5-loop QED correction to muon/electron g-2

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Muon and electron g-2: current status

Experiment:

a_e=0.00115965218073(28) [**2011**, D. Hanneke, S. Fogwell Hoogerheide, G. Gabrielse, Phys. Rev. A 83, 052122]
a_e=0.00115965218059(13) [**2022**, X. Fan, T. G. Myers, B. A. D. Sukra, G. Gabrielse, Phys. Rev. Lett. 130, 071801]
a_μ=0.00116592061(41) [**2021**, Muon g-2 collaboration, Phys. Rev. Lett. 126, 141801]
a_μ=0.00116592059(22) [**2023**, Muon g-2 collaboration, Phys. Rev. Lett. 131, 161802]

Theory:

$$a_\mu = a_\mu(\mathrm{QED}) + a_\mu(\mathrm{hadronic}) + a_\mu(\mathrm{electroweak}),$$
$$a_\mu(\mathrm{QED}) = \sum_{n\geq 1} \left(\frac{\alpha}{\pi}\right)^n a_\mu^{2n},$$
$$a_\mu^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_\mu/m_e) + A_2^{(2n)}(m_\mu/m_\tau) + A_3^{(2n)}(m_\mu/m_e, m_\mu/m_\tau)$$

a_e=0.001159652181606(11)(12)(299)
2019, T. Aoyama, T. Kinoshita, M. Nio, Atoms, 7, 28
Uncertainties come from: $A_1^{(10)}$, hadronic+electroweak, α
α^{-1}=137.035999046(27) [**2018**, R. H. Parker et al., Science, V. 360, Is. 6385, pp. 191-195]
$A_1^{(10)}$[Aoyama,Hayakawa,Kinoshita,Nio(AHKN-2019)] = **6.737(159)**
$A_1^{(10)}$[S.Volkov,2019] = **5.855(90)** (**4.8σ discrepancy**)

a_μ=0.0011659180953(4386)(100)(10) [**5.1σ difference with the experiment**]
Uncertainties come from: hadronic, electroweak, QED [very optimistic]
2020, T. Aoyama et al., Physics Reports, V. 887, pp. 1-166
$A_2^{(10)}(m_\mu/m_e)$=**742.18(87)** - **not double-checked**, a_μ^{12} ~ **5400** - **estimations only** (not confirmed by calculations)

The history of the universal QED contributions $A_1^{(2n)}$ calculations (n=1,2,3,4)

$$a_e(QED)=\sum_{n\geqslant 1}\left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n}=A_1^{(2n)}+A_2^{(2n)}(m_e/m_\mu)+A_2^{(2n)}(m_e/m_\tau)+A_3^{(2n)}(m_e/m_\mu,m_e/m_\tau)$$

- J.Schwinger [**1948**], analytically: $A_1^{(2)}$=**0.5**
- R. Karplus, N. Kroll [**1949**] – $A_1^{(4)}$ with a mistake

  A.Petermann [**1957**], C. Sommerfield [**1958**], analytically: $A_1^{(4)}$=**-0.328478966...**
- ~**1970**...~**1975**, $A_1^{(6)}$, numerically:

  1. M.Levine, J. Wright.
  2. R. Carroll, Y. Yao.
  3. T. Kinoshita, P. Cvitanović.

     T. Kinoshita, P. Cvitanović [**1974**]: $A_1^{(6)}$=**1.195±0.026**
- E. Remiddi, S. Laporta et al., ~**1965**..**1996**, analytically: $A_1^{(6)}$=**1.181241456**...
- T. Kinoshita, M. Nio et al., numerically, **2015**: $A_1^{(8)}$=**-1.91298(84)**

  (first estimations in 1980-x)
- S. Laporta, semianalytically, **2017**: $A_1^{(8)}$=**-1.9122457649...**

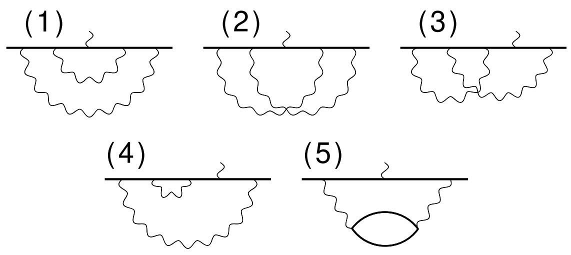
10-th order universal QED contributions: old and new results

- AHKN = T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio

  $A_1^{(10)}$[AHKN,with lepton loops] = -0.933(17) [**2012**]
  $A_1^{(10)}$[AHKN,no lepton loops] = 7.670(159) [**2019**]
  $A_1^{(10)}$[AHKN] = **6.737(159)** [**2019**]

- My results

  $A_1^{(10)}$[Volkov,no lepton loops] = 6.793(90) [**2019**]
  $A_1^{(10)}$[Volkov,no lepton loops] = 6.828(60) [**2024**]
  $A_1^{(10)}$[Volkov,with lepton loops] = -0.9377(35) [**2024**]
  $A_1^{(10)}$[Volkov] = **5.891(61)** [**2024**]
  [S. Volkov, arXiv:2404.00649 (2024)]



Feynman graphs of 4-th order:
(1)-(4) are without lepton loops,
(5) is with lepton loops

The history of the mass-dependent QED contributions calculations

$$a_\mu(\text{QED}) = \sum_{n\geq 1} \left(\frac{\alpha}{\pi}\right)^n a_\mu^{2n}, \quad a_\mu^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_\mu/m_e) + A_2^{(2n)}(m_\mu/m_\tau) + A_3^{(2n)}(m_\mu/m_e, m_\mu/m_\tau)$$

- H.H.Elend [**1966**], analytically (**recently obtained masses are substituted into the analytical results**):
  for electron $A_2^{(4)}(m_e/m_\mu)=0.519738676(24)\cdot 10^{-6}$, $A_2^{(4)}(m_e/m_\tau)=0.183790(25)\cdot 10^{-8}$,
  for muon $A_2^{(4)}(m_\mu/m_e)=1.0942583093(76)$, $A_2^{(4)}(m_\mu/m_\tau)=0.000078076(11)$.
- ~**1970**...~**1990**, numerically, different research groups, increasing precision:
  J. Aldins, S. J. Brodsky, C. Chlouber, A. J. Dufner, T. Kinoshita, W. J. Marciano, B. Nizic, Y. Okamoto, M. A. Samuel...
  T. Kinoshita, W. J. Marciano [**1990**]: for muon $A_2^{(6)}(m_\mu/m_e)=\mathbf{22.8671(22)}$.
- M. A. Samuel, G. Li, S. Laporta, E. Remiddi [**1991-1993**], analytically:
  for electron $A_2^{(6)}(m_e/m_\mu)=\mathbf{-0.737394164(24)\cdot 10^{-5}}$, $A_2^{(6)}(m_e/m_\tau)=\mathbf{-0.658273(79)\cdot 10^{-7}}$,
  for muon $A_2^{(6)}(m_\mu/m_e)=\mathbf{22.86837998(20)}$, $A_2^{(6)}(m_\mu/m_\tau)=\mathbf{0.000360671(94)}$.
- S. Laporta [**1993**], semianalytically: for muon $A_3^{(6)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{0.0005238(19)}$.
- A. Czarnecki, M. Skrzypek, B. Krause [**1999**], analytically: for muon $A_3^{(6)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{0.000527738(75)}$.
- T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio [**2012**], numerically:
  for electron $A_2^{(8)}(m_e/m_\mu)=\mathbf{0.0009222(66)}$, $A_2^{(8)}(m_e/m_\tau)=\mathbf{8.24(12)\cdot 10^{-6}}$,
  for muon $A_2^{(8)}(m_\mu/m_e)=\mathbf{132.6852(60)}$, $A_2^{(8)}(m_\mu/m_\tau)=\mathbf{0.04234(12)}$, $A_3^{(8)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{0.06272(4)}$,
  $A_2^{(10)}(m_\mu/m_e)=\mathbf{742.18(87)}$ [**NOT DOUBLE-CHECKED**], $A_2^{(10)}(m_\mu/m_\tau)=\mathbf{-0.068(5)}$, $A_3^{(10)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{2.011(10)}$.
- A. Kurz, T. Liu, P. Marquard, M. Steinhauser [**2013**], semianalytically:
  for electron $A_2^{(8)}(m_e/m_\mu)=\mathbf{0.009161970703(372)}$, $A_2^{(8)}(m_e/m_\tau)=\mathbf{7.42924(118)\cdot 10^{-6}}$,
  for muon $A_2^{(8)}(m_\mu/m_\tau)=\mathbf{0.0424941(53)}$.
- A. Kurz, T. Liu, P. Marquard, V. A. Smirnov, A. V. Smirnov, M. Steinhauser [**2016**], semianalytically:
  for muon $A_2^{(8)}(m_\mu/m_e)=\mathbf{132.86(48)}$, $A_3^{(8)}(m_\mu/m_e,m_\mu/m_\tau)=\mathbf{0.0627220(100)}$.

$A_1^{(10)}$, a_e and α

- $A_1^{(10)}[\text{AHKN}] = 6.737(159)$
  $A_1^{(10)}[\text{Volkov}] = 5.891(61)$

5σ

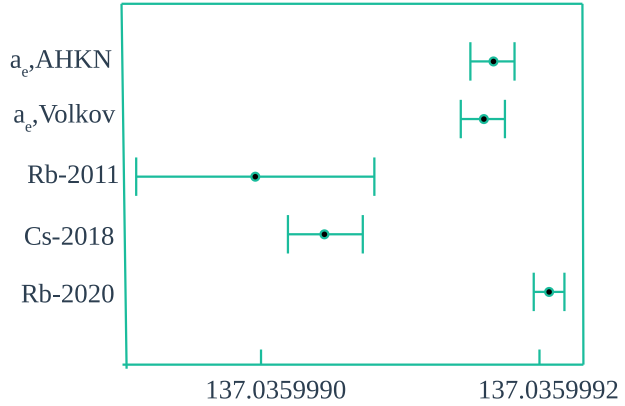
- $\alpha^{-1}[a_e, \text{AHKN}] = 137.0359991663(155)$
  $\alpha^{-1}[a_e, \text{Volkov}] = 137.0359991595(155)$

- $\alpha^{-1}[\text{Rb-2011}] = 137.035998996(85)$
  (PRL 106, 080801, 2011 + CODATA-2014)
  $\alpha^{-1}[\text{Cs-2018}] = 137.035999046(27)$
  (Science 360, 191, 2018)
  $\alpha^{-1}[\text{Rb-2020}] = 137.035999206(11)$
  (Nature 588, 61, 2020)

3.86σ

3.65σ

5.4σ



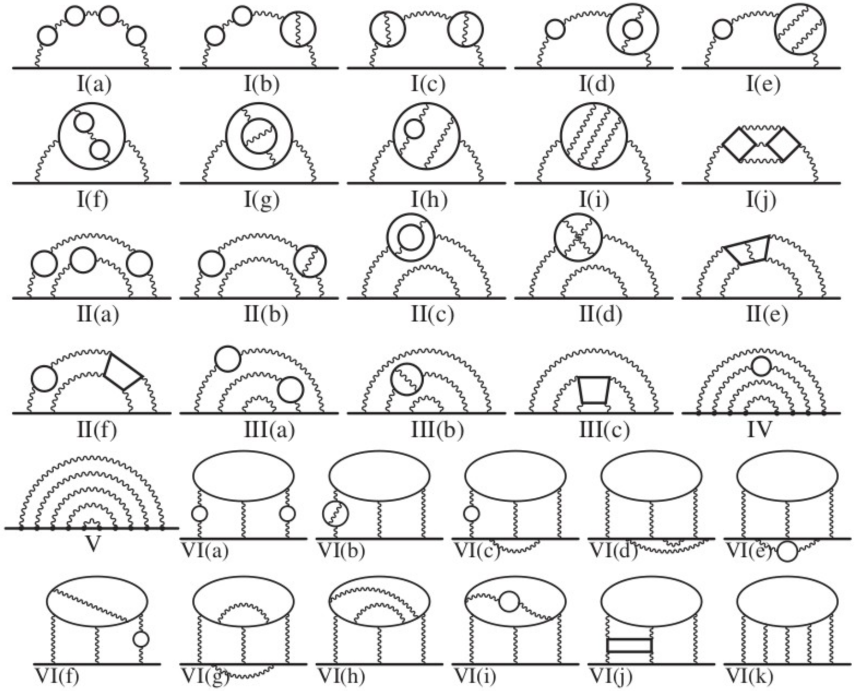
α, a_e, testing quantum field theory

$$\alpha^2 = \frac{2R_\infty}{c}\frac{m_{\text{atom}}}{m_e}\frac{h}{m_{\text{atom}}} \qquad\qquad R_\infty = \frac{e^4 m_e}{8(\varepsilon_0)^2 h^3 c}$$

- Measurements and calculations of α and a_e are used to check the validity of QED, standard model and quantum field theory in general.
- R_∞/c, m_{atom}/m_e, h/m_{atom} are extracted from different experiments. The interpretation of the experimental results of first two ones **depend on bound-state quantum electrodynamics**.
- h/m_{atom}:
  α^{-1}[Rb-2011] = 137.035998996(85)
  [PRL 106, 080801, 2011 + CODATA-2014]
  α^{-1}[Cs-2018] = 137.035999046(27)
  [Science 360, 191, 2018]
  α^{-1}[Rb-2020] = 137.035999206(11)
  [Nature 588, 61, 2020]

$A_1^{(10)}$ in large gauge-invariant classes

Red: **2019+2024** (5σ discrepancy). Blue: **2024** (FULL AGREEMENT!!!)



Each class is obtained by moving photons along lepton lines and by relocating photon self-energy insertions into arbitrary photons. The external photon should be inserted consistently.
The picture is from T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, PRL 109, 110807 (2012).

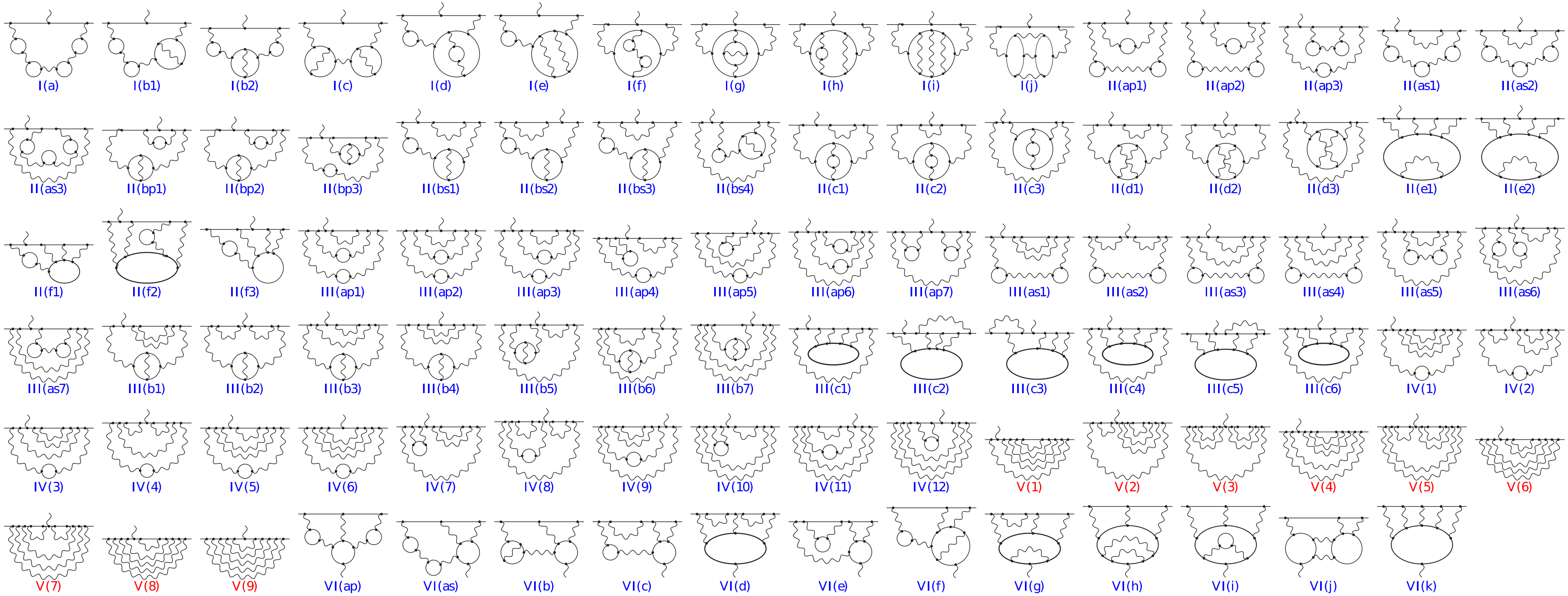
Class	My	AHKN	Class	My	AHKN
I(a)	0.00047105(16)	0.000470940(60)	III(a)	2.12726(14)	2.12733(17)
I(b)	0.0070081(12)	0.00701080(70)	III(b)	3.32730(32)	3.32712(45)
I(c)	0.0234643(26)	0.0234680(20)	III(c)	4.9199(15)	4.921(11)
I(d)	0.00380370(61)	0.00380170(50)	IV	-7.7303(16)	-7.7296(48)
I(e)	0.010289(11)	0.0102960(40)	V	6.828(60)	7.670(159)
I(f)	0.00757106(47)	0.0075684(20)	VI(a)	1.041537(82)	1.04132(19)
I(g)	0.0285696(18)	0.0285690(60)	VI(b)	1.34697(11)	1.34699(28)
I(h)	0.0016826(63)	0.001696(13)	VI(c)	-2.53312(49)	-2.5289(28)
I(i)	0.01726(29)	0.01747(11)	VI(d)	1.8468(22)	1.8467(70)
I(j)	0.0004038(63)	0.0003975(18)	VI(e)	-0.43129(17)	-0.43120(70)
II(a)	-0.1094945(50)	-0.109495(23)	VI(f)	0.77154(23)	0.7703(22)
II(b)	-0.473625(27)	-0.473559(84)	VI(g)	-1.5965(10)	-1.5904(63)
II(c)	-0.116506(12)	-0.116489(32)	VI(h)	0.18554(68)	0.1792(39)
II(d)	-0.24291(15)	-0.24300(29)	VI(i)	-0.04396(10)	-0.0438(12)
II(e)	-1.34235(54)	-1.3449(10)	VI(j)	-0.22920(43)	-0.2288(18)
II(f)	-2.43553(30)	-2.4336(15)	VI(k)	0.67974(39)	0.6802(38)

$A_1^{(10)}$ in small gauge-invariant classes

95 gauge-invariant classes. Red: **2019**. Blue: **2024**.

Each class is obtained by moving photons along lepton paths and loops, but **without jumping over the external photon**

(The idea: P. Cvitanović, Nuclear Physics B127 (1977), 176-188)



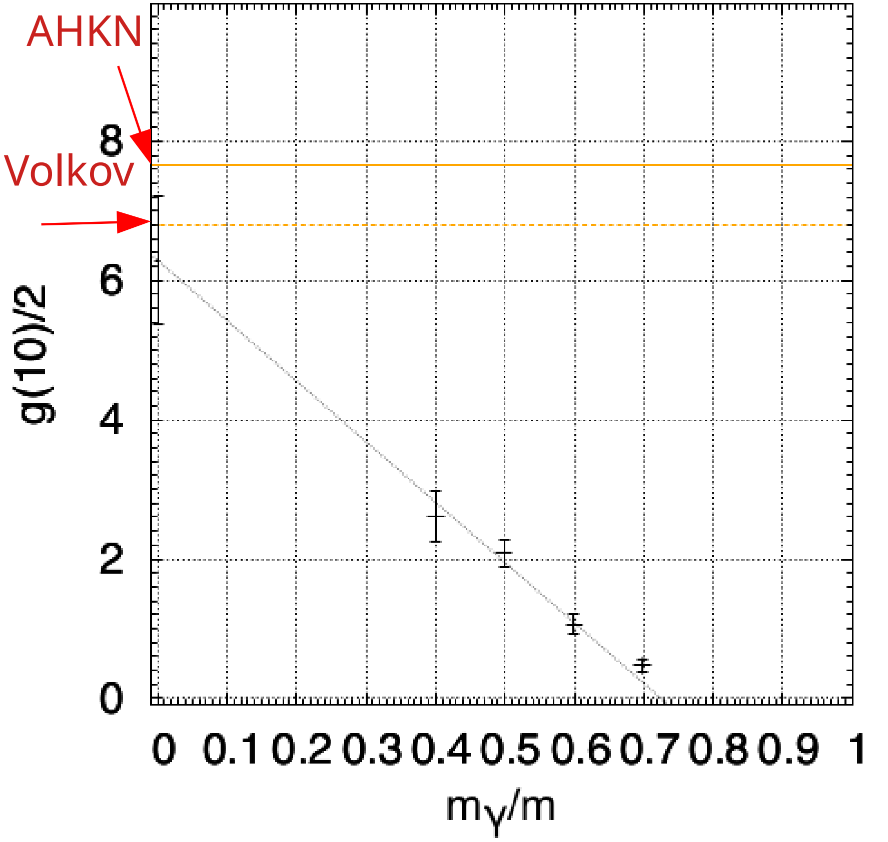
$A_1^{(10)}$ in small gauge-invariant classes

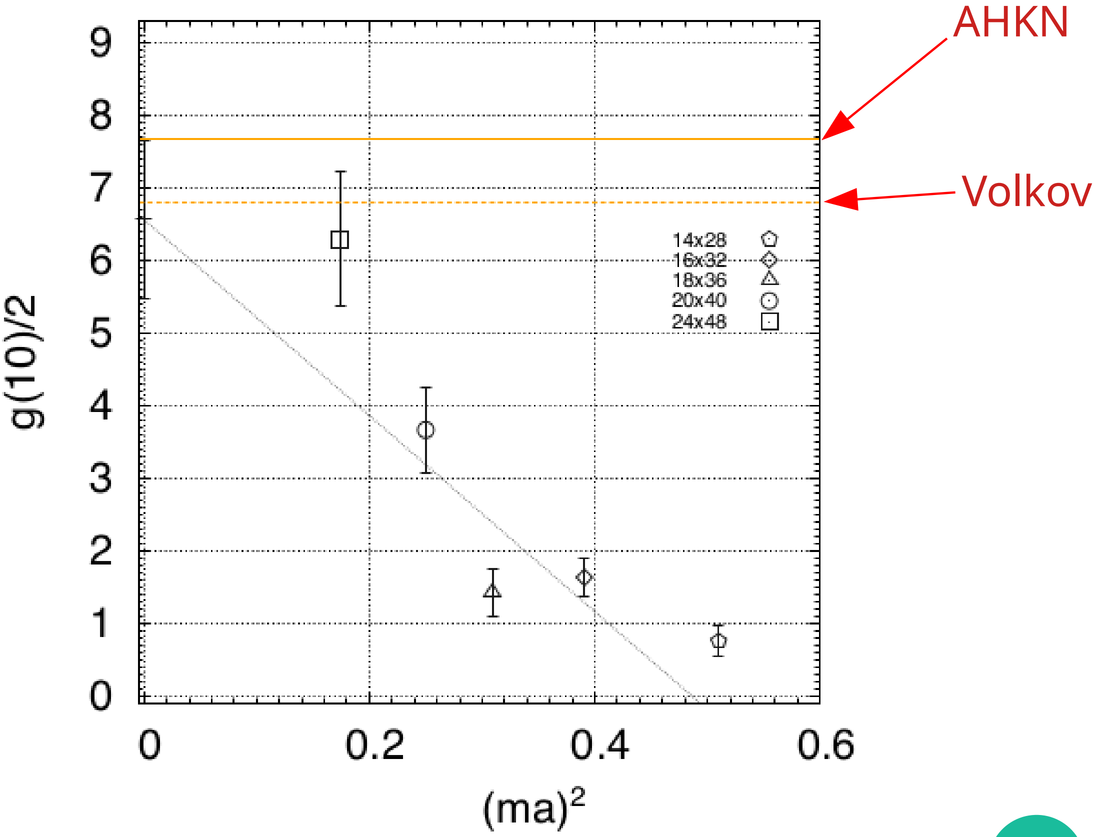
95 gauge-invariant classes. Red: **2019+2024**. Blue: **2024**.

Class	Value	Class	Value	Class	Value	Class	Value	Class	Value
I(a)	0.00047105(16)	II(bp3)	-0.0836017(85)	III(ap4)	0.143139(17)	III(c2)	3.28050(91)	V(3)	0.325(15)
I(b1)	0.0046715(11)	II(bs1)	-0.125057(11)	III(ap5)	0.071170(37)	III(c3)	0.27166(48)	V(4)	-0.774(28)
I(b2)	0.00233656(54)	II(bs2)	-0.125056(14)	III(ap6)	0.164795(20)	III(c4)	-0.75943(79)	V(5)	-2.161(25)
I(c)	0.0234643(26)	II(bs3)	0.088110(10)	III(ap7)	0.0275451(27)	III(c5)	0.22308(51)	V(6)	-0.409(24)
I(d)	0.00380370(61)	II(bs4)	-0.180629(14)	III(as1)	0.054599(55)	III(c6)	0.04392(46)	V(7)	2.635(17)
I(e)	0.010289(11)	II(c1)	-0.0869389(86)	III(as2)	0.225730(17)	IV(1)	-0.51586(43)	V(8)	-0.993(15)
I(f)	0.00757106(47)	II(c2)	0.0375573(46)	III(as3)	0.055588(66)	IV(2)	-0.64878(28)	V(9)	1.0855(44)
I(g)	0.0285696(18)	II(c3)	-0.0671244(72)	III(as4)	0.289427(35)	IV(3)	-1.14824(60)	VI(ap)	0.482955(59)
I(h)	0.0016826(63)	II(d1)	-0.18988(11)	III(as5)	0.440024(55)	IV(4)	1.19593(38)	VI(as)	0.558582(57)
I(i)	0.01726(29)	II(d2)	0.111341(75)	III(as6)	0.102979(21)	IV(5)	-1.52785(60)	VI(b)	1.34697(11)
I(j)	0.0004038(63)	II(d3)	-0.164378(83)	III(as7)	0.017022(35)	IV(6)	0.50531(30)	VI(c)	-2.53312(49)
II(ap1)	0.00924758(84)	II(e1)	-1.13757(46)	III(b1)	0.35850(13)	IV(7)	-0.19295(26)	VI(d)	1.8468(22)
II(ap2)	-0.0297767(11)	II(e2)	-0.20478(28)	III(b2)	0.550730(74)	IV(8)	-0.78444(38)	VI(e)	-0.43129(17)
II(ap3)	-0.0262183(11)	II(f1)	-0.239896(88)	III(b3)	-0.21825(16)	IV(9)	-4.49029(73)	VI(f)	0.77154(23)
II(as1)	-0.0466111(41)	II(f2)	-1.91510(23)	III(b4)	0.916024(91)	IV(10)	0.19698(32)	VI(g)	-1.5965(10)
II(as2)	0.0137916(15)	II(f3)	-0.28054(18)	III(b5)	0.43819(10)	IV(11)	0.05233(55)	VI(h)	0.18554(68)
II(as3)	-0.0299276(18)	III(ap1)	0.056161(42)	III(b6)	1.35323(15)	IV(12)	-0.37250(31)	VI(i)	-0.04396(10)
II(bp1)	0.0326223(31)	III(ap2)	0.251184(26)	III(b7)	-0.07113(10)	V(1)	6.166(17)	VI(j)	-0.22920(43)
II(bp2)	-0.0800129(67)	III(ap3)	0.227894(47)	III(c1)	1.86018(48)	V(2)	0.967(23)	VI(k)	0.67974(39)

First independent check of $A_1^{(10)}$[no lepton loops] [2022]

R. Kitano, H. Takaura, Prog. Theor. Exp. Phys. 2023, 103B02 (2023)





Method of calculation

- Reduction to finite integrals (**1 diagram=1 integral**, **no master integrals**, **no dimensional regularization** at all).
- Monte-Carlo integration (**non-adaptive**, probability density functions based on the combinatorics of the diagram)
- Parallel realization (simultaneous computation on many graphics accelerators).

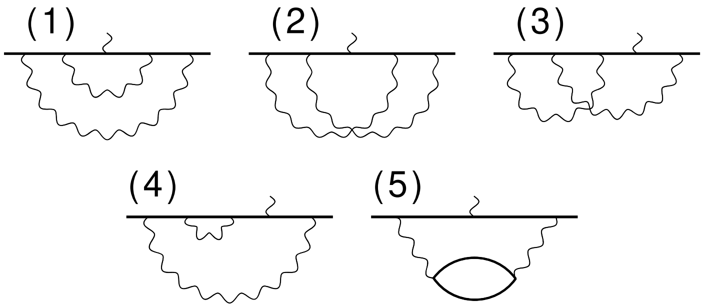
Method of calculation: reduction to finite integrals

S. Volkov, J. Exp. Theor. Phys. 122, 1008-1031 (2016); S. Volkov, Phys. Rev. D 109, 036012 (2024)

- Subtraction of divergences directly at the level of integrands in Feynman parametric space (**1 diagram=1 integral**, **no dimensional regularization**, **no master integrals, no residual renormalization**).
- A nontrivial modification of **BPHZ** (the Bogoliubov-Parasiuk-Hepp-Zimmermann renormalization).
- Linear operators applied to the Feynman amplitudes of subdiagrams are used for subtraction of ultraviolet, infrared and mixed divergences. The usage of linear operators for the IR divergence subtraction is an **innovation of my method**; this allows us to avoid residual renormalization after subtraction.

UV and IR divergences

- UV divergences correspond to subdiagrams; there exists a well-developed universal subtraction procedure that removes them in each Feynman diagram and is equivalent to renormalization (BPHZ).
- Renormalization also removes all IR divergences in the QED contributions to the lepton *g-2*. However, all direct modifications of BPHZ lead to an inter-diagram cancellation.
- UV and IR divergences can mix with each other in Feynman diagrams; this makes their handling more complicated.



Diagrams for $A_1^{(4)}$

№	Value
1	0.77747802
2	-0.46764544
3	$0.564021-(1/2)\log(\lambda^2/m^2)$
4	$-0.089978+(1/2)\log(\lambda^2/m^2)$
5	0.0156874

Contributions to $A_1^{(4)}$ (Petermann, 1957)

Point-by-point subtractions of divergences under the integral sign

- Methods of this kind are rarely used (because of the necessity to understand, how the divergences work), but I am not the first one who utilized it in calculations:

  M. J. Levine, J. Wright, Phys. Rev. D 8, 3171 (1973).
  R. Carroll, Y.-P. Yao, Phys. Lett. 48B, 125 (1974).
  P. Cvitanović, T. Kinoshita, Phys. Rev. D 10, 3991 (1974).
  T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Nucl. Phys. B 796, 184 (2008).
  L. Ts. Adzhemyan, M. V. Kompaniets, J. Phys.: Conf. Ser. 523 012049, 2014 **[not *g-2*!!!]**.

- My old method:

  S. Volkov, J. Exp. Theor. Phys. 122, 1008-1031 (2016)

- My new method (first publications in 2021):

  S. Volkov, Phys. Rev. D 109, 036012 (2024)
  (more flexibility, **there is a hope that the ideas are extendable beyond QED and *g-2***)

Monte-Carlo integration

- Suppose we have a Feynman-parametric integral $\int_{z_1,\ldots,z_M>0} f(z_1, z_2, \ldots, z_M)\delta(z_1+\ldots+z_M-1)dz$
- To make the Monte-Carlo convergence faster, we use different **predefined** probability density functions for different Feynman diagrams.
- Hepp sectors: $z_{j_1} \geq z_{j_2} \geq \ldots \geq z_{j_M}$
- Probability density: $C \times \frac{\prod_{l=2}^{M} \left(z_{j_l}/z_{j_{l-1}}\right)^{\mathrm{Deg}(\{j_l, j_{l+1}, \ldots, j_M\})}}{z_1 z_2 \ldots z_M}$

  [the idea of E. Speer, J. Math. Phys. 9, 1404 (1968)]
- The numbers Deg are defined **on all subsets** of *{1,2,…,M}*. For example, 2^{15}=32768 real numbers for one diagram at 5 loops.
- <u>My ideas are</u>: 0) The usage of these functions can be very efficient!
  1) How to calculate Deg(s) for each set to make the convergence fast. It is based on a **theory** [S. Volkov, Nucl. Phys. B 961, 115232 (2020)] and **adjustment**.
  2) How to generate random samples fastly
  S. Volkov, Phys. Rev. D 96, 096018 (2017); S. Volkov, Phys. Rev. D 98, 076018 (2018)

My results: electron and muon g-2, 8-th order [2023]

Contribution	My value	Previously known	Ref.
$A_1^{(8)}$	-1.9118(41)	-1.91224576	[1]
$A_2^{(8)}(m_e/m_\mu)$	0.000924(11)	0.0009141970703(372)	[2]
$A_2^{(8)}(m_e/m_\tau)$	0.00000710(60)	0.00000742924(118)	[2]
$A_3^{(8)}(m_e/m_\mu, m_e/m_\tau)$	0.000000745(24)	0.00000074687(28)	[2]
$A_2^{(8)}(m_\mu/m_e)$	132.673(84)	132.6852(60)	[3]
$A_2^{(8)}(m_\mu/m_\tau)$	0.04252(11)	0.0424941(53)	[2]
$A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau)$	0.0622(33)	0.062722(10)	[4]

[1] S. Laporta, Phys. Lett. B 772, 232 (2017).
[2] A. Kurz, T. Liu, P. Marquard, and M. Steinhauser, Nucl. Phys. B 879, 1 (2014).
[3] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, Phys. Rev. Lett. 109, 111808 (2012).
[4] A. Kurz, T. Liu, P. Marquard, A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, Phys. Rev. D 93, 053017 (2016).

Computations 2019 and 2024

- Integrands on GPU, different precisions, with round-off error control (most part of the time is the integrand values calculation).
- Computation of **2019**
  3213 undirected Feynman diagrams without lepton loops
  GPU NVidia V100, supercomputer "Govorun" (Dubna)
  ≈40000 GPU-hours
  compiled integrand code size ≈ 500 GB
  3.2×10^{14} Monte Carlo samples
- Computation of **2024**
  2323 undirected Feynman diagrams with lepton loops
  + 3213 without lepton loops
  GPU NVidia A100, supercomputer "HoreKa" (Karlsruhe)
  ≈45000+50000 GPU-hours
  compiled integrand code size ≈ 200+400 GB
  $3.2\times10^{14}+2.4\times10^{14}$ Monte Carlo samples

Summary

- QED uncertainty in a_μ is relatively small, but the 10-order coefficient has not been double-checked yet; its large value makes it important. Higher-order corrections also require a more careful examination.
- I finished the calculation of the **total** 10-th order universal QED contribution to lepton magnetic moments.
- The results are in full agreement with the ones obtained by T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio in 31 of 32 large gauge-invariant classes.
- A significant discrepancy (5σ) remains in Set V. A recalculation confirmed my old result.
- The reason of the discrepancy is unknown, but independent calculations are coming.
- The calculation results can also be split into **small** gauge-invariant classes. Set V gives 9 classes; the remaining 31 large classes are split into 86 small classes. My calculation is the first one giving such a detailization. This makes independent checks easier!

Thank you for your attention!