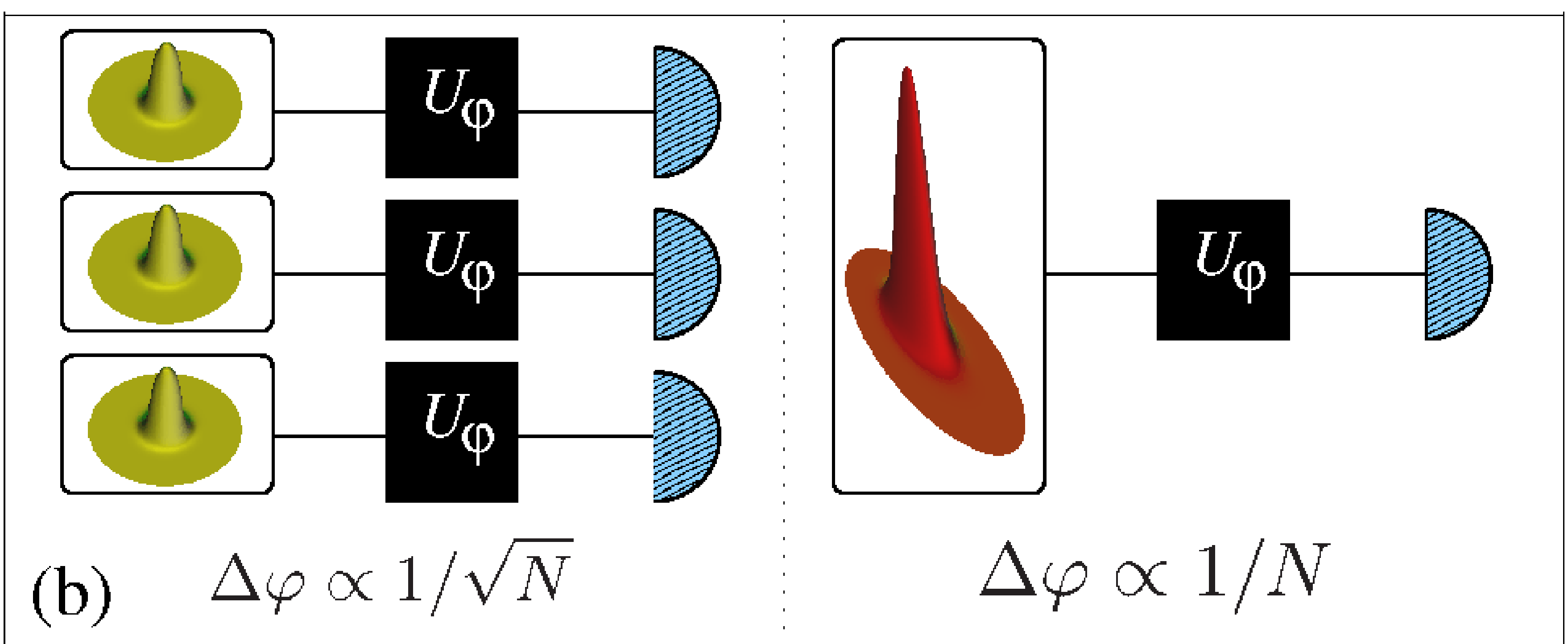
Main idea

•Take the energy used by N coherent (classical) probes



Main idea

- Take the energy used by N coherent (classical) probes
- Use it to squeeze *one* probe



Main idea

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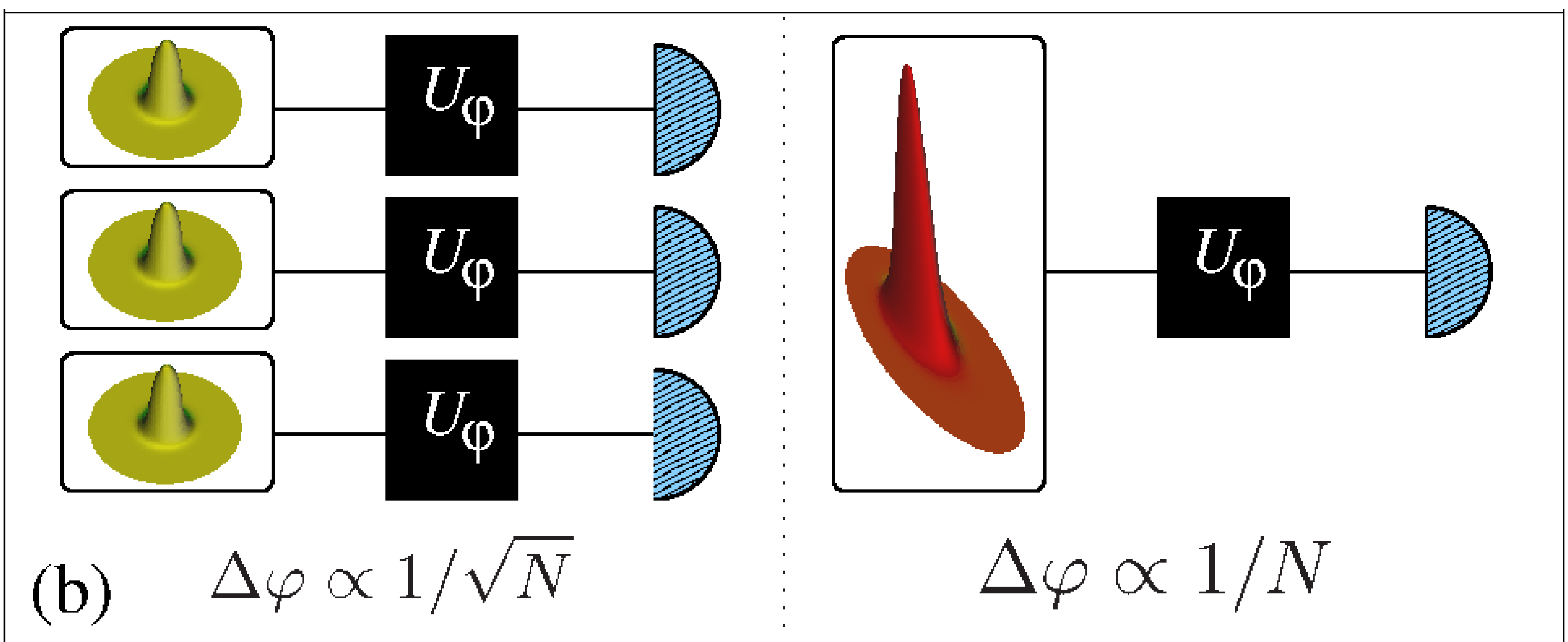
- A quadratic enhancement!! $\frac{1}{\sqrt{N}} \rightarrow \frac{1}{N}$



Main idea

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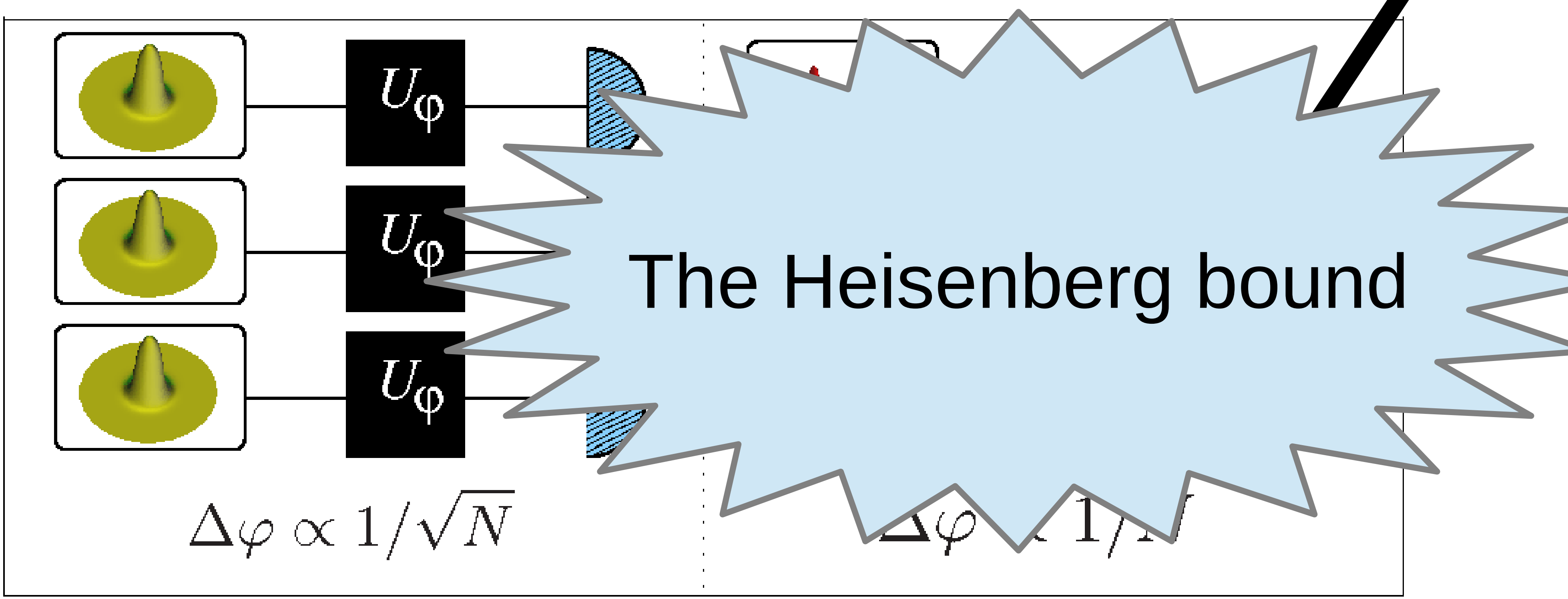
$$\frac{1}{\sqrt{N}} \rightarrow \frac{1}{N}$$



What did I say?

1. Quantum metrology parallel, sequential strategies

2. Role of entanglement

3. infinite dimensions: different bounds

4. Squeezing

Take home message

Quantum metrology: a framework to increase the precision of measurements using quantum effects!

Nat. Photon. 5, 222;
PRL 96, 010401

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