Quantum electrodynamics of composite particles

Krzysztof Pachucki

University of Warsaw

PREN2023
June 29, 2023

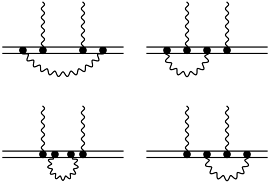
Leading finite nuclear size

Atomic energy levels are shifted due to the finite nuclear size

- $E_{\rm fs}(Z\,\alpha)$ is a function of the nuclear charge Z
- expansion in α: $E_{\rm fs} = E_{\rm fs}^{(4)} + E_{\rm fs}^{(5)} + E_{\rm fs}^{(6)} + \ldots$,
  lepton/nuclear mass ratio dependence
- $E_{\rm fs}^{(4)} = \frac{2\,\pi}{3}\,\phi^2(0)\,Z\,\alpha\,r_C^2$,
  it includes complete dependence on the nuclear mass through
  $\phi^2(0) = (\mu\,Z\,\alpha)^3/(\pi\,n^3)\,\delta_{l0}$
- definition of the charge radius r_C depend on the nuclear spin
  $\delta H = -Z\,e\left(\frac{r_C^2}{6} + \frac{\delta_I}{M^2}\right)\vec{\nabla}\cdot\vec{E}$, where $\delta_0 = 0, \delta_{1/2} = 1/8, \delta_1 = 0\ \ldots$
- there is no established definition of the nuclear charge radii for the spin $I > 1$
- what is the values of the self-energy correction ?,
  overlap of r_C^2 with the so-called nuclear self-energy

Nuclear self-energy

How to define the nuclear charge radius in the presence of self-energy ?



- $\Delta T^{00} = \frac{q^2\, M}{p^2 - M^2}\left(\frac{4\, Z^2 \alpha}{3\,\pi\, M^2}\ \ln\frac{M^2}{M^2 - p^2} + \frac{2}{3}\, r_C^2\right) + (q \to -q)$

- $E(n,l) = \frac{2}{3\, n^3}\,(Z\,\alpha)^4\,\mu^3\, r_C^2\,\delta_{l0} + \frac{4\, Z\,(Z\,\alpha)^5}{3\,\pi\, n^3}\ \frac{\mu^3}{M^2}\left[\ln\left(\frac{M}{\mu\,(Z\,\alpha)^2}\right)\delta_{l0} - \ln k_0(n,l)\right]$

Elastic two-photon exchange with the nuclear size

- $E^{(5)}_{\rm fs} = -\frac{\pi}{3}\,\phi^2(0)\,(Z\,\alpha)^2\,m\,r_F^3$, where $r_F^3 = \int d^3r_1 \int d^3r_2\,\rho(r_1)\,\rho(r_2)\,|\vec{r}_1 - \vec{r}_2|$
- $E^{(5)}_{\rm fs}$ is (very) important for muonic atoms
- This result is valid in the nonrecoil limit, thus what are the nuclear recoil corrections?
- $E^{(5)}_{\rm recfs} = -\frac{m}{M}\,\phi^2(0)\,(Z\,\alpha)^2\left[\frac{7}{6} - 2\,\gamma - 2\,\ln(m\,r_L)\right] r_C^2$, where

  $$\int d^3r_1 \int d^3r_2\,\rho(\vec{r}_1)\,\rho(\vec{r}_2)\,|\vec{r}_1 - \vec{r}_2|^2\,\ln(m\,|\vec{r}_1 - \vec{r}_2|) = 2\,r_C^2\,\ln(m\,r_L)$$

- it is significantly enhanced r_F^3 versus r_C^2
- higher order in mass ratio corrections depend on the nuclear spin
- inclusion of the (electron-nucleus) Breit interaction leads to spurious terms that are linear in r_C

Elastic three-photon exchange

In the infinite nuclear mass limit

$$
\begin{aligned}
E^{(6)}_{\rm fns}(nS) &= -(Z\,\alpha)^6\, m^3\, r_C^2\, \frac{2}{3\,n^3}\left[\frac{9}{4n^2} - 3 - \frac{1}{n} + 2\,\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m\, r_{C2}\, Z\,\alpha)\right] \\
&\quad + (Z\,\alpha)^6\, m^5\, r_C^4\, \frac{4}{9\,n^3}\left[-\frac{1}{n} + 2 + 2\,\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m\, r_{C1}\, Z\,\alpha)\right] \\
&\quad + (Z\,\alpha)^6\, m^5\, r_{CC}^4\, \frac{1}{15\,n^5}\,, \\
E^{(6)}_{\rm fns}(nP_{1/2}) &= (Z\,\alpha)^6\, m\left(\frac{m^2\, r_C^2}{6} + \frac{m^4\, r_{CC}^4}{45}\right)\frac{1}{n^3}\left(1 - \frac{1}{n^2}\right), \\
E^{(6)}_{\rm fns}(nP_{3/2}) &= (Z\,\alpha)^6\, m^5\, r_{CC}^4\, \frac{1}{45\,n^3}\left(1 - \frac{1}{n^2}\right), \\
E^{(6)}_{\rm fns}(nL_J) &= 0 \text{ for } L > 1\,,
\end{aligned}
$$

where $r_{CC}^4 = \langle r^4 \rangle$ and the effective nuclear charge radii r_{C1} and r_{C2} encode the high-momentum contributions and are expected to be of the order of r_C.

What are the finite nuclear size recoil corrections ?

Nonperturbative nuclear recoil correction

Exact nonperturbative formula (a'la Shabaev) for pure recoil corrections:

$$E_{\rm rec} = \frac{m^2}{M}\,\frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega\,\langle a|\big[p^j - D^j(\omega)\big]\,G(\omega+E_a)\,\big[p^j - D^j(\omega)\big]|a\rangle$$

where

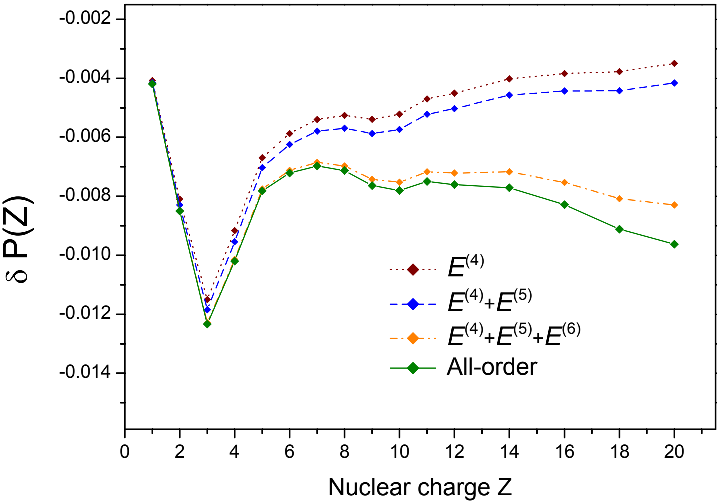
- $G(E) = [E - H_D(1 - i\epsilon)]^{-1}$ is the Dirac-Coulomb Green function
- $D^j(\omega) = -4\pi Z\alpha\,\alpha^i\,G_C^{ij}(\omega,\vec r)$, and α^i are the Dirac matrices.
- Photon propagator in the modified Coulomb gauge

$$G_C^{ij}(\omega,\vec r) = \delta^{ij}\,\mathcal{D}(\omega,r) + \frac{\nabla^i\nabla^j}{\omega^2}\Big[\mathcal{D}(\omega,r) - \mathcal{D}(0,r)\Big]$$

$$\mathcal{D}(\omega,r) = \int\frac{d^3k}{(2\pi)^3}\,e^{i\vec k\cdot\vec r}\,\frac{\rho(\vec k^2-\omega^2)}{\omega^2-\vec k^2}\,.$$

- The nuclear charge density is a function of an invariant $\rho(\vec k^2 - \omega^2)$
- Radiative recoil corrections at $\alpha\,(Z\,\alpha)^5$ are not known !!!

Numerical results for the finite size recoil



where $\delta P = E_{\rm recfs}/[(m^2/M)(Z\,\alpha)^5/\pi]$ and $E^{(6)}_{\rm recfs} \approx -\frac{m^3}{M}\,(Z\,\alpha)^6\, r_C$ for electronic atoms

Point nucleus two-photon exchange $(Z\,\alpha)^5$

$$E(n,l) = \frac{\mu^3}{m\,M}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\left\{\frac{2}{3}\,\delta_{l0}\,\ln\left(\frac{1}{Z\,\alpha}\right) - \frac{8}{3}\,\ln k_0(n,l) - \frac{1}{9}\,\delta_{l0} - \frac{7}{3}\,a_n - 2\,\delta_{l0}\,\ln\left(1+\frac{m}{M}\right)\right.$$
$$\left. + \frac{m^2}{M^2-m^2}\,\ln\left(\frac{M}{m}\right)\delta_{l0}\,[2 + I\,(2\,I-1)]\right\} \qquad (1)$$

where

$$a_n = -2\left[\ln\left(\frac{2}{n}\right) + \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) + 1 - \frac{1}{2\,n}\right]\delta_{l0} + \frac{1-\delta_{l0}}{l\,(l+1)\,(2\,l+1)}\,. \qquad (2)$$

- It depends on the nuclear spin I
- EVP correction to TPE is barely knowns and is quite difficult to calculate !!!
- It is unknown for $I > 1$

TPE

$$E_{\rm TPE} = E_{\rm nucl1} + E_{\rm nucl2} + E_{\rm pol} + \dots\,,$$
$$E_{\rm nucl1} = -\frac{\pi}{3}\, m\,\alpha^2 \phi^2(0) \Big[Z\, \tilde{R}_F^3(p) + (A-Z)\, \tilde{R}_F^3(n) \Big]\,,$$
$$E_{\rm nucl2} = -\frac{\pi}{3}\, m\,\alpha^2 \phi^2(0) \sum_{i,j=1}^{Z} \langle \phi_N | |\vec{r}_i - \vec{r}_j|^3 | \phi_N \rangle\,,$$
$$E_{\rm pol} = -\frac{4\,\pi\,\alpha^2}{3}\, \phi^2(0) \int_{E_T} dE\, \sqrt{\frac{2\,\mu}{E}}\, |\langle \phi_N | \vec{d} | E \rangle|^2,$$

- Instead of the (nuclear) Zemach radius r_Z the TPE contribution involves effective Zemach radii of individual nucleons,

  $R_F^3(p) = 2.876(246)\ {\rm fm}^3,\ \ R_F^3(n) = 0.712(223)\ {\rm fm}^3,$

  the inter-proton $|\vec{r}_i - \vec{r}_j|^3$ and a kind of the electric dipole polarizability
- the difference between the elastic (Zemach) and the complete TPE is significant
- $E_{\rm TPE}$ requires subtraction of the point nucleus $(Z\,\alpha)^5$ contribution to be consistent with the QED contribution

3PE

The total nuclear structure correction $E^{(6)}$ is represented as a sum of several parts: $E^{(6)} = E_1^{(6)} + E_2^{(6)} + E_C + E_{\rm np}^{(6)}$, which have been calculated only for μD, $E_{\rm np}^{(6)}$ is the tree-photon exchange with individual nucleons,

$$
\begin{aligned}
E_C(n,l) = & -\frac{1}{3}\left\langle\frac{1}{r^4}\right\rangle\left\langle\phi_N\middle|\vec{R}\,\frac{1}{H_N-E_N}\,\vec{R}\middle|\phi_N\right\rangle \\
& -\frac{4\,\pi}{3}\,\phi^2(0)\left\langle\phi_N\middle|\,\vec{R}\,\frac{1}{H_N-E_N}\left[1+\ln\left(\frac{2\,m\,\alpha^2}{H_N-E_N}\right)\right]\vec{R}\middle|\phi_N\right\rangle \\
E_2^{(6)}(nS) = & -\alpha^6\,m^3\,\frac{2}{3\,n^3}\left[r_d^2\left(\frac{9}{4n^2}-3-\frac{1}{n}+2\,\gamma-\ln\frac{n}{2}+\Psi(n)+\ln\alpha\right)\right. \\
& \left.+\,r_s^2\left(\frac{47}{12}-\frac{13}{2}\ln 2-\frac{5}{2}\ln\frac{\langle E\rangle_2}{m}-\gamma\right)+r_p^2\,\ln(r_{p2}\,m)\right] \\
E_1^{(6)}(nS) = & \,\alpha^6\,m^5\,\frac{4}{9\,n^3}\left[r_d^4\left(-\frac{1}{n}+\gamma-\ln\frac{n}{2}+\Psi(n)+\ln\alpha\right)+r_{dd}^4\,\frac{3}{20\,n^2}\right. \\
& -\,r_{ss}^4\left(\frac{3}{5}\ln 2-\frac{9}{10}\right)+r_s^4\left(\frac{1}{3}-\frac{3}{2}\ln 2-\frac{1}{2}\ln\frac{\langle E\rangle_1}{m}+2\,\beta\right) \\
& \left.+\,r_p^4\Big(2+\gamma+\ln(r_{p1}\,m)\Big)+r_s^2\,r_p^2\left(\frac{10}{3}-5\,\ln 2-\ln\frac{\langle E\rangle_2}{m}\right)\right].
\end{aligned}
$$

Charge radii of light muonic atoms

Order	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
$(Z\alpha)^5$	TPE	0.029 2(25)	1.979(20)	16.38(31)	9.76(40)
$\alpha^2(Z\alpha)^4$	Coulomb distortion	0.0	$-$0.261	$-$1.010	$-$0.536
$(Z\alpha)^6$	3PE	$-$0.001 3(3)	0.002 2(9)	$-$0.214(214)	$-$0.165(165)
$\alpha(Z\alpha)^5$	eVP$^{(1)}$ with TPE	0.000 6(1)	0.027 5(4)	0.266(24)	0.158(12)
$\alpha(Z\alpha)^5$	μSE$^{(1)}$ + μVP$^{(1)}$ with TPE	0.000 4	0.002 6(3)	0.077(8)	0.059(6)
$E_{\rm QED}$	point nucleus	206.034 4(3)	228.774 0(3)	1644.348(8)	1668.491(7)
$\mathcal{C}\, r_C^2$	finite size	$-5.225\,9\, r_p^2$	$-6.107\,4\, r_d^2$	$-103.383\, r_h^2$	$-106.209\, r_\alpha^2$
$E_{\rm NS}$	nuclear structure	0.028 9(25)	1.750 3(200)	15.499(378)	9.276(433)
E_L(exp)	experiment[1]	202.370 6(23)	202.878 5(34)	1258.598(48)	1378.521(48)
r_C	this work	0.840 60(39)	2.127 58(78)	1.970 07(94)	1.678 6(12)
r_C	previous[a]	0.840 87(39)	2.125 62(78)	1.970 07(94)	1.678 24(83)

[1] Presented in Antognini *et al (2013), Pohl* et al *(2016), Shuhmann* et al *(:2023), Krauth* et al *(2021)*

Summary

Deuteron-proton charge radii diference: perfect agreement

$$r_d^2 - r_p^2|_{\text{muonic}} = 3.820\,0(7)_{\text{exp}}(30)_{\text{theo}}\ \text{fm}^2$$
$$r_d^2 - r_p^2|_{\text{electronic}} = 3.820\,7(3)\ \text{fm}^2\,.$$

Helion-alpha charge radii diference: 3.6 σ disagreement

$$r_h^2 - r_\alpha^2|_{\text{muonic}} = 1.063\,6(6)_{\text{exp}}(30)_{\text{theo}}\ \text{fm}^2$$
$$r_h^2 - r_\alpha^2|_{\text{electronic}} = 1.075\,7(15)\,\text{fm}^2$$

Conclusions

- the nuclear mass effects has to be accounted for (beyond the Dirac equation)
- the finite nuclear size effects $\tilde{\rho}(\vec{q}^2 - \omega^2)$ require inclusion of the exchange energy
- the main limitation comes from inelastic TPE and 3PE
- α^6 (pure and radiative) recoil finite size correction are not yet known
- results for Li and Be ???

Nuclear structure effects in hyperfine splitting

- $\delta E_{\rm nucl} = \delta^{(1)} E_{\rm nucl} + \delta^{(2)} E_{\rm nucl} + \ldots$ where
  $\delta^{(1)} E_{\rm nucl}$ is the two-photon exchange correction of order $(Z\,\alpha)\,E_F$,
  $\delta^{(2)} E_{\rm nucl}$ is the three-photon exchange correction of order $(Z\,\alpha)^2\,E_F$,
  $E_F = -\frac{2}{3}\,\psi^2(0)\,\vec{\mu}\cdot\vec{\mu}_e$
- $\delta^{(1)} E_{\rm nucl} = -2\,m_r Z\alpha\, r_Z E_F$ where
  r_Z is the Zemach radius defined by $r_Z = \int d^3 r_1 \int d^3 r_2\, \rho_M(r_1)\,\rho_E(r_2)\,|\vec{r}_1 - \vec{r}_2|$
- nuclear recoil correction

$$\delta^{(1)} E_{\rm fns,rec} = -\,E_F\,\frac{Z\,\alpha}{\pi}\,\frac{m}{M}\,\frac{3}{8}\left\{g\left[\gamma - \frac{7}{4} + \ln(m\,r_{M^2})\right]\right.$$
$$\left. -\,4\left[\gamma + \frac{9}{4} + \ln(m\,r_{EM})\right] - \frac{12}{g}\left[\gamma - \frac{17}{12} + \ln(m\,r_{E^2})\right]\right\}$$

$O(\alpha^2)$ corrections to hfs

- $\delta^{(2)} E_{\rm fns} = \frac{4}{3} E_F (m r_p Z\alpha)^2 \Big[-\frac{1}{n} + 2\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m r_{pp} Z\alpha) + \frac{r_m^2}{4 r_p^2 n^2} \Big]$
- $O(\alpha^2)$ recoil corrections are unknown
- nonperturbative formula for the recoil correction to hfs has not yet been derived (ongoing project)
- the use elastic formfactors in description of hfs is very much approximate

More accurate picture

$$\delta^{(1)} E_{\rm hfs} = E_{\rm Low} + E_{\rm 1nuc} + E_{\rm pol}$$

$$E_{\rm 1nuc} = -\frac{8\,\pi}{3}\,\alpha^2\,\frac{\psi^2(0)}{m_p + m}\,\vec{s}\cdot\Big\langle\sum_a g_a\,\vec{s}_a\,r_{a{\rm Z}}\Big\rangle$$

$$E_{\rm Low} = \frac{\alpha}{16}\,\psi^2(0)\,\vec{\sigma}\sum_{a\neq b}\frac{e_a\,e_b}{m_b}\left\langle 4\,r_{ab}\,\vec{r}_{ab}\times\vec{p}_b + \frac{g_b}{r_{ab}}\left[\vec{r}_{ab}\,(\vec{r}_{ab}\cdot\vec{\sigma}_b) - 3\,\vec{\sigma}_b\,r_{ab}^2\right]\right\rangle$$

Let us consider the special case of a spherically symmetric nucleus and neglect the proton-neutron mass difference.

$$E_{\rm Low} = -\frac{8\,\pi}{3}\,\alpha^2\,\frac{\psi^2(0)}{m_n}\sum_{a-{\rm protons}}\sum_b\langle r_{ab}\,g_b\,\vec{s}_b\rangle\,\vec{s}$$

Much better description for hfs in μD

Discrepancies in μD hfs

- the “experimental value” of the nuclear-structure correction in μD(2S) hfs

  $\delta E_{\rm nucl,exp} = E_{\rm hfs}(\rm exp) - E_{\rm hfs}(\rm point) = 0.0966(73)$ meV

- the numerical value of the Zemach correction with $r_Z = 2.593(16)$ fm is

  $\delta E_{\rm Zem} = -0.1177(33)$ meV, opposite sign !

- including the nuclear vector polarizability and
  the inelastic three-photon exchange (10% effect)

  $\delta E_{\rm nucl,theo} = 0.028\,3(86)$ meV

- the difference

  $\delta E_{\rm nucl,theo} - \delta E_{\rm nucl,exp} = 0.058\,3(113)$

- Nuclear structure effects in hfs are not yet well known