

# Hard Three-Loop Spin-Independent Corrections in Muonium and Positronium

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*M.E., Valery Shelyuto, PRA **105**, 012803 (2022);  
PLB **832**, 137247 (2022); arXiv:2306.07748*



# Outline

- 1 Research on Energy levels in Muonium and Positronium
- 2 Hard three-loop corrections in muonium and positronium
- 3 Skeleton integrals
- 4 Gauge Invariant Contributions
  - One-loop polarization insertions
  - Two-Loop Polarization Insertions
  - One-loop electron factor and polarization
  - Radiatively corrected electron factor
- 5 Conclusions



# Muonium ( $\text{MU} = \mu^+ e^-$ ) and Positronium ( $\text{Ps} = e^+ e^-$ )

- Muonium: for many years main emphasis of theoretical and experimental research was on hyperfine splitting, see, e.g., reviews: *M.E. et al. (2007)*, *Tiesinga et al. (2021)*
- New measurements of  $1S - 2S$ , Lamb shift and fine structure in muonium are going on and forthcoming: *MU-Mass at PSI*; *J-PARC MUSE*, etc.
- Positronium: for many years main emphasis of theoretical and experimental research was on hyperfine splitting, see, e.g., review *Adkins et al. (2022)*
- New measurements of  $1S - 2S$ , Lamb shift and fine structure are going on and forthcoming positronium experiments: *ETH Zurich*, *UC Riverside*, *University College London*



# Hard three-loop corrections in muonium and positronium



**Hard three-loop corrections arise as radiative insertions in two-photon exchange graphs**

## Muonium

- Some hard three-loop corrections were calculated long time ago (*reviews: M.E. et al. (2007); Tiesinga et al. (2021)*)
- Spin-dependent corrections
  - ① Nonrecoil corrections of order  $\alpha^2(Z\alpha)^5 m$
  - ② Recoil corrections of order  $\alpha^2(Z\alpha)^5 (m/M)m$
- Spin-independent corrections
  - ① Nonrecoil corrections of order  $\alpha^2(Z\alpha)^5 m$
  - ② Recoil corrections of order  $\alpha^2(Z\alpha)^5 (m/M)m$  - *this work*

## Positronium

- Hard three-loop spin-dependent and spin-independent corrections, see review *Adkins et al. (2022)*
- We consider some hard spin-independent radiative-recoil corrections of order  $\alpha^7 m$

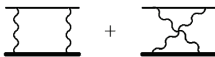


Figure: Skeleton diagrams.

## Skeleton integrals

- We consider corrections which arise as two-loop radiative insertions in skeleton diagrams

# Skeleton integrals

- Recoil muonium skeleton integral

$$\begin{aligned}\Delta E_{skel-rec}^{(Mu)} &= \frac{16(Z\alpha)^5 m}{\pi n^3(1-\mu^2)} \left(\frac{m_r}{m}\right)^3 \int_0^\infty \frac{kdk}{(k^2 + \lambda^2)^2} \\ &\times \left[ \mu \sqrt{1 + \frac{k^2}{4}} \left(\frac{1}{k} + \frac{k^3}{8}\right) - \sqrt{1 + \frac{\mu^2 k^2}{4}} \left(\frac{1}{k} + \frac{\mu^4 k^3}{8}\right) \right. \\ &\left. - \frac{\mu k^2}{8} \left(1 + \frac{k^2}{2}\right) + \frac{\mu^3 k^2}{8} \left(1 + \frac{\mu^2 k^2}{2}\right) + \frac{1}{k} \right] \delta_{\ell 0}\end{aligned}$$

- $m$  - electron mass,  $M$  - muon mass,  $m_r = mM/(m+M)$  - reduced mass,  $\mu = m/M$ ,  $\lambda$  - IR mass of the exchanged photon,  $n$  - principal quantum number,  $\ell$  - orbital momentum,  $k$  - dimensionless momentum in units of  $m$

- Positronium skeleton integral

$$\Delta E_{skel-rec}^{(Ps)} = \frac{2\alpha^5 m}{\pi n^3} \int_0^\infty dk \left[ \frac{k^2}{8\sqrt{k^2+4}} + \frac{3}{8\sqrt{k^2+4}} - \frac{1}{\sqrt{k^2+4}k^4} - \frac{k}{8} - \frac{1}{8k} \right] \delta_{\ell 0}$$

- No separation into recoil and nonrecoil contributions



# One-loop polarization insertions



- Substitution in the skeleton integral

$$\frac{1}{k^2} \rightarrow 3 \left( \frac{\alpha}{\pi} \right)^2 k^2 l_1^2(k), \quad l_1(k) = \int_0^1 dv \frac{v^2(1-v^2/3)}{4 + (1-v^2)k^2}$$

## Muonium

$$\begin{aligned} \Delta E_1^{(Mu)} = & \frac{48(Z\alpha)^5 m}{\pi n^3(1-\mu^2)} \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{m_r}{m} \right)^3 \int_0^\infty k dk l_1^2(k) \left\{ \mu \sqrt{1 + \frac{k^2}{4}} \right. \\ & \times \left( \frac{1}{k} + \frac{k^3}{8} \right) - \sqrt{1 + \frac{\mu^2 k^2}{4}} \left( \frac{1}{k} + \frac{\mu^4 k^3}{8} \right) - \frac{\mu k^2}{8} \left( 1 + \frac{k^2}{2} \right) \\ & \left. + \frac{\mu^3 k^2}{8} \left( 1 + \frac{\mu^2 k^2}{2} \right) + \frac{1}{k} \right\} \delta_{\ell 0}, \end{aligned}$$



# One-loop polarization insertions

## Muonium

- Numerically with account of all orders in  $\mu = m/M$

$$\Delta E_1^{(Mu)} = 0.959540854(3) \dots \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m}{M} \delta_{\ell 0}$$

- Analytic result, first order in  $\mu$

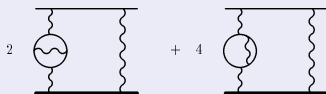
$$\begin{aligned} \Delta E_1^{(Mu)} &\approx \frac{48(Z\alpha)^5 m}{\pi n^3} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_r}{m}\right)^3 \mu \int_0^\infty k dk l_1^2(k) \left[ \sqrt{1 + \frac{k^2}{4}} \right. \\ &\quad \times \left. \left( \frac{1}{k} + \frac{k^3}{8} \right) - \frac{k^2}{8} \left( 1 + \frac{k^2}{2} \right) \right] \delta_{l0} \\ &= \left( \frac{1541}{486} - \frac{172}{2835} \pi^2 - \frac{4}{3} \zeta(3) \right) \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m}{M} \left(\frac{m_r}{m}\right)^3 m \delta_{l0} \\ &= 0.9692 \dots \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m}{M} \left(\frac{m_r}{m}\right)^3 m \delta_{\ell 0}. \end{aligned}$$

# One-loop polarization insertions

## Positronium

$$\Delta E_1^{(Ps)} = \left( -\frac{\zeta(3)}{6} + \frac{1709}{3888} - \frac{11\pi^2}{405} \right) \frac{\alpha^7}{\pi^3 n^3} m \delta_{\ell 0} = -0.028848 \dots \frac{\alpha^7}{\pi^3 n^3} m \delta_{\ell 0}$$

## Two-Loop Polarization Insertions



- Substitution in the skeleton integral

$$\frac{1}{k^2} \rightarrow 2 \left( \frac{\alpha}{\pi} \right)^2 I_2(k), \quad I_2(k) = \int_0^1 dv \frac{\frac{3}{4} v^2 \left( 1 - \frac{v^2}{3} \right) + R(v)}{4 + (1 - v^2) k^2}$$

# Two-loop polarization insertions

## Muonium

$$\begin{aligned} \Delta E_2^{(Mu)} = & \frac{32(Z\alpha)^5 m}{\pi n^3(1-\mu^2)} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_r}{m}\right)^3 \int_0^\infty \frac{dk}{k} \left\{ I_2(k) \left[ \mu \sqrt{1 + \frac{k^2}{4}} \right. \right. \\ & \times \left( \frac{1}{k} + \frac{k^3}{8} \right) - \sqrt{1 + \frac{\mu^2 k^2}{4}} \left( \frac{1}{k} + \frac{\mu^4 k^3}{8} \right) - \frac{\mu k^2}{8} \left( 1 + \frac{k^2}{2} \right) \\ & \left. \left. + \frac{\mu^3 k^2}{8} \left( 1 + \frac{\mu^2 k^2}{2} \right) + \frac{1}{k} \right] - \frac{41}{162} \frac{\mu}{k} \right\} \delta_{\ell 0}. \end{aligned}$$

- Numerically with account of all orders in  $\mu = m/M$

$$\Delta E_2^{(Mu)} = -3.133412(3) \dots \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m}{M} \left(\frac{m_r}{m}\right)^3 m \delta_{\ell 0}$$



# Two-loop polarization insertions

## Muonium

- Analytic result, first order in  $\mu = m/M$

$$\begin{aligned}\Delta E_2^{(Mu)} &\approx \frac{32(Z\alpha)^5 m}{\pi n^3} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_r}{m}\right)^3 \mu \int_0^\infty \frac{dk}{k} \left\{ l_2(k) \left[ \sqrt{1 + \frac{k^2}{4}} \right. \right. \\ &\times \left. \left. \left( \frac{1}{k} + \frac{k^3}{8} \right) - \frac{k^2}{8} \left( 1 + \frac{k^2}{2} \right) \right] - \frac{41}{162k} \right\} \delta_{\ell 0} \\ &= \left( \frac{6589}{7560} + \frac{145756\pi^2}{99225} + \frac{7\pi^4}{270} - \frac{296\pi^2}{315} \ln 2 + \frac{4\pi^2}{9} \ln^2 2 \right. \\ &\left. - \frac{4}{9} \ln^4 2 - \frac{32}{3} \text{Li}_4\left(\frac{1}{2}\right) - \frac{11597}{1260} \zeta(3) \right) \frac{\alpha^2(Z\alpha)^5 m}{\pi^3 n^3} \frac{m}{M} \left(\frac{m_r}{m}\right)^3 m \delta_{\ell 0} \\ &= -3.1121 \dots \frac{\alpha^2(Z\alpha)^5 m}{\pi^3 n^3} \frac{m}{M} \left(\frac{m_r}{m}\right)^3 m \delta_{\ell 0}.\end{aligned}$$

# Two-loop polarization insertions

## Positronium

$$\begin{aligned}\Delta E_2^{(Ps)} &= \frac{4\alpha^7 m}{\pi^3 n^3} \int_0^\infty dk l_2(k) \left[ \frac{k^4}{8\sqrt{k^2+4}} + \frac{3k^2}{8\sqrt{k^2+4}} \right. \\ &\quad \left. - \frac{k^3+k}{8} - \frac{1}{\sqrt{k^2+4}k^2} + \frac{41}{324k^2} \right] \delta_{\ell 0} \\ &= \left( -\frac{4\text{Li}_4\left(\frac{1}{2}\right)}{3} - \frac{17921\zeta(3)}{10080} + \frac{26347}{60480} + \frac{311233\pi^2}{793800} \right. \\ &\quad \left. + \frac{7\pi^4}{2160} + \frac{1}{18}\pi^2 \ln^2 2 - \frac{\ln^4 2}{18} - \frac{76}{315}\pi^2 \ln 2 \right) \frac{\alpha^7 m}{\pi^3 n^3} \delta_{\ell 0} \\ &= 0.393966 \dots \frac{\alpha^7}{\pi^3 n^3} m \delta_{\ell 0}.\end{aligned}$$



# One-loop electron factor and polarization

## Previous order



$$\Delta E = -\frac{(Z\alpha)^5}{\pi n^3} m_r^3 \int \frac{d^4 k}{i\pi^2 k^4} \frac{1}{4} \text{Tr}[(1 + \gamma_0)L_{\mu\nu}] \frac{1}{4} \text{Tr}[(1 + \gamma_0)H_{\mu\nu}] \delta_{\ell 0}$$

- $L_{\mu\nu} = L_{\mu\nu}^{\Sigma} + 2L_{\mu\nu}^{\Lambda} + L_{\mu\nu}^{\Xi}$

$$\begin{aligned} \frac{1}{4} \text{Tr}[(1 + \gamma_0)L_{\mu\nu}] &\equiv \frac{\alpha}{\pi m} \mathcal{L}_{\mu\nu} \left( \frac{k}{m} \right) \\ &= \frac{\alpha}{\pi m} \left[ \mathcal{L}_{\mu\nu}^{\Sigma} \left( \frac{k}{m} \right) + 2\mathcal{L}_{\mu\nu}^{\Lambda} \left( \frac{k}{m} \right) + \mathcal{L}_{\mu\nu}^{\Xi} \left( \frac{k}{m} \right) \right] \end{aligned}$$

# One-loop electron factor and polarization

## Previous order

- $H_{\mu\nu} = \gamma_\mu \frac{\not{p} + \not{k} + M}{k^2 + 2Mk_0 + i0} \gamma_\nu + \gamma_\nu \frac{\not{p} - \not{k} + M}{k^2 - 2Mk_0 + i0} \gamma_\mu$
- $\frac{1}{4} \text{Tr} \left[ (1 + \gamma_0) H_{\mu\nu} \right] =$   
 $-\frac{1}{M} \left[ k^2 g_{\mu 0} g_{\nu 0} - k_0 (g_{\mu 0} k_\nu + g_{\nu 0} k_\mu) + k_0^2 g_{\mu\nu} \right] \frac{1}{k_0^2 - \frac{k^4}{4M^2}}$

## Muonium

- **Principal value definition**  $k^2 \wp \left( \frac{1}{k_0^2} \right) = k^2 \lim_{\frac{k}{M} \rightarrow 0} \frac{k_0^2 + \frac{k^4}{4M^2}}{\left( k_0^2 - \frac{k^4}{4M^2} \right)^2}$
- **Linear in  $\mu = m/M$  approximation**

$$\frac{1}{4} \text{Tr} \left[ (1 + \gamma_0) H_{\mu\nu} \right] \rightarrow -\frac{1}{M} \left[ k^2 g_{\mu 0} g_{\nu 0} \wp \left( \frac{1}{k_0^2} \right) - (g_{\mu 0} k_\nu + g_{\nu 0} k_\mu) \frac{1}{k_0} + g_{\mu\nu} \right]$$
$$\equiv -\frac{1}{M} \mathcal{H}_{\mu\nu}(k)$$

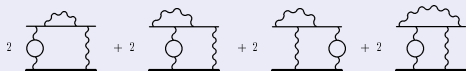
# One-loop electron factor and polarization

## Muonium

- Linear in  $\mu = m/M$  radiative-recoil contribution of previous order

$$\Delta E_{rec} = \frac{\alpha(Z\alpha)^5}{\pi^2 n^3} \frac{m_r^3}{Mm} \int \frac{d^4 k}{i\pi^2 k^4} \mathcal{L}_{\mu\nu} \left( \frac{k}{m} \right) \mathcal{H}_{\mu\nu}(k)$$

Muonium: linear in  $\mu = m/M$  contribution, order  $\alpha^2(Z\alpha)^5(m/M)m$



- Insert  $(\alpha/\pi)k^2 I_1(k)$  in  $\Delta E_{rec}$

$$\begin{aligned} \Delta E_3^{(Mu)} &= \frac{\alpha^2(Z\alpha)^5}{\pi^2 n^3} \frac{m_r^3}{Mm} \int \frac{d^4 k}{i\pi^2 k^2} 2I_1(k) \mathcal{L}_{\mu\nu} \left( \frac{k}{m} \right) \mathcal{H}_{\mu\nu}(k) \\ &= (J_{\Sigma P} + 2J_{\Lambda P} + J_{\Xi P}) \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m}{M} \left( \frac{m_r}{m} \right)^3 m\delta_{\ell 0} \\ &= -9.2569(2) \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m}{M} \left( \frac{m_r}{m} \right)^3 m\delta_{\ell 0} \end{aligned}$$



# One-loop electron factor and polarization

## Positronium

- **Unexpected connection between the total (recoil + nonrecoil) & linear in  $\mu = m/M$  heavy particle factor at  $m = M$**

$$\mathcal{H}_{\mu\nu}(k)_{tot} = \mathcal{H}_{\mu\nu}(k)_{rec} \frac{k_0^2}{k_0^2 - \frac{k^4}{4m^2}} = \frac{k^2 g_{\mu 0} g_{\nu 0} - (g_{\mu 0} k_\nu + g_{\nu 0} k_\mu) k_0 + g_{\mu\nu} k_0^2}{k_0^2 - \frac{k^4}{4m^2}}$$

- Dimensionless interpolating factor

$$\tilde{G}(k, M) = \frac{k_0^2 \left( \frac{k^4}{4M^2} + k_0^2 \right)}{\left( k_0^2 - \frac{k^4}{4M^2} \right)^2} - \frac{k_0^2 k^4 m^2}{2M^4 \left( k_0^2 - \frac{k^4}{4M^2} \right)^2}$$

$$\tilde{G}(k, m) = k_0^2 / (k_0^2 - k^4 / 4m^2), \quad \tilde{G}(k, M)|_{k/M \rightarrow 0} \rightarrow 1$$

- Substitution  $\mathcal{H}_{\mu\nu}(k)_{rec} \rightarrow \mathcal{H}_{\mu\nu}(k)_{rec} \tilde{G}(k, M)$
- $\mathcal{H}_{\mu\nu}(k)_{rec} \tilde{G}(k, M)|_{k/M \rightarrow 0}$  – integrand for linear in  $\mu$  corrections for muonium
- $\mathcal{H}_{\mu\nu}(k)_{rec} \tilde{G}(k, m)$  – integrand for total (recoil + nonrecoil) corrections for positronium

# One-loop electron factor and polarization

## Positronium

- Universal expression

$$\begin{aligned}\Delta E_{pol} &= \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m_r^3}{mM} \int \frac{d^4 k}{i\pi^2 k^4} \mathcal{L}_{\mu\nu}(k) \mathcal{H}_{\mu\nu}(k)_{rec} 2k^2 I_1(k) \tilde{G}(km, M) \delta_{\ell 0} \\ &\equiv \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m_r^3}{mM} \Delta \mathcal{E}_{pol}(M)\end{aligned}$$

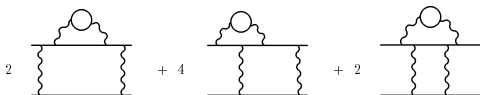
- Once again linear in  $\mu = m/M$  contribution for muonium

$$\Delta E_3^{(Mu)} = \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m_r^3}{mM} \Delta \mathcal{E}_{pol}(M \rightarrow \infty)$$

- Total (recoil and nonrecoil) spin-independent contribution in positronium

$$\begin{aligned}\Delta E_3^{(Ps)} &= \frac{\alpha^2(Z\alpha)^5}{\pi^3 n^3} \frac{m}{4} \Delta \mathcal{E}_{pol}(M = m) \equiv (J_{\Sigma P} + 2J_{\Lambda P} + J_{\Xi P}) \frac{\alpha^7 m}{\pi^3 n^3} \delta_{\ell 0} \\ &= 0.5701(2) \frac{\alpha^7}{\pi^3 n^3} m \delta_{\ell 0}\end{aligned}$$

# Radiatively corrected electron factor



## Muonium

- Again

$$\Delta E_{rec} = \frac{\alpha^2 (Z\alpha)^5}{\pi^3 n^3} \frac{m_r^3}{Mm} \int \frac{d^4 k}{i\pi^2 k^4} \mathcal{L}_{\mu\nu} \left( \frac{k}{m} \right) \mathcal{H}_{\mu\nu}(k)_{rec}$$

- Linear in mass ratio contribution in muonium

$$\Delta E_4^{(Mu)} = 0.10332(3) \frac{\alpha^2 (Z\alpha)^5}{\pi^3 n^3} \frac{m}{M} \left( \frac{m_r}{m} \right)^3 m \delta \ell_0$$



# Radiatively corrected electron factor

## Positronium

- Substitution

$$\mathcal{H}_{\mu\nu}(k)_{rec} \rightarrow \mathcal{H}_{\mu\nu}(k)_{tot} = \mathcal{H}_{\mu\nu}(k)_{rec} \frac{k_0^2}{k_0^2 - \frac{k^4}{4m^2}}$$

- In positronium  $m = M$  and an extra factor 2

$$\Delta E_{tot} = 2 \frac{\alpha^2 (Z\alpha)^5}{\pi^3 n^3} \frac{m_r^3}{Mm} \int \frac{d^4 k}{i\pi^2 k^4} \mathcal{L}_{\mu\nu} \left( \frac{k}{m} \right) \mathcal{H}_{\mu\nu}(k)_{tot}$$

- Numerically

$$\Delta E_4^{(Ps)} = -0.12947(3) \frac{\alpha^7 m}{\pi^3 n^3} \delta_{\ell 0}$$



# Summary

$$\Delta E^{(Mu)} = \sum_{i=1}^4 \Delta E_i^{(Mu)} = -11.3275(2) \frac{\alpha^2 (Z\alpha)^5 m}{\pi^3 n^3} \frac{m}{M} \left(\frac{m_r}{m}\right)^3 m \delta_{10}$$

$$\Delta E^{(Ps)} = \sum_{i=1}^4 \Delta E_i^{(Ps)} = 0.8057(2) \frac{\alpha^7 m}{\pi^3 n^3} \delta_{10}$$

- Numerically the contributions to the Lamb shift in muonium calculated above are at the level of a few tenths of kHz
- Numerically the contributions to the Lamb shift in positronium calculated above are at the level of a few kHz
- These contributions are too small to be relevant for the results of the ongoing experiments, but will hopefully become relevant in the future



# Summary

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# Thank you!

