The recoil–finite-size correction for the hyperfine splitting in μH and eH

Yong-Hui Lin, HISKP, Universität Bonn

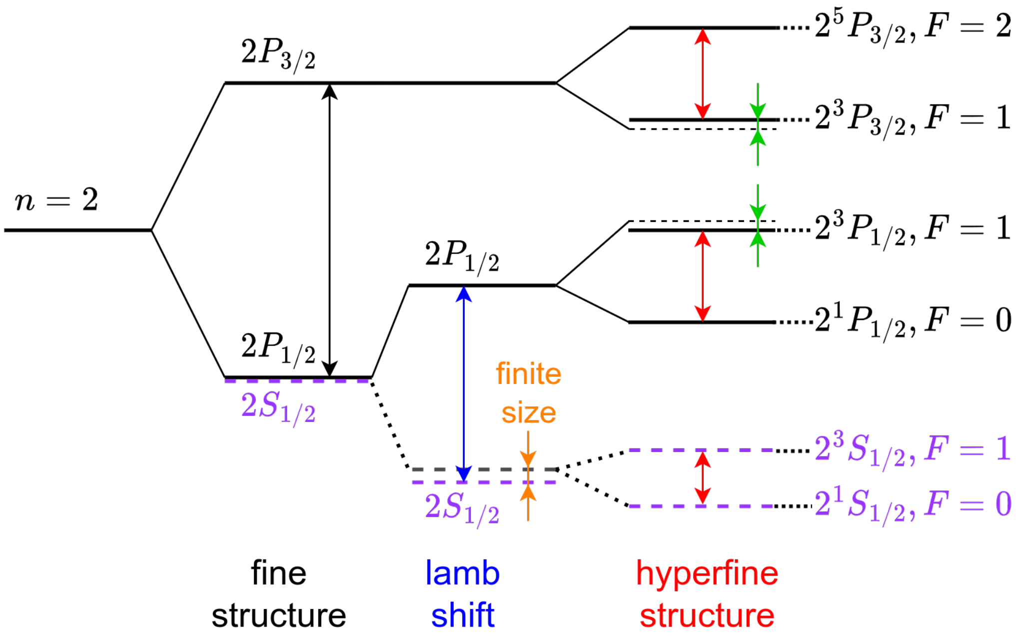
-- Aldo Antognini, YHL, Ulf-G. Meißner, Phys.Lett.B 835 (2022) 137575

PREN +μASTI 2023

HIM Helmholtz-Institut Mainz

The Hydrogen Spectrum

- Fine structure
  - ☞ Degeneracy respect to $\boldsymbol{n}$ and $\boldsymbol{j}$.
- Lamb shift
  - ☞ Lifts levels with $\boldsymbol{l}$=$\boldsymbol{j}$±1/2, contains proton finite size effect (LO ∝ r_p^2).
- Hyperfine structure
  - ☞ interaction with **nucleus spin** $\boldsymbol{s_p}$, mixes states with same $\boldsymbol{F}$=$\boldsymbol{j}$+$\boldsymbol{s_p}$ but different $\boldsymbol{j}$.



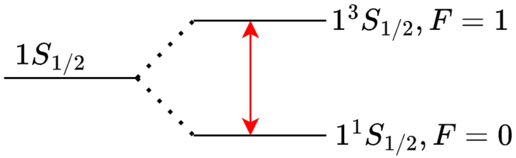
The Hyperfine Splitting

HFS	$eH(1S)$	$\mu H(2S)$
Main (E_F)	1	1
nonrecoil QED $(\delta_{\mathrm{QED}} E_F)$	0.0011360896(19)	0.0011752(1) + 0.0025432(1)(VP)
LO proton size $(-2(Z\alpha)mr_Z E_F)$	$-42.4(1.1) \times 10^{-6}$	$-0.007410(80)$
LO recoil $(\delta_{\mathrm{recoil}} E_F)$	$5.22(1) \times 10^{-6}$	0.000931(7)
proton polarizability $(\delta_{\mathrm{pol}} E_F)$	$\leq 4 \times 10^{-6}$	0.0004(1)
$\vdots$	$\vdots$	$\vdots$
total	1420399.3(1.6) kHz	22.8148(20) meV

I. Eides et al., Phys.Rept. 342 (2001) 63-261; O. Tomalak, Eur.Phys.J.C 77 (2017) 12, 858; A. Antognini et al., Annals Phys. 331 (2013) 127-145

The 1S-HFS of μH and eH

A. Antognini et al., Ann.Rev.Nucl.Part.Sci. 72 (2022) 389

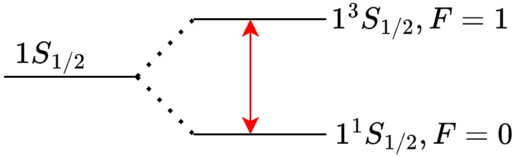


μH

$$
\begin{aligned}
E^{\rm th}_{\rm HFS}(\boldsymbol{\mu}\mathbf{p}) &= E_F + \Delta E_{\rm QED} + \Delta E^{2\gamma} \\
&= 182.443 + 1.354(7) + E_F\left(\underline{1.01958(13)}\Delta_{\rm Z}{}^{\boldsymbol{\mu}\mathbf{p}} + \underline{1.01656(4)}\Delta_{\rm recoil}{}^{\boldsymbol{\mu}\mathbf{p}} + \underline{1.00402}\Delta_{\rm pol}{}^{\boldsymbol{\mu}\mathbf{p}}\right) \quad [\text{meV}]
\end{aligned}
$$

- Δ_Z : the Zemach correction C. Zemach 1956
- Δ_{recoil} : the Recoil correction
- Δ_{pol} : the Proton polarizability correction
- Coefficients contain all relevant higher order corrections

The 1S-HFS of μH



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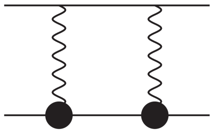
\end{aligned}
$$

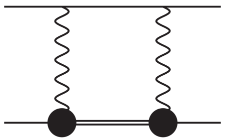
<u>elastic</u>



+Xbox

Determined completely by the **EMFFs** of the proton.

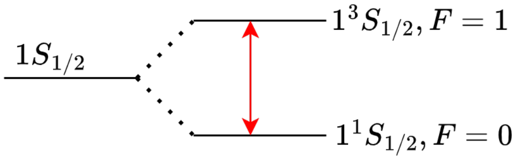
<u>inelastic</u>



+Xbox

Both the **EMFFs** and **Inelastic structure functions** (no enough data) are needed.

Zemach (Δ_Z) and Recoil (Δ_{recoil}) Corrections



$$E^{\rm th}_{\rm HFS}(\mu p) = E_F + \Delta E_{\rm QED} + \Delta E^{2\gamma}$$

$$= 182.443 + 1.354(7) + E_F\left(1.01958(13)\Delta_Z{}^{\mu p} + 1.01656(4)\Delta_{\rm recoil}{}^{\mu p} + 1.00402\Delta_{\rm pol}{}^{\mu p}\right) \quad [\text{meV}]$$

$$\Delta_Z = -2Z\alpha m_r \, r_Z$$

$$= -2Z\alpha m_r \left(-\frac{4}{\pi}\int_0^\infty \frac{dQ}{Q^2}\left[\frac{G_E(Q^2)G_M(Q^2)}{1+\kappa} - 1\right]\right)$$

$$\Delta_{\rm recoil} = \frac{Z\alpha}{\pi(1+\kappa)}\int_0^\infty \frac{dQ}{Q}\left\{\frac{G_M(Q^2)}{Q^2}\frac{8mM}{v_l+v}\left(2F_1(Q^2) + \frac{F_1(Q^2)+3F_2(Q^2)}{(v_l+1)(v+1)}\right) - \frac{8m_r G_M(Q^2)G_E(Q^2)}{Q} - \frac{mF_2^2(Q^2)}{M}\frac{5+4v_l}{(1+v_l)^2}\right\}$$

$$v_{(l)} = \sqrt{1+4M^2(m^2)/Q^2}$$

A. Antognini et al., Ann.Rev.Nucl.Part.Sci. 72 (2022) 389

The 1S-HFS of μH and eH

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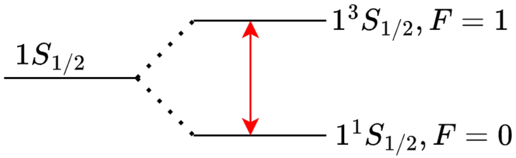
μH



$$\begin{aligned} E^{\rm th}_{\rm HFS}(\boldsymbol{\mu}\mathbf{p}) &= E_F + \Delta E_{\rm QED} + \Delta E^{2\gamma} \\ &= 182.443 + 1.354(7) + E_F\left(\underline{1.01958(13)}\Delta_{\rm Z}{}^{\boldsymbol{\mu}\mathbf{p}} + \underline{1.01656(4)}\Delta_{\rm recoil}{}^{\boldsymbol{\mu}\mathbf{p}} + \underline{1.00402}\Delta_{\rm pol}{}^{\boldsymbol{\mu}\mathbf{p}}\right) \quad [\rm meV] \end{aligned}$$

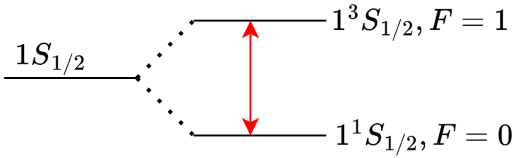
eH

$$E^{\rm th}_{\rm HFS}(\mathbf{H}) = 1418840.082(9) + 1613.024(3) + E_F^{\mathbf{H}}\left(\underline{1.01558(13)}\Delta_{\rm Z}{}^{\mathbf{H}} + \underline{0.99807(13)}\Delta_{\rm recoil}{}^{\mathbf{H}} + \underline{1.00002}\Delta_{\rm pol}{}^{\mathbf{H}}\right) \quad [\rm kHz]$$

The 1S-HFS of μH and eH

A. Antognini et al., Ann.Rev.Nucl.Part.Sci. 72 (2022) 389

μH



δE_{exp}=1ppm E_F

CREMA, FAMU, J-PARC in future

$$E^{\text{th}}_{\text{HFS}}(\boldsymbol{\mu}\mathbf{p}) = E_F + \Delta E_{\text{QED}} + \Delta E^{2\gamma}$$
$$= 182.443 + 1.354(7) + E_F\left(\underline{1.01958(13)}\Delta_Z{}^{\boldsymbol{\mu}\mathbf{p}} + \underline{1.01656(4)}\Delta_{\text{recoil}}{}^{\boldsymbol{\mu}\mathbf{p}} + \underline{1.00402}\Delta_{\text{pol}}{}^{\boldsymbol{\mu}\mathbf{p}}\right) \quad [\text{meV}]$$

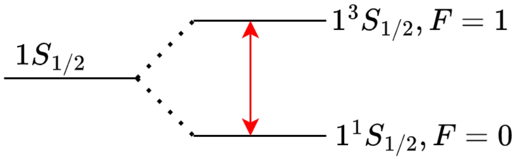
eH

E_{exp}=1420405.7517667(9) kHz

L. Essen et al. Nature (1971)

$$E^{\text{th}}_{\text{HFS}}(\mathbf{H}) = 1418840.082(9)+1613.024(3)+E^{\mathbf{H}}_F\left(\underline{1.01558(13)}\Delta_Z{}^{\mathbf{H}}+\underline{0.99807(13)}\Delta_{\text{recoil}}{}^{\mathbf{H}}+\underline{1.00002}\Delta_{\text{pol}}{}^{\mathbf{H}}\right) \quad [\text{kHz}]$$

Motivation



$$E^{\mathrm{th}}_{\mathrm{HFS}}(\mu\mathrm{p}) = E_F + \Delta E_{\mathrm{QED}} + \Delta E^{2\gamma}$$
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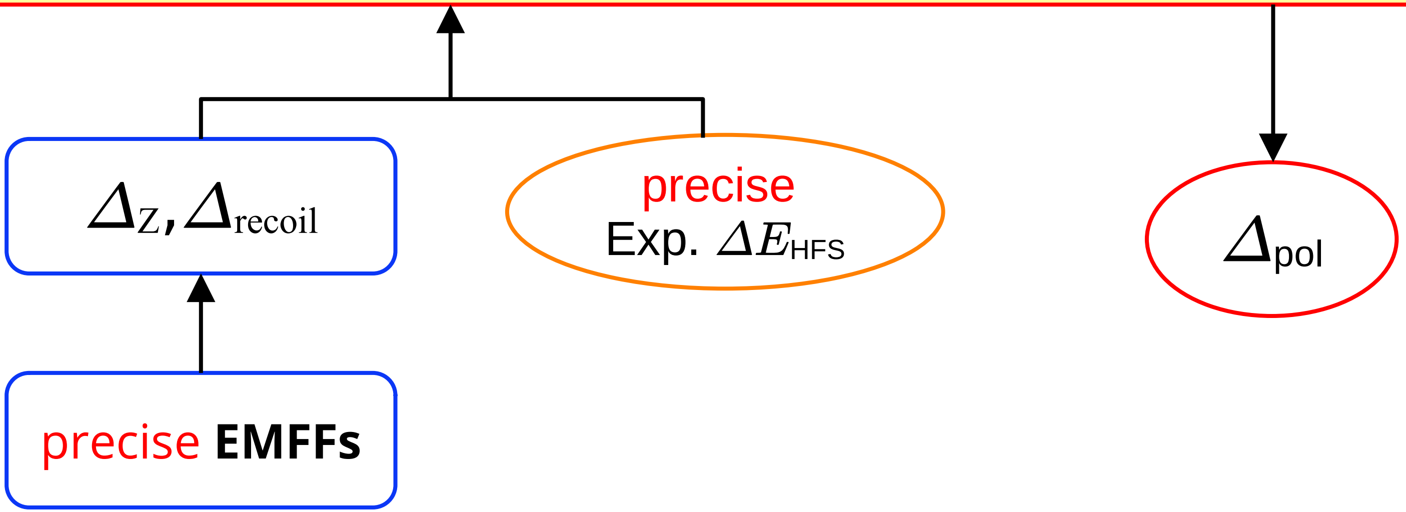


The EMFFs of the Proton

- Definition: Nucleon matrix elements of the EM vector current J_μ^{em}

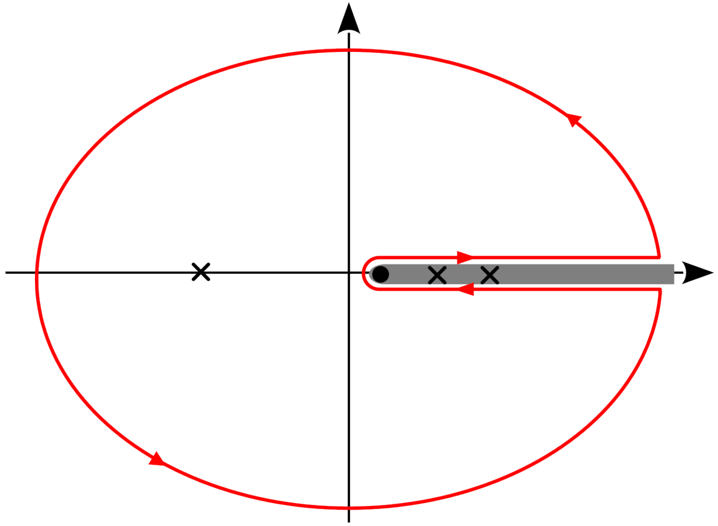
$$\langle N(p')|J_\mu^{\text{em}}|N(p)\rangle = \bar{u}(p')\left[F_1(t)\gamma_\mu + i\frac{F_2(t)}{2m}\sigma_{\mu\nu}q^\nu\right]u(p)$$

- F_1 = Dirac form factor, F_2 = Pauli form factor
- Normalization: $F_1^p(0) = 1, F_1^n(0) = 0, F_2^p(0) = \kappa_p, F_2^n(0) = \kappa_n$
- four-momentum transfer $t \equiv q^2 = (p' - p)^2 \equiv -Q^2$
- isospin basis: $F_i^S = (F_i^p + F_i^n)/2, \quad F_i^V = (F_i^p - F_i^n)/2, i = 1, 2$
- Sachs form factors: $G_E = F_1 + t/(4m^2)F_2, G_M = F_1 + F_2$
- Nucleon radii: $F(t) = F(0)(1 + t\langle r^2\rangle/6 + \cdots)$
- Model-independent approach: Dispersion relation

Dispersion Relations

- Unsubtracted dispersion relations for $F_i(t)$ $(i = 1, 2)$

$$F_i(t) = \frac{1}{\pi} \int_{t_0}^{\infty} dt' \frac{\mathrm{Im}\ F_i(t')}{t' - t}$$



☞ convergence proven in perturbation theory
[S. D. Drell et al. 1965]

☞ analyticity: the form factors have **cuts** ($[t_i, \infty[$) and also **poles** (t_j) in the positive real-t axis.

☞ unitarity: cuts → multi-meson continua; poles → vector mesons → spectral functions $\mathbf{Im}\ \boldsymbol{F_i}$

Dispersion Relations

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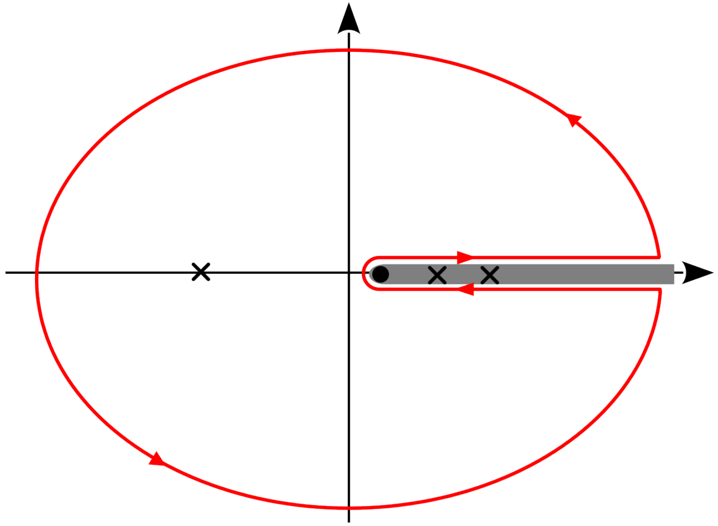

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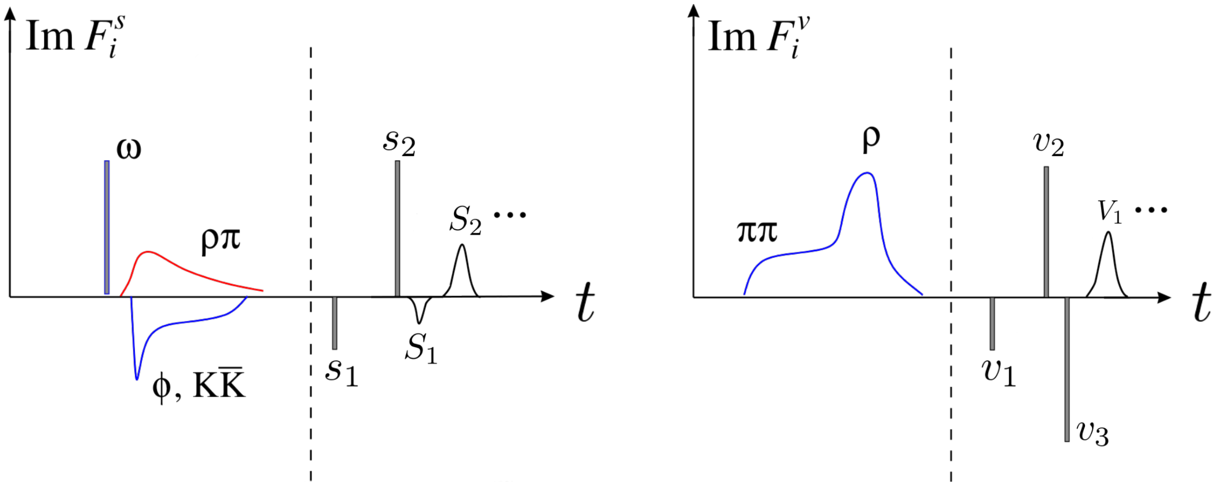
☞ unitarity: cuts → multi-meson continua; poles → vector mesons → spectral functions **Im** $\boldsymbol{F_i}$

- Connect data from small to large & positive to negative momentum transfer
- Encode perturbative (asymptotic behaviors) and non-perturbative (vector meson couplings) physics

Spectral Functions

- Below 1 GeV
  - ☞ I=0: $\rho\pi, K\bar{K}, \omega(a_i), \phi(b_i)$
  - ☞ I=1: $\pi\pi$
- Above 1 GeV
  - ☞ I=0: effective vector mesons w/o width $(m_{S_n}, \Gamma_{S_n}, a^i_{S_n})$
  - ☞ I=1: effective vector mesons w/o width $(m_{V_n}, \Gamma_{V_n}, a^i_{V_n})$



- $\pi\pi$ given by unitarity [Frazer and Fulco 1960]
- $\rho\pi, K\bar{K}$ also constrained [MMSO 1997, HRM 1999]

Experimental Inputs

Experimental data					Theoretical Constraints
Region	Observables	Souce	t GeV^2	number	
spacelike $t<0$	$d\sigma/d\Omega$	MAMI	0.00384-0.977	1422	$F_1^p(0)=1$
		PRad	0.000215-0.058	71	$F_1^n(0)=0$
	$\mu_p G_E^p/G_M^p$	JLAP	1.18-8.49	16	$F_2^p(0)=\kappa_p$
	$\mu_n G_E^n/G_M^n$	world	1.58-3.41	4	$F_2^n(0)=\kappa_n$
	G_E^n	world	0.14-3.41	29	$\int_{t_0}^{\infty} \mathrm{Im}\, F_1(t)dt=0$
	G_M^n	world	0.071-10.0	49	$\int_{t_0}^{\infty} \mathrm{Im}\, F_2(t)dt=0$
timelike $t>0$	$\lvert G_{\text{eff}}^p\rvert$	world	3.52-20.25	153	$\int_{t_0}^{\infty} \mathrm{Im}\, F_2(t)\boldsymbol{t}dt=0$
	$\lvert G_{\text{eff}}^n\rvert$	world	3.52-9.49	32	
	$\lvert G_E^p/G_M^p\rvert$	BaBar	3.52-9.0	6	$\langle r_n^2\rangle = -0.105^{+0.005}_{-0.006}$ fm^2
	$d\sigma/d\Omega$	BESIII	3.52-3.8	10	

Normalizations

Asymptotic behavior from pQCD

[Lepage and Brodsky 1980]

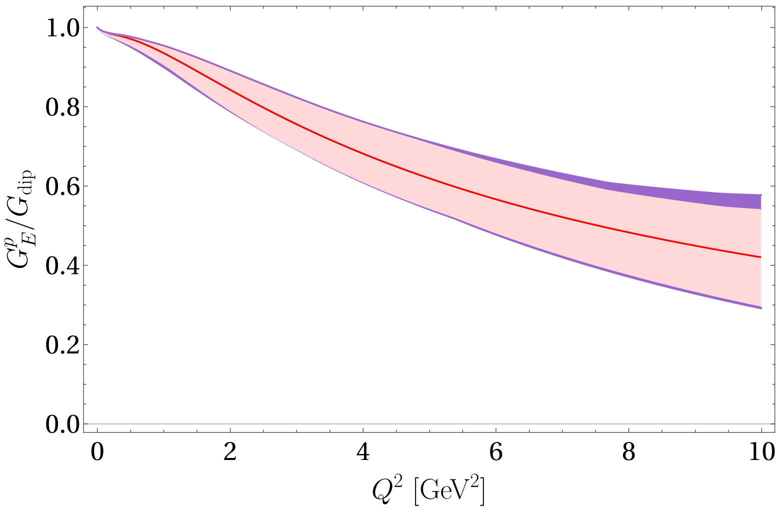
Filin et al. 2020

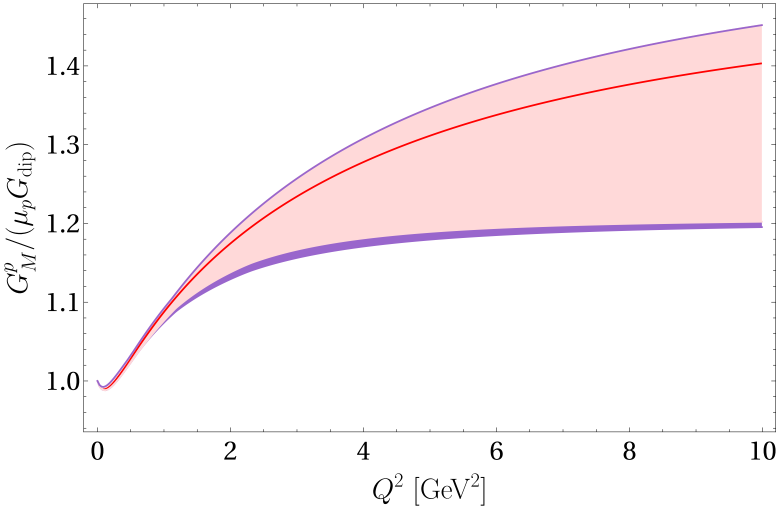
Fitted DR EMFFs of the Nucleon

YHL, HWH and UGM, PLB 2021,EPJA 2021,PRL 2022

#data=1792,

#fitpara.=4+3×(N_s+N_v)+4×(N_S+N_V)+31+2−11





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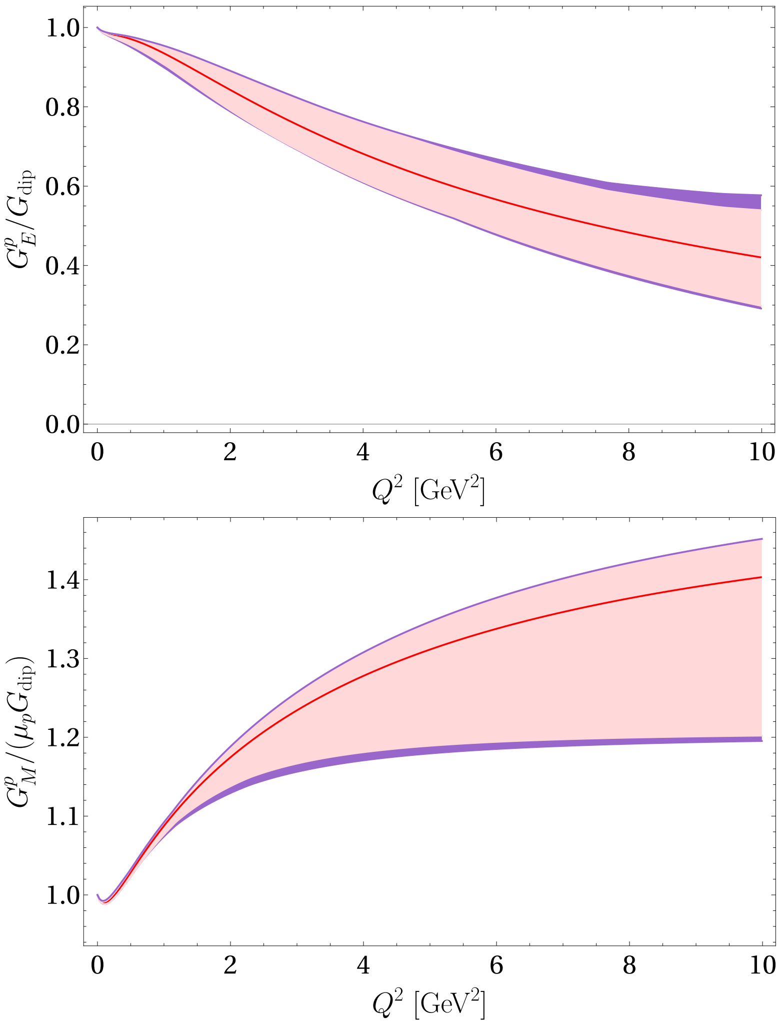
The best fit: w,ϕ 3+5 o Γ 3+3 w Γ norm.

χ^2/dof=1.238



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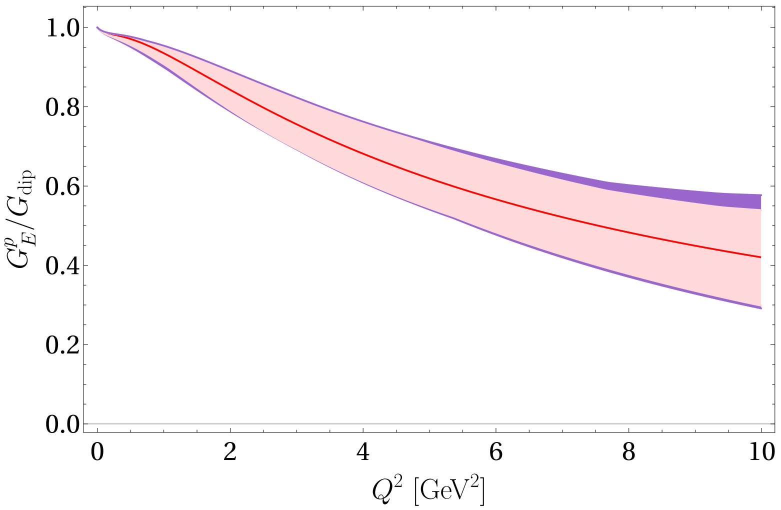
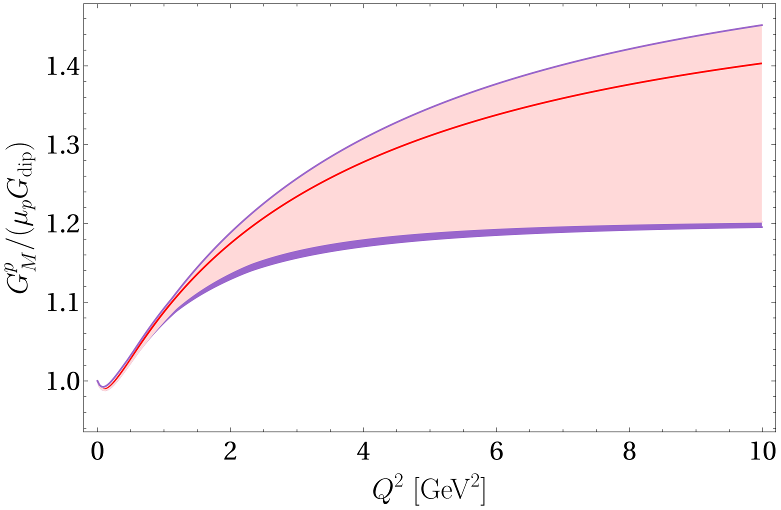
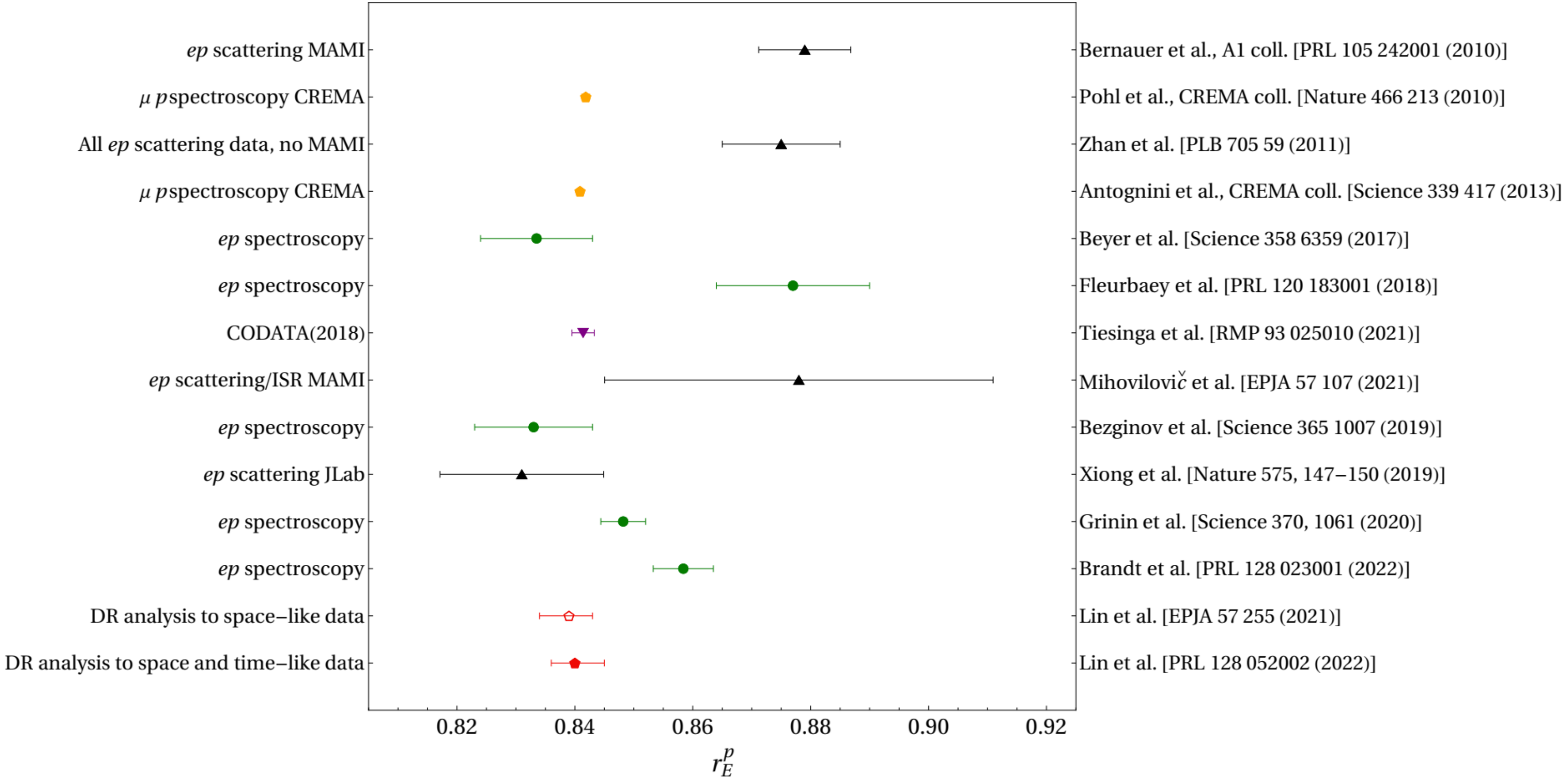
- Errors of data → bootstrap → Stat. error
- Configuration → Variations → Sys. error





Proton charge radius

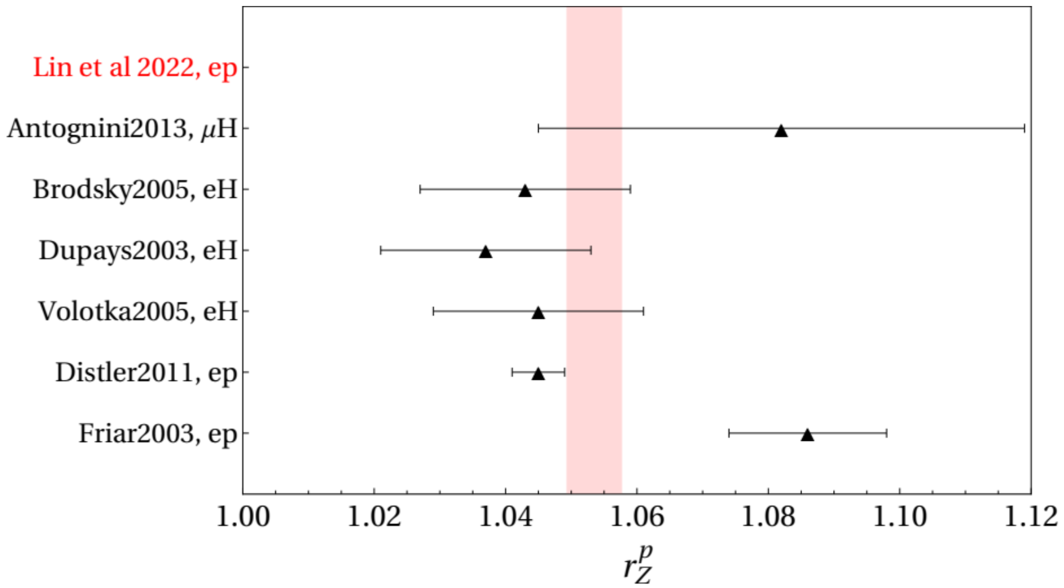
$$r_E^p = 0.840^{+0.003}_{-0.002}{}^{+0.002}_{-0.002}\ \text{fm},\ r_M^p = 0.849^{+0.003}_{-0.003}{}^{+0.001}_{-0.004}\ \text{fm}$$



Improved Zemach (Δ_Z) and Recoil (Δ_{recoil}) Corrections

- Zemach radius

$$r_Z = 1.054^{+0.003+0.000}_{-0.002-0.001}\ \text{fm}$$



- Recoil corrections

Δ_{recoil}	$\mu H(\times 10^{-6})$	$eH(\times 10^{-8})$
This work	$837.6^{+2.8}_{-1.0}$	$526.9^{+1.7}_{-0.4}$
[O. Tomalak 2017]	844(7)	532.8(4.9)

Summary

- Dispersion theory provides a model-independent tool to analyze the nucleon EMFFs
- Simultaneous description of space- and time-like data
- Always a small proton charge radius (0.84 fm), agrees with recent experiments
- Improve the determinations of the Zemach (Δ_Z) and recoil (Δ_{recoil}) corrections to HFS

  ↪ Bridge the proton polarizability effects to the experimental HFS

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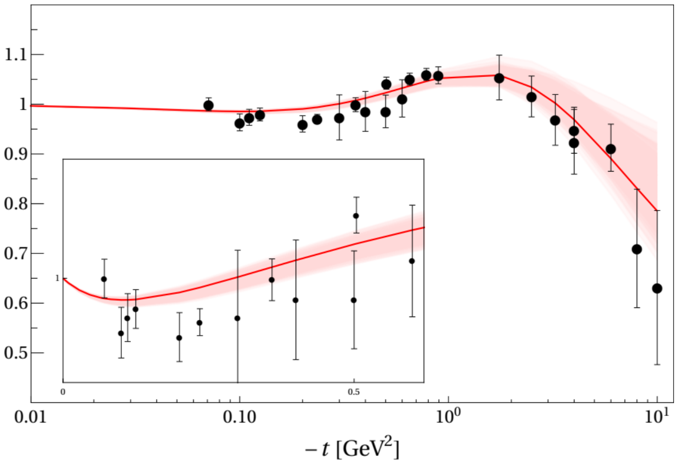
  ↪Bridge the proton polarizability effects to the experimental HFS

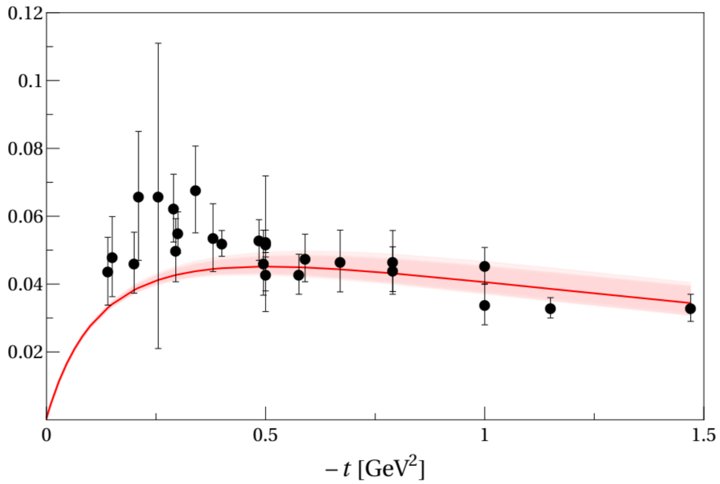
Thanks for your attention!

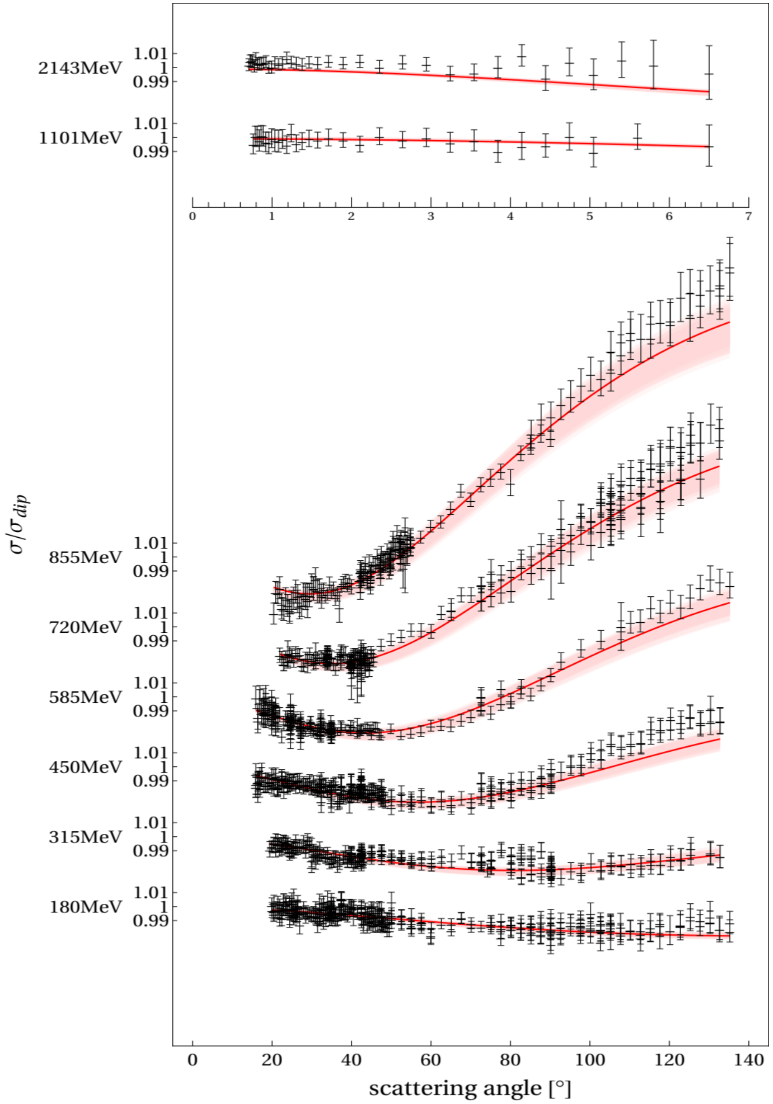
Back up

$G_M^n/(\mu_n G_{\rm dip})$



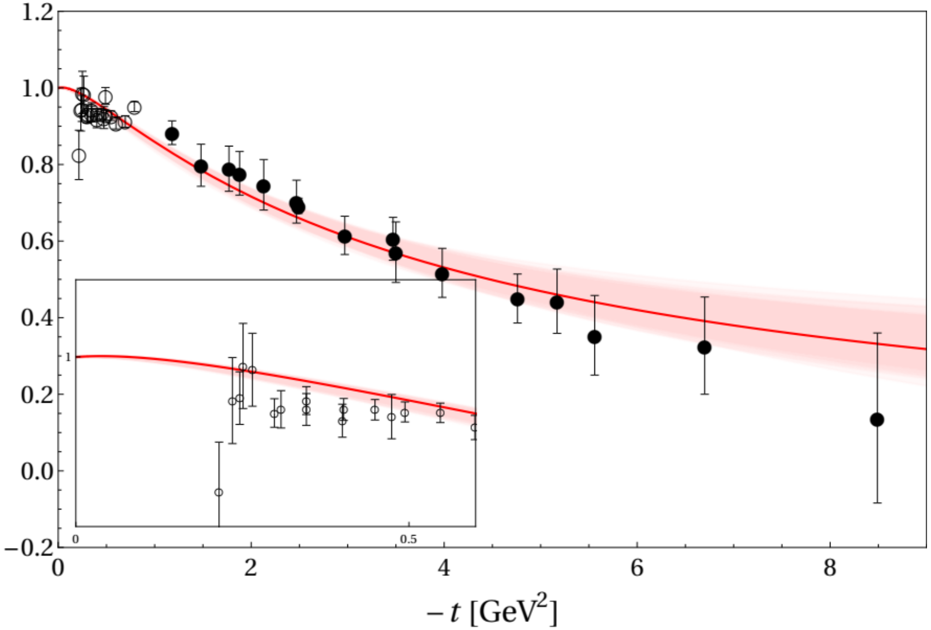
G_E^n





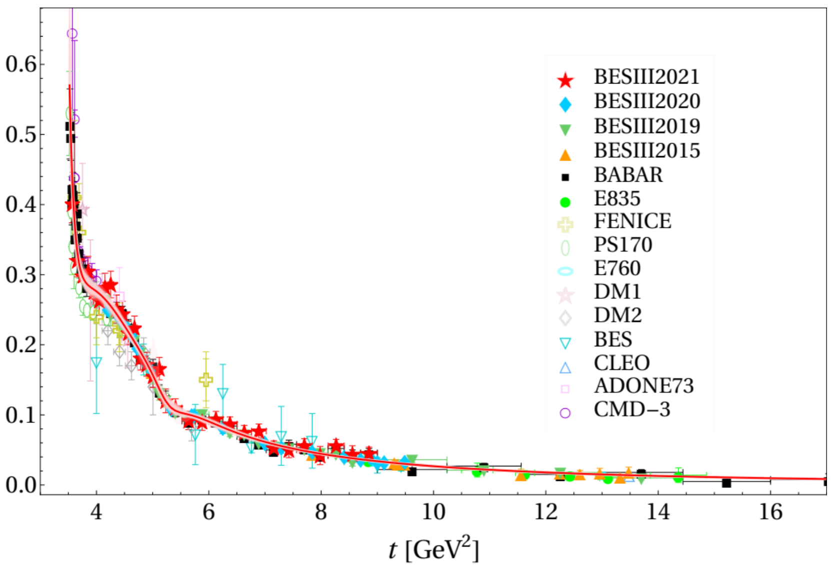
Back up

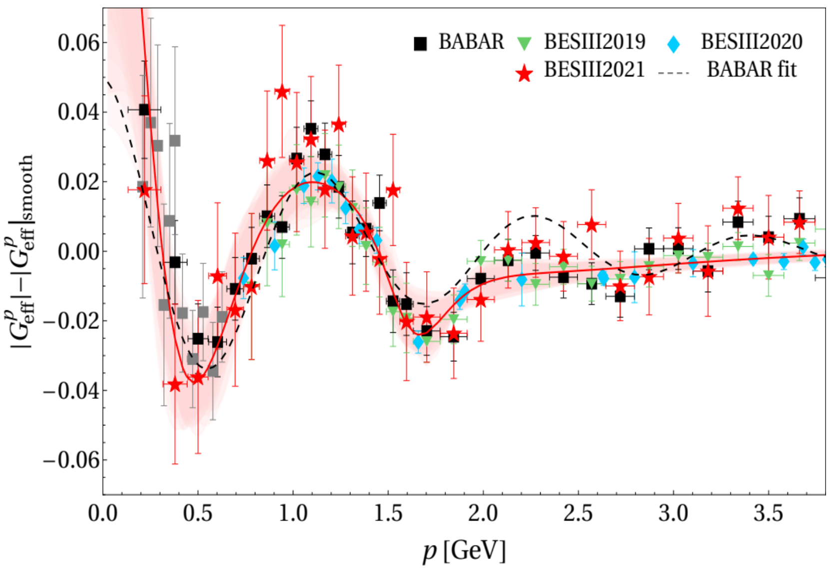
$$\mu_p G_E^p / G_M^p$$



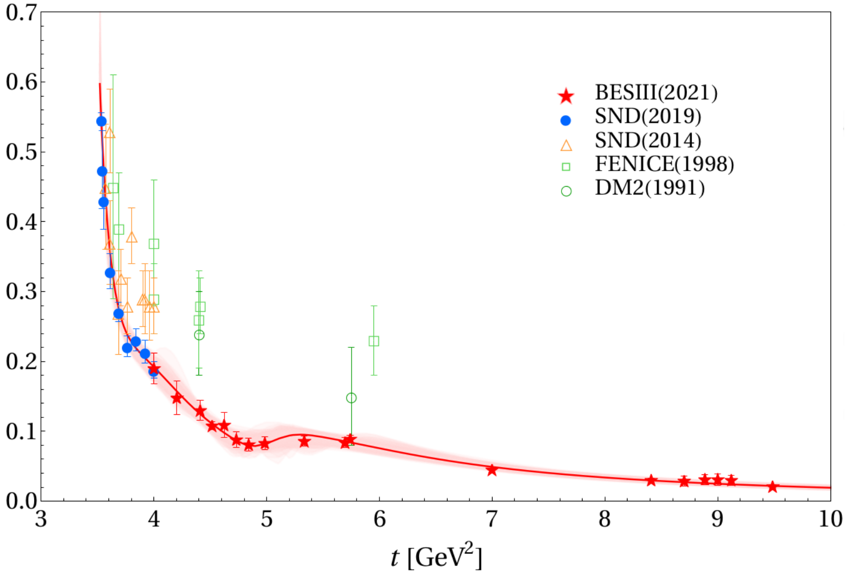
Empty symbols not fitted

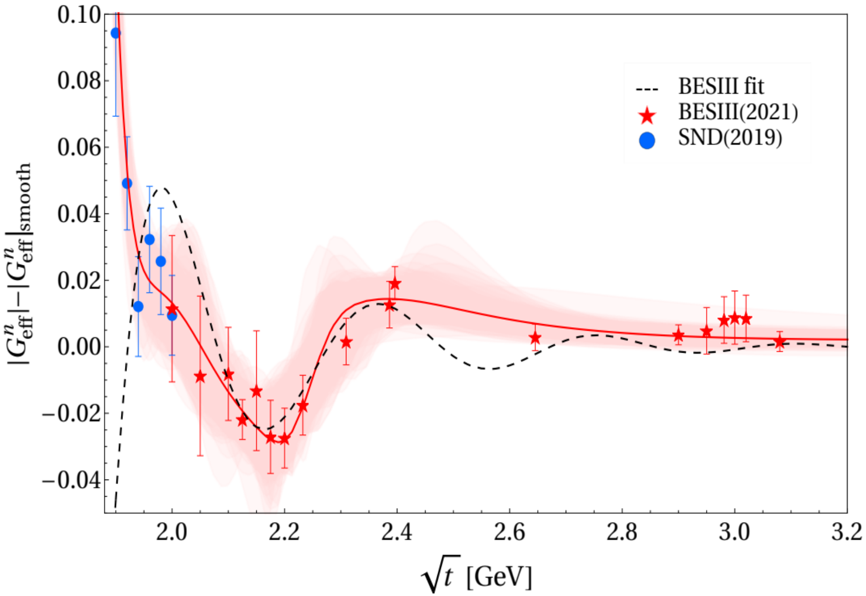
Back up


BESIII2021
BESIII2020
BESIII2019
BESIII2015
BABAR
E835
FENICE
PS170
E760
DM1
DM2
BES
CLEO
ADONE73
CMD−3
0.6
0.5
0.4
0.3
0.2
0.1
0.0
4
6
8
10
12
14
16
t [GeV²]


BABAR
BESIII2019
BESIII2020
BESIII2021
BABAR fit
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
p [GeV]
|G^p_eff|−|G^p_eff|smooth

Back up


BESIII(2021)
SND(2019)
SND(2014)
FENICE(1998)
DM2(1991)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
3
4
5
6
7
8
9
10
t [GeV^2]


BESIII fit
BESIII(2021)
SND(2019)
0.10
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
$|G_{\rm eff}^n|-|G_{\rm eff}^n|_{\rm smooth}$
2.0
2.2
2.4
2.6
2.8
3.0
3.2
$\sqrt{t}$ [GeV]