HPQED: Heavy Particle Quantum Electrodynamics

Krzysztof Pachucki (University of Warsaw & ETH)

In collaboration with V. Yerokhin (MPI, Heidelberg) & V. Patkos (Charles University, Praha)

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Pure recoil correction

- Expansion of the binding energy in powers of the electron nuclear mass ratio:

  $$E(m/M, Z\,\alpha) = E^{(0)} + \tfrac{m}{M}\,E^{(1)} + \left(\tfrac{m}{M}\right)^2 E^{(2)} + \ldots$$

  and derives a formula for each $E^{(i)}$ using the QED theory

- $E^{(0)}$ is the Dirac energy

- An approach to derive formulas for $E^{(i)}$ is the HPQED

- The starting point is the Hamiltonian of the nucleus:

  $$H_{\rm nuc} = \frac{\vec{\Pi}^2}{2\,M} + q\,A^0 - \frac{q}{2\,M}\,g\,\vec{I}\cdot\vec{B} - \frac{q\,\delta_I}{8\,M^2}\,\vec{\nabla}\cdot\vec{E} - \frac{q}{4\,M^2}\,(g-1)\,\vec{I}\cdot[\vec{E}\times\vec{\Pi} - \vec{\Pi}\times\vec{E}] + \ldots$$

  where $\vec{\Pi} = \vec{P} - q\,\vec{A}$, $q = -Z\,e$, $\delta_0 = 0$, $\delta_{1/2} = 1$.

- Finite nuclear size encoded in $G_E(-q^2)$ and $G_M(-q^2)$ is moved to the photon propagator

HPQED formalism

- First order correction

$$E^{(1)} = \left\langle \Psi \left| \frac{(\vec{P} - q\,\vec{A})^2}{2\,M} \right| \Psi \right\rangle_{\text{QED}}$$

- How to interpret the derivative over Coulomb center ?

$$\hat{\psi}(x) = \sum_s^{+} a_s \phi_s(\vec{x})\, e^{-i\,E_s t} + \sum_s^{-} b_s \phi_s(\vec{x})\, e^{-i\,E_s t}\,,$$

$$\vec{\nabla}_R = -\int d^3 r\, \hat{\psi}^+(\vec{r})\, \vec{\partial}_r\, \hat{\psi}(\vec{r}) + \vec{\partial}_R$$

- as a test

$$\vec{\nabla}_R \hat{\psi}(0,\vec{x}) = -\int d^3 r\, \hat{\psi}^+(\vec{r})\, \vec{\partial}_r\, \hat{\psi}(\vec{r})\; \hat{\psi}(0,\vec{x}) - \vec{\partial}_x \hat{\psi}(0,\vec{x}) = 0\,,$$

- Important observation: $\vec{p} - q\,\vec{A} i$ in the Coulomb gauge $\to$ $-q\,\vec{A}$ in the temporal gauge.

1: Leading order pure recoil corrections

Exact nonperturbative formula (a'la Shabaev) for the leading m/M pure recoil corrections with including the finite nuclear size

Important: the elastic contribution in the two-photon exchange should be consistent with this pure recoil correction

$$E^{(1)} = \langle\phi|\Sigma^{(1)}(E_D)|\phi\rangle$$
$$\Sigma^{(1)}(E) = \frac{i}{M}\int_s \frac{d\omega}{2\,\pi}\, D^j(\omega)\, G(E+\omega)\, D^j(\omega)$$

where

- $G(E) = [E - H_D(1 - i\epsilon)]^{-1}$ is the Dirac-Coulomb Green function
- $D^j(\omega) = -4\pi Z\alpha\,\alpha^i\, G_T^{ij}(\omega,\vec{r})$, and α^i are the Dirac matrices.
- Photon propagator in the temporal gauge

$$G_T^{ij}(\omega,\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, \frac{\rho(\vec{k}^2-\omega^2)}{\omega^2-\vec{k}^2}\left(\delta^{ij} - \frac{k^i\,k^j}{\omega^2}\right).$$

- Breit interactions is modified by the finite nuclear size !
- numerical calculations & Vladimir Yerokkhin (in the Coulomb gauge)

Expansion in $Z\alpha$

- for a point nucleus

$$E_{\rm rec} = \frac{m^2 - E_D^2}{2\,M} + E_{\rm rec}^{(5)} + E_{\rm rec}^{(6)} + \ldots$$

$$E_{\rm rec}^{(5)} = -\frac{7}{6\,\pi}\,\frac{(Z\,\alpha)^2}{M\,m}\left\langle\frac{1}{r^3}\right\rangle - \frac{8}{3\,\pi}\,\frac{Z\,\alpha}{M\,m}\left\langle\vec{p}\,(H_0 - E_0)\,\ln\left[\frac{2\,(H_0 - E_0)}{m\,(Z\,\alpha)^2}\right]\vec{p}\right\rangle$$
$$+\left(\frac{62}{9} - \frac{2}{3}\,\ln(Z\,\alpha)\right)\frac{(Z\,\alpha)^2}{m\,M}\,\langle\delta^{(3)}(r)\rangle$$

$$E_{\rm rec}^{(6)} = \frac{(Z\,\alpha)^2}{2\,M\,m^2}\,\langle\phi|\frac{\vec{L}^2}{r^4}|\phi\rangle + \left(4\,\ln 2 - \frac{7}{2}\right)\frac{(Z\,\alpha)^3}{M\,m}\,\langle\pi\,\delta^3(r)\rangle$$

$$E_{\rm rec}^{(7)} = \text{??? only numerically}$$

- for the finite size nucleus (instead of r_F^3)

$$\delta E_{\rm rec,fns} = -\frac{m}{M}\,\phi^2(0)\,(Z\,\alpha)^2\left[\frac{7}{6} - 2\,\gamma - 2\,\ln(m\,\tilde{r})\right]r_C^2 + \ldots$$

important in muonic atoms

Radiative recoil correction

$$E_{\rm vprec} = \delta_{\rm vp} \frac{i}{M} \int_s \frac{d\omega}{2\,\pi} \langle\phi| D^j(\omega)\, G(E_D+\omega)\, D^j(\omega)|\phi\rangle$$

Vacuum polarization can effectively be implemented by modification of $\rho(-k^2)$ and is important for muonic atoms.

$$E_{\rm selfrec} = \langle\phi|\Sigma_{\rm radrec}(E_D)|\phi\rangle + 2\,\langle\phi|\Sigma_{\rm rad}(E_D)\,\frac{1}{(E_D-H_D)'}\,\Sigma_{\rm rec}(E_D)|\phi\rangle$$
$$+ \langle\phi|\Sigma'_{\rm rad}(E_D)|\phi\rangle\,\langle\phi|\Sigma_{\rm rec}(E_D)|\phi\rangle + \langle\phi|\Sigma'_{\rm rec}(E_D)|\phi\rangle\,\langle\phi|\Sigma_{\rm rad}(E_D)|\phi\rangle$$

$$\Sigma_{\rm radrec}(E) = \frac{i}{M}\int_s \frac{d\omega'}{2\,\pi}\, e^2 \int \frac{d^4k}{(2\,\pi)^4\, i}\,\frac{1}{k^2}$$
$$\times \Big[\alpha^\mu\, e^{-i\,\vec{k}\cdot\vec{r}}\, G(E+\omega)\, D^j(\omega')\, G(E+\omega+\omega')^{\,j}(\omega')\, G(E+\omega)\,\alpha_\mu\, e^{i\,\vec{k}\cdot\vec{r}}$$
$$+ D^j(\omega')\, G(E+\omega')\,\alpha^\mu\, e^{-i\,\vec{k}\cdot\vec{r}}\, G(E+\omega+\omega')\,\alpha_\mu\, e^{i\,\vec{k}\cdot\vec{r}}\, G(E+\omega')\, D^j(\omega')$$
$$+ \alpha^\mu\, e^{-i\,\vec{k}\cdot\vec{r}}\, G(E+\omega)\, D^j(\omega')\, G(E+\omega+\omega')\;\alpha_\mu\, e^{i\,\vec{k}\cdot\vec{r}}\, G(E+\omega')\, D^j(\omega')$$
$$+ D^j(\omega')\, G(E+\omega')\,\alpha^\mu\, e^{-i\,\vec{k}\cdot\vec{r}}\, G(E+\omega+\omega')\; D^j(\omega')\, G(E+\omega)\,\alpha_\mu\, e^{i\,\vec{k}\cdot\vec{r}}\Big]$$

It has not yet been calculated numerically, and is known only within $Z\,\alpha$ expansion.

Second order pure recoil correction

For the scalar nucleus

$$E^{(2)} = \langle\phi|\Sigma^{(1)}(E_D)\,\frac{1}{(E_D - H_D)'}\,\Sigma^{(1)}(E_D)|\phi\rangle$$

$$+ \left(\frac{d}{dE}\bigg|_{E=E_D}\langle\phi|\Sigma^{(1)}(E)|\phi\rangle\right)\langle\phi|\Sigma^{(1)}(E_D)|\phi\rangle + \langle\phi|\Sigma^{(2)}(E_D)|\phi\rangle$$

where

$$\Sigma^{(1)}(E) = \frac{i}{M}\int_s \frac{d\omega}{2\,\pi}\,D^j(\omega)\,G(E+\omega)\,D^j(\omega)\,,$$

$$\Sigma^{(2)}(E) = \left(\frac{i}{M}\int_s \frac{d\omega_1}{2\,\pi}\right)\left(\frac{i}{M}\int_s \frac{d\omega_2}{2\,\pi}\right)$$

$$\Big[D^j(\omega_1)\,G(E_D+\omega_1)\,D^k(\omega_2)\,G(E_D+\omega_1+\omega_2)\,D^j(\omega_1)\,G(E_D+\omega_2)\,D^k(\omega_2)$$

$$+ D^j(\omega_1)\,G(E_D+\omega_1)\,D^k(\omega_2)\,G(E_D+\omega_1+\omega_2)\,D^k(\omega_2)\,G(E_D+\omega_1)\,D^j(\omega_1)\Big]\,.$$

This second order recoil correction has not yet been calculated,
but is important for muonic atoms !

HFS in the nonrecoil limit

- Hyperfine splitting for the infinitely heavy nucleus is obtained from the expectation value

$$E_{\rm hfs} = \langle\phi|\, V_{\rm hfs}\,|\phi\rangle\;.$$

where

$$V_{\rm hfs} = -\, e\,\vec{\alpha}\cdot\vec{A}_I\,,$$

$$e\,\vec{A}_I(\vec{r}) = \frac{e}{4\,\pi}\,\vec{\mu}\times\left[\frac{\vec{r}}{r^3}\right]_{\rm fs}\,,$$

$$\frac{1}{4\,\pi}\left[\frac{\vec{r}}{r^3}\right]_{\rm fs} = -\,\vec{\nabla}\int\frac{d^3q}{(2\,\pi)^3}\,\frac{\rho_M(\vec{q}^{\,2})}{\vec{q}^{\,2}}\,e^{i\,\vec{q}\,\vec{r}}\,.$$

- The finite nuclear size contribution can be expanded in $Z\,\alpha$

$\delta E_{\rm nucl} = \delta^{(1)} E_{\rm nucl} + \delta^{(2)} E_{\rm nucl} + \ldots$

- $\delta^{(1)} E_{\rm nucl} = -2\, m_r Z\alpha\, r_Z E_F$ where $r_Z = \int d^3r_1 \int d^3r_2\, \rho_M(r_1)\, \rho_E(r_2)\, |\vec{r}_1 - \vec{r}_2|$

- $\delta^{(2)} E_{\rm fns} = \frac{4}{3} E_F (m r_C Z\alpha)^2 \left[-\frac{1}{n} + 2\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m\,\tilde{r}\,Z\alpha) + \frac{r_M^2}{4 r_C^2 n^2} \right]$

Finite nuclear mass correction to the hyperfine splitting

Exact in $Z\alpha$ formula: $H_{\rm nuc} = \frac{\vec{\Pi}^2}{2M} - \frac{q}{2M}\, g\, \vec{I}\cdot\vec{B} - \frac{q}{4M^2}\,(g-1)\,\vec{I}\cdot[\vec{E}\times\vec{\Pi} - \vec{\Pi}\times\vec{E}]$

$$
\begin{aligned}
E_{\rm hfsrec} &= E_{\rm kin} + E_{\rm so} + E_{\rm sec} \\
E_{\rm kin} &= \frac{1}{M}\int_s \frac{d\omega}{2\pi}\,\frac{1}{\omega}\,\big[\langle\phi|D_T^j(\omega)\,G(E_D+\omega)\,\partial^j(V_{\rm hfs}(\omega))|\phi\rangle - \langle\phi|\partial^j(V_{\rm hfs}(\omega))\,G(E_D+\omega)\,D_T^j(\omega)|\phi\rangle\big] \\
&\quad + \delta_{\rm hfs}\,\frac{i}{M}\int_s \frac{d\omega}{2\pi}\,\langle\phi|D_T^j(\omega)\,G(E_D+\omega)\,D_T^j(\omega)|\phi\rangle\,, \\
E_{\rm so} &= -\frac{(g-1)}{M^2}\,\epsilon^{ijk}\,I^i\int_s \frac{d\,\omega}{2\pi}\,\omega\,\langle\phi|D_T^j(\omega)\,G(E_D+\omega)\,D_T^k(\omega)|\phi\rangle\,, \\
E_{\rm sec} &= \left(\frac{4\pi Z\alpha}{2M}\,g\right)^2\epsilon^{ijk}\,I^k\int_s\frac{d\,\omega}{2\pi}\,\frac{1}{\omega}\,\langle\phi|(\vec{\alpha}\times\vec{\nabla})^i\,D(\omega)\,G(E_D+\omega)\,(\vec{\alpha}\times\vec{\nabla})^j\,D(\omega)\,|\phi\rangle\,,
\end{aligned}
$$

where

$$
V_{\rm hfs}(\omega,\vec{r}) = e\,\vec{\mu}\cdot\vec{\alpha}\times\vec{\nabla}D(\omega,r)\,,
$$

such that $V_{\rm hfs}(0,r) = V_{\rm hfs}(r)$, and

$$
D(\omega,r) = \int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{r}}\,\frac{\rho(\vec{k}^2-\omega^2)}{\omega^2-\vec{k}^2}\,.
$$

Recoil HFS: expansion in $Z\alpha$

$$\delta E_{\rm rec} = \delta^{(1)} E_{\rm rec} + \delta^{(2)} E_{\rm rec} + \ldots$$

$$\delta^{(1)} E_{\rm rec} = -E_F \frac{Z\alpha}{\pi} \frac{m}{M} \frac{3}{8} \left\{ g\left[\gamma - \frac{7}{4} + \ln(m\, r_{M2})\right] - 4\left[\gamma + \frac{9}{4} + \ln(m\, r_{EM})\right] - \frac{12}{g}\left[\gamma - \frac{17}{12} + \ln(m\, r_{E2})\right]\right\}$$

$$\delta^{(2)} E_{\rm rec} = E_F\,(Z\alpha)^2 \frac{m_{\rm r}^2}{m\,M} \left\{ -\frac{\ln(Z\alpha)}{4}\left[-6 + \frac{7}{2}g + \frac{14}{g}\right] - \frac{\ln 2}{4}\left[-2 + \frac{11}{2}g + \frac{46}{g}\right] + \frac{1}{36}\left[-\frac{81}{2} + \frac{31}{2}g + \frac{279}{g}\right]\right\}.$$

$\delta^{(1)} E_{\rm rec}$ is about 10% of the leading Zemach contribution for light elements, however the elastic form-factor assumption is not necessarily a good approximation !

$\delta^{(2)} E_{\rm rec}$ is an of result of complicated calculations by Bodwin and Yennie (1988) in the point nucleus limit and has not yet been verified.

We aim to verify their result using exact formulas analytically and numerically (to all orders in $Z\alpha$), and obtain the finite nuclear size contribution (important for μH).

HPQED

- There exists a compact formula for $(m/M)^n$ recoil correction for any n
- Hamiltonian of the nucleus:

$$H_{\rm nuc} = \frac{\vec{\Pi}^2}{2\,M} + q\,A^0 - \frac{q}{2\,M}\,g\,\vec{I}\cdot\vec{B} - \frac{q\,\delta_I}{8\,M^2}\,\vec{\nabla}\cdot\vec{E} - \frac{q}{4\,M^2}\,(g-1)\,\vec{I}\cdot[\vec{E}\times\vec{\Pi} - \vec{\Pi}\times\vec{E}] + \ldots$$

  where $\vec{\Pi} = \vec{P} - q\,\vec{A}$
- In the temporal gauge $\vec{\Pi} \to -q\,\vec{A}$
- Potential applications: hydrogen Lamb shift, muonic atoms
- Extension to a few electron ions using $1/Z$ expansion and two-time Green function a'la Shabaev