Nuclear recoil effect without expansion in $Z\alpha$

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Introduction

- Approach without expansion in $Z\alpha$ => electronic and muonic atoms can be considered on the same footing.
- However, for electronic atoms the point-nucleus limit is (generally) a good approximation; for muonic atoms it is NOT.
- Limitation: we assume that the nucleus is described by form-factors == elastic part of nuclear effects.
- Applicable for relatively heavy nuclei, for which one cannot calculate both elastic and inelastic nuclear effects on the same footing.

Two-body atom: nonrelativistic picture

Energy levels from Schrödinger equation

$$E_{nl} = -\frac{m}{1+\frac{m}{M}}\frac{(Z\alpha)^2}{2n^2} + \left(\frac{m}{1+\frac{m}{M}}\right)^3 \frac{2(Z\alpha)^4}{3n^3} r_C^2\, \delta_{l,0}$$

$$E_{\rm pnt} = -m\,\frac{(Z\alpha)^2}{2n^2}\,,$$

Point-nucleus nonrecoil energy

$$\delta E_{\rm fns} = m^3\,\frac{2(Z\alpha)^4}{3n^3}\,r_C^2\,\delta_{l,0}\,,$$

Finite nuclear size

$$\delta E_{\rm rec,pnt} = \frac{m^2}{M}\,\frac{(Z\alpha)^2}{2n^2}\,,$$

Recoil point nucleus

$$\delta E_{\rm rec,fns} = -3\,\frac{m^4}{M}\,\frac{2(Z\alpha)^4}{3n^3}\,r_C^2\,\delta_{l,0}\,.$$

Recoil finite nuclear size

Two-body atom: leading relativistic correction

Point-nucleus nonrecoil energy: known analytically (Dirac energy),

Finite nuclear size: can be obtained numerically from Dirac equation,

Recoil point nucleus: can be obtained from Breit equation,

Recoil finite nuclear size: CANNOT be obtained from Breit equation.

[Shabaev et al. PRA 57, 4235 (1998)]:
Recoil finite nuclear size within Breit approximation contains terms of order $(m^3/M)\,(Z\alpha)^3\, r_C$, which are numerically dominant but are exactly cancelled by the QED contribution.

Recoil with finite nuclear size can be addressed only within QED

Nuclear recoil to all orders in Zα, point nucleus

$$E_{\rm rec} = \frac{i}{2\pi M} \int_{-\infty}^{\infty} d\omega \sum_n \frac{1}{\varepsilon_a + \omega - \varepsilon_n(1 - i0)} \langle a | \vec{p} - \vec{D}(\omega) | n \rangle \cdot \langle n | \vec{p} - \vec{D}(\omega) | a \rangle$$

where

$$D^j(\omega) = -4\pi Z\alpha\, \alpha^i \left(\delta^{ij}\, \mathcal{D}(\omega, r) + \frac{\nabla^i \nabla^j}{\omega^2} \left[\mathcal{D}(\omega, r) - \mathcal{D}(0, r) \right] \right)$$

and

$$\mathcal{D}(\omega, r) = -\frac{e^{i|\omega|r}}{4\pi r}$$

Transverse part of the photon propagator in the Coulomb gauge

Derivation:

V. M. Shabaev, Theor. Math. Phys. 63, 588 (1985)
K. Pachucki and H. Grotch, Phys. Rev. A 51, 1854 (1995)

Numerical calculations:

A.N. Artemyev, V.M. Shabaev, V.A. Yerokhin, PRA 52, 1884 (1995)

Point nucleus => results are not applicable for muonic atoms

Nuclear recoil to all orders in Zα, extended nucleus

The generalized photon propagator for one extended-size vertex

$$\mathcal{D}(\omega, r) = -\frac{e^{i|\omega|r}}{4\pi r}$$

Point nucleus

⇒

$$\mathcal{D}(\omega, r) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, \frac{\rho(\vec{k}^2-\omega^2)}{\omega^2-\vec{k}^2}\,.$$

Extended-size nucleus

For the exponential model of the nuclear charge distribution

$$\rho(r) = \frac{\lambda^3}{8\pi}\, e^{-\lambda r}\,, \quad \rho(\vec{k}^2) = \frac{\lambda^4}{(\lambda^2+\vec{k}^2)^2}$$

we obtain

$$\mathcal{D}(\omega, r) = -\frac{1}{4\pi}\left[\frac{e^{i|\omega|r}}{r} - \frac{e^{i\sqrt{\omega^2-\lambda^2}\,r}}{r} - \frac{i\lambda^2}{2}\,\frac{e^{i\sqrt{\omega^2-\lambda^2}\,r}}{\sqrt{\omega^2-\lambda^2}}\right]$$

Formulas for nuclear recoil become valid for the extended-nucleus case if we use the generalized photon propagator

K. Pachucki and V.A. Yerokhin, PRL 130, 053002 (2023)

Nuclear recoil to all orders in Zα, evaluation #1

$$E_{\rm rec} = \frac{i}{2\pi M}\int_{-\infty}^{\infty} d\omega \sum_n \frac{1}{\varepsilon_a+\omega-\varepsilon_n(1-i0)}\Big[\langle a|\vec{p}|n\rangle\cdot\langle n|\vec{p}|a\rangle - 2\langle a|\vec{p}|n\rangle\cdot\langle n|\vec{D}(\omega)|a\rangle + \langle a|\vec{D}(\omega)|n\rangle\cdot\langle n|\vec{D}(\omega)|a\rangle\Big] = E_{CC}+E_{CB}+E_{BB}$$

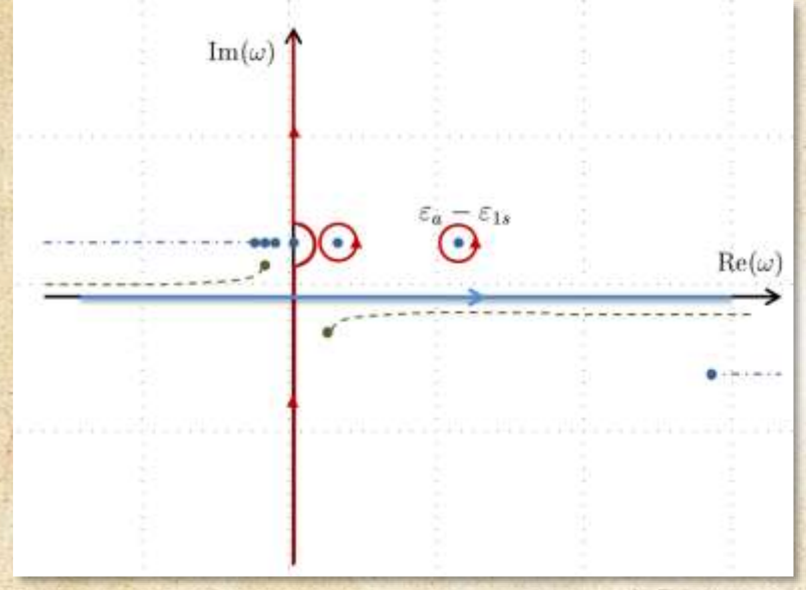
$$\frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega\,\frac{1}{\varepsilon_a+\omega-\varepsilon_n(1-i0)} = \frac{i}{2\pi}\,\text{v.p.}\int_{-\infty}^{\infty} d\omega\,\frac{1}{\varepsilon_a+\omega-\varepsilon_n} + \frac{1}{2}\text{sign}(\varepsilon_n)$$

$$E_{CC} = \frac{1}{2M}\sum_n \text{sign}(\varepsilon_n)\,\langle a|\vec{p}|n\rangle\cdot\langle n|\vec{p}|a\rangle = \langle a|\,\frac{\vec{p}^{\,2}}{2M}\,|a\rangle - \frac{1}{M}\sum_{\varepsilon_n<0}\langle a|\vec{p}|n\rangle\cdot\langle n|\vec{p}|a\rangle$$

Nuclear recoil to all orders in Zα, evaluation #2

$$E_{BB} = \frac{i}{2\pi M} \int_{-\infty}^{\infty} d\omega \sum_n \frac{1}{\varepsilon_a + \omega - \varepsilon_n(1 - i0)} \langle a | \vec{D}(\omega) | n \rangle \cdot \langle n | \vec{D}(\omega) | a \rangle$$

Wick rotation of integration contour



$$E_{BB} = \frac{1}{M} \sum_{0 < \varepsilon_n \le \varepsilon_a} a_n \langle a | \vec{D}(\varepsilon_a - \varepsilon_n) | n \rangle \langle n | \vec{D}(\varepsilon_a - \varepsilon_n) | a \rangle - \frac{1}{\pi M} \int_0^{\infty} d\omega \sum_n \frac{\varepsilon_a - \varepsilon_n}{(\varepsilon_a - \varepsilon_n)^2 + \omega^2} \langle a | \vec{D}(i\omega) | n \rangle \langle n | \vec{D}(i\omega) | a \rangle$$

Results for muonic atoms

TABLE I. Nuclear recoil correction to the energies of muonic atoms, in keV.

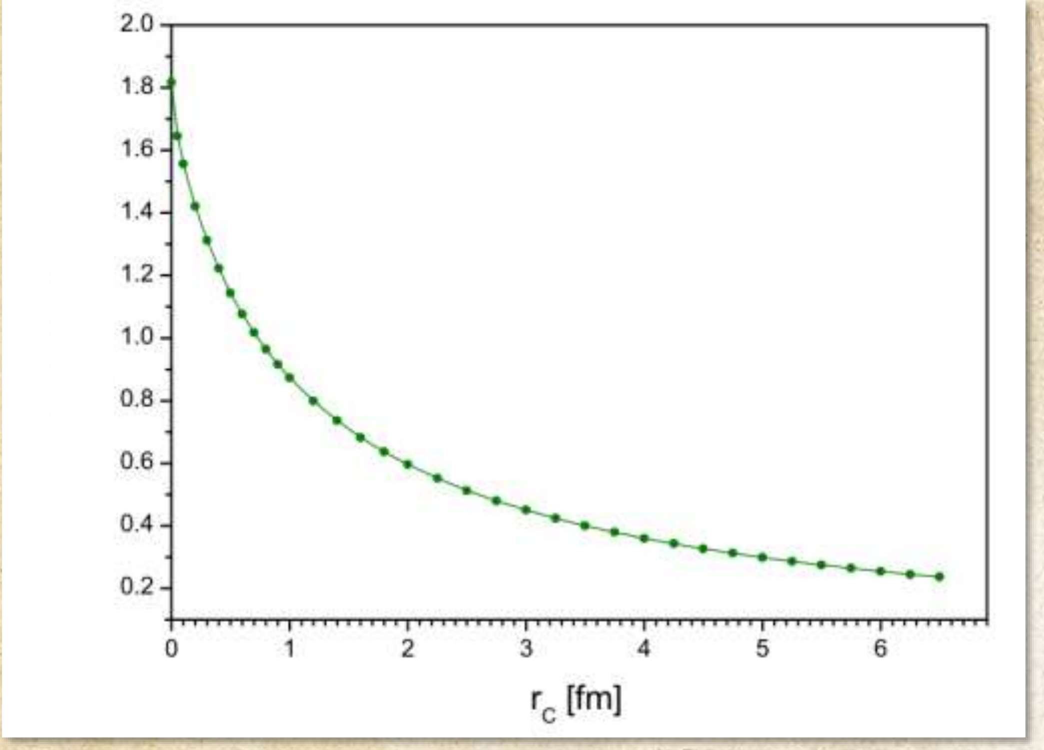
Atom	r_C [fm]	State	E_{rec}	Previous†
$^{89}_{40}$Zr	4.2706	$1s_{1/2}$	3.499	3.21(15)
		$2s_{1/2}$	1.146	1.09(2)
		$2p_{1/2}$	1.432	1.42(1)
		$2p_{3/2}$	1.414	1.40(1)
		$3s_{1/2}$	0.552	0.53(1)
		$3p_{1/2}$	0.639	0.64
		$3p_{3/2}$	0.634	0.63
		$3d_{3/2}$	0.641	0.64
		$3d_{5/2}$	0.638	0.63
$^{147}_{62}$Sm	4.9892	$1s_{1/2}$	3.438	2.88(8)
		$2s_{1/2}$	1.402	1.26(5)
		$2p_{1/2}$	1.947	1.92(5)
		$2p_{3/2}$	1.930	1.92(4)
		$3s_{1/2}$	0.719	0.66(2)
		$3p_{1/2}$	0.890	0.88(1)
		$3p_{3/2}$	0.885	0.88(1)
		$3d_{3/2}$	0.936	0.92(1)
		$3d_{5/2}$	0.926	0.91(1)
$^{205}_{83}$Bi	5.5008	$1s_{1/2}$	3.179	2.41(6)
		$2s_{1/2}$	1.554	1.33(4)
		$2p_{1/2}$	2.212	2.12(3)
		$2p_{3/2}$	2.226	2.26(1)
		$3s_{1/2}$	0.842	0.75(3)
		$3p_{1/2}$	1.062	1.02(3)
		$3p_{3/2}$	1.065	1.03(3)
		$3d_{3/2}$	1.199	1.19(2)
		$3d_{5/2}$	1.180	1.17(2)

† Michel et al. Phys. Rev. A **96**, 032510 (2017), following E. Borie and G. A. Rinker, Rev. Mod. Phys. **54**, 67 (1982).

V.A. Yerokhin and N.S. Oreshkina , PRA 108, 052824 (2023)

Results for muonic atoms: recoil with finite nuclear size

Nuclear recoil correction for Z = 90 and 1s state, as a function of the nuclear radius



Finite nuclear size reduces the recoil effect by an order of magnitude

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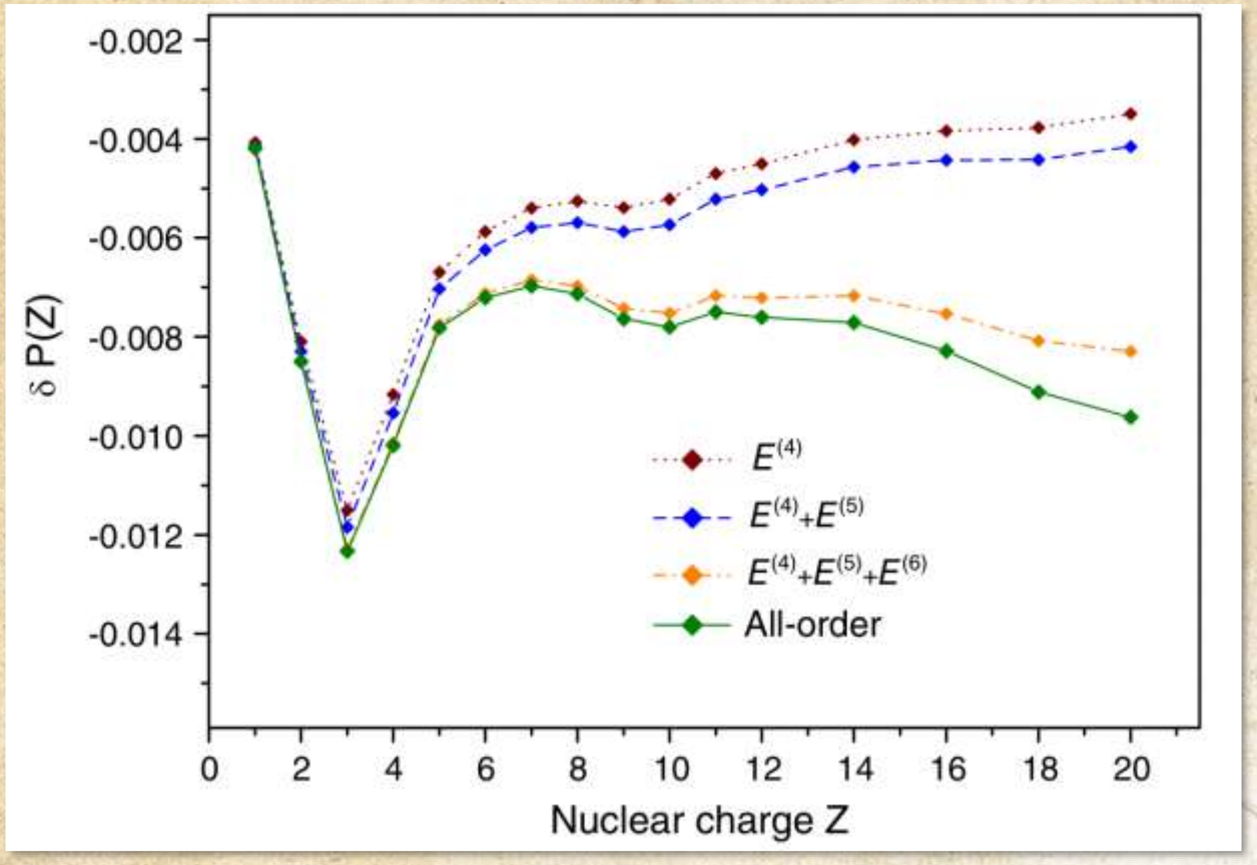
Results for electronic atoms: recoil with finite nuclear size



$$E_{\rm rec,fns} = -\frac{m^2}{M}\,\psi^2(0)\,\Big[2\pi(Z\alpha)\,(mr_C)^2 + (Z\alpha)^2\left(\frac{7}{6} - 2\gamma - 2\ln(mr_L)\right)(mr_C)^2 + \pi(Z\alpha)^3\,a_6\,(mr_C)\Big]$$

$$a_6 \approx 1.0$$

K. Pachucki and V.A. Yerokhin, PRL 130, 053002 (2023)

Outlook

All-order calculations of other nuclear recoil effects are possible:

- Nuclear recoil + vacuum polarization
- Radiative recoil
- Second-order nuclear recoil (of order m^2/M^2)
- Nuclear recoil for hyperfine splitting