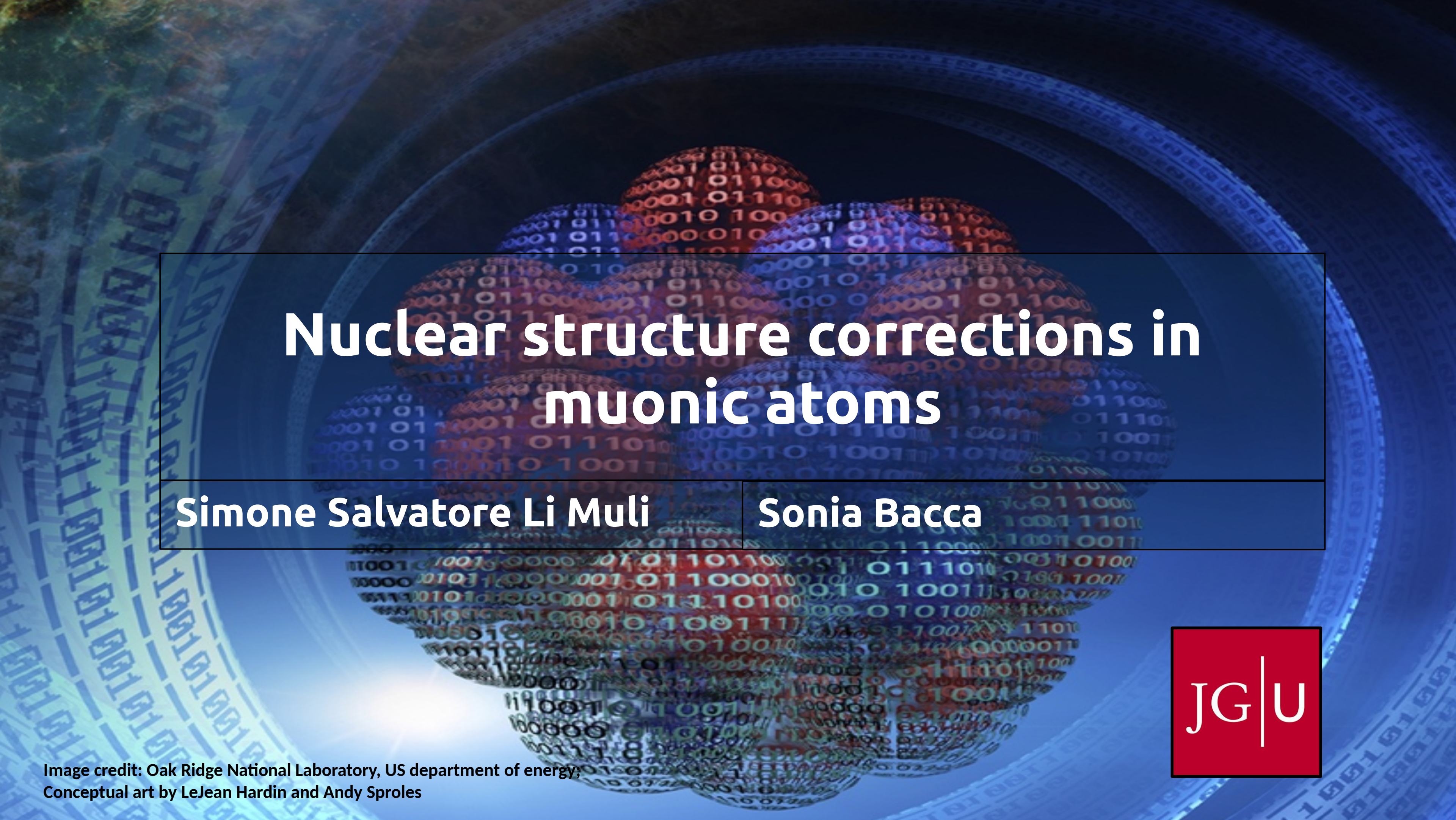


Nuclear structure corrections in muonic atoms

Simone Salvatore Li Muli
Sonia Bacca

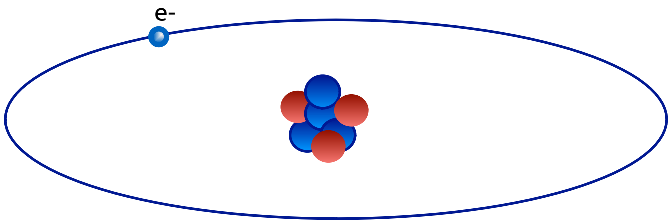
JG|U

Image credit: Oak Ridge National Laboratory, US department of energy;
Conceptual art by LeJean Hardin and Andy Sproles

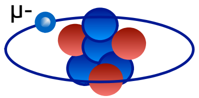
Muonic atoms

Hydrogen-like systems

Ordinary atoms



Muonic atoms



The muon is more sensitive to the nucleus

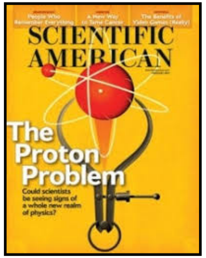
Excellent precision probe for the nucleus

Experimental program at **PSI** of the **CREMA** collaboration

Muonic Hydrogen
- Pohl et al., Nature (2010)
- Antognini et al., Science (2013)

Muonic Deuterium
- Pohl et al., Science (2016)

Muonic Helium isotopes
- Krauth et al., Nature (2021)
- Schuhmann et al., Arxiv (2023)



A matter of precision

$$\delta_{\mathrm{LS}} = \delta_{\mathrm{QED+NR}} + \delta_{\mathrm{FS}}^{(4)} \times r_c^2 + \delta_{\mathrm{TPE}}^{(5)} + \delta_{\mathrm{3PE}}^{(6)} + \ldots$$

For the muonic Helium-4 ion

$$\begin{aligned}
\delta_{\mathrm{QED+NR}} &= +1,668.489(14)\ \mathrm{meV} \\
\delta_{\mathrm{FS}}^{(4)} &= -106.220(8)\ \mathrm{meV\ fm}^{-2} \\
\delta_{\mathrm{TPE}}^{(5)} &= +9.340(250)\ \mathrm{meV} \\
\delta_{\mathrm{3PE}}^{(6)} &= -0.150(150)\ \mathrm{meV}
\end{aligned}$$

→ $r_c = 1.67824(13)_{\mathrm{ex}}(82)_{\mathrm{th}}\ \mathrm{fm}$

J. J. Krauth et. al. Nature 589,527 (2021)

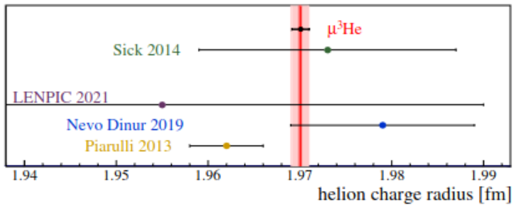
A matter of precision

$$\delta_{\rm LS} = \delta_{\rm QED+NR} + \delta^{(4)}_{\rm FS} \times r_c^2 + \delta^{(5)}_{\rm TPE} + \delta^{(6)}_{\rm 3PE} + \ldots$$

For the muonic Helium-3 ion

$$\begin{aligned} \delta_{\rm QED+NR} &= +1,644.348(8)\ \text{meV} \\ \delta^{(4)}_{\rm FS} &= -103.383\ \text{meV fm}^{-2} \\ \delta^{(5)}_{\rm TPE} &= +15.499(378)\ \text{meV} \\ \delta^{(6)}_{\rm 3PE} &= -0.214(214)\ \text{meV} \\ \delta^{(5)}_{\rm HO} &= -0.667(25)\ \text{meV} \end{aligned}$$

$$r_c = 1.97007(12)_{\rm ex}(93)_{\rm th}\ \text{fm}$$

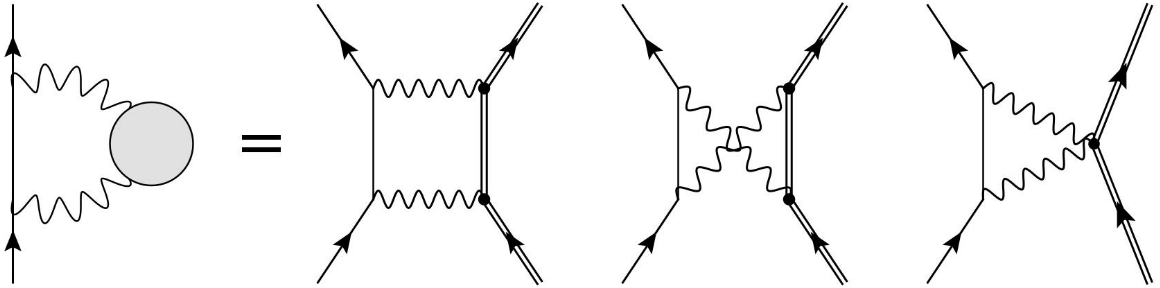


K. Schuhmann et. al. Arxiv 2305.11679 (2023)

Outline

Theory

Evaluation of the NS effects

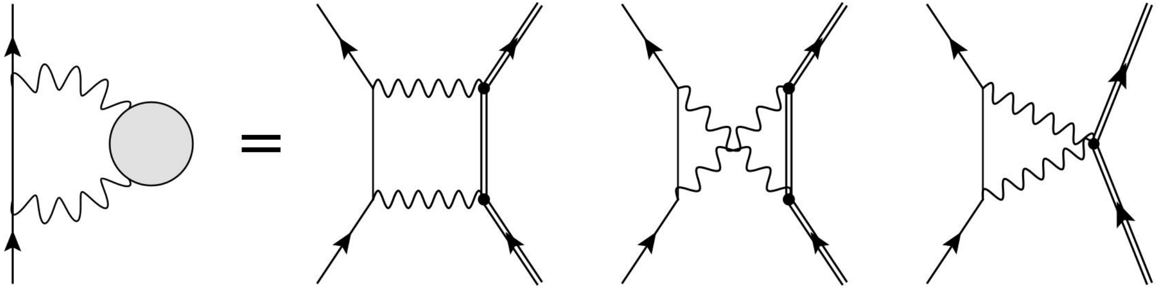


$$\Delta E_{nl} = -8\alpha^2 m\ \phi_{nL}^2 \int \frac{d^3q}{4\pi} \left\{ \sum_{N\neq 0} |\langle N|\,\rho_{\rm ch}(\mathbf{q})\,|0\rangle|^2\ \frac{(2E_q+\omega_N)}{q^4 E_q\left[(E_q+\omega_N)^2 - m^2\right]} \right.$$

$$+ \sum_{N\neq 0} |\langle 0|\,\mathbf{J}_\perp(\mathbf{q})\,|N\rangle|^2 \left[\frac{q^2}{4m^2}\ \frac{(2E_q+\omega_N)}{q^4 E_q\left[(E_q+\omega_N)^2 - m^2\right]} - \frac{1}{4m^2q^3}\ \frac{\omega_N + 2q}{(\omega_N+q)^2} \right]$$

$$\left. + B_\perp(\mathbf{q})\ \frac{1}{8m^2q^2} \left(\frac{1}{q} - \frac{1}{E_q} \right) \right\}$$

Evaluation of the NS effects

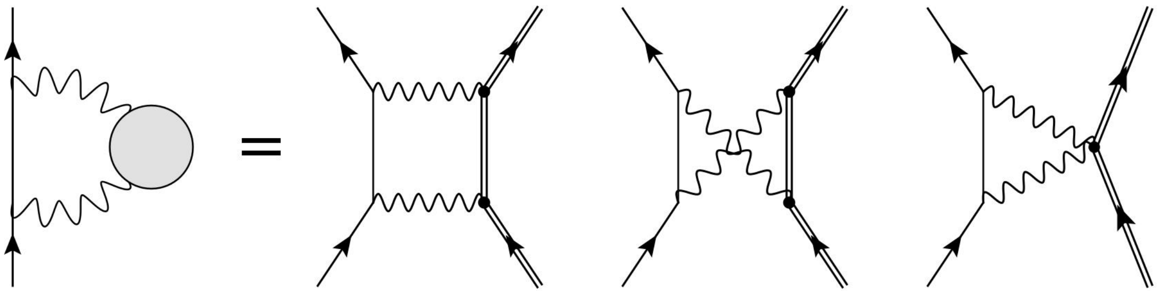


$$\Delta E_{nl} = -8\alpha^2 m\ \phi^2_{nL} \int \frac{d^3q}{4\pi} \Bigg\{ \sum_{N\neq 0} |\langle N|\,\rho_{\text{ch}}(\mathbf{q})\,|0\rangle|^2 \frac{(2E_q+\omega_N)}{q^4 E_q\left[(E_q+\omega_N)^2 - m^2\right]}$$

$$+ \sum_{N\neq 0} |\langle 0|\,\mathbf{J}_\perp(\mathbf{q})\,|N\rangle|^2 \left[\frac{q^2}{4m^2}\,\frac{(2E_q+\omega_N)}{q^4 E_q\left[(E_q+\omega_N)^2-m^2\right]} - \frac{1}{4m^2q^3}\,\frac{\omega_N+2q}{(\omega_N+q)^2}\right]$$

$$+ B_\perp(\mathbf{q})\,\frac{1}{8m^2q^2}\left(\frac{1}{q} - \frac{1}{E_q}\right)\Bigg\}$$

Evaluation of the NS effects

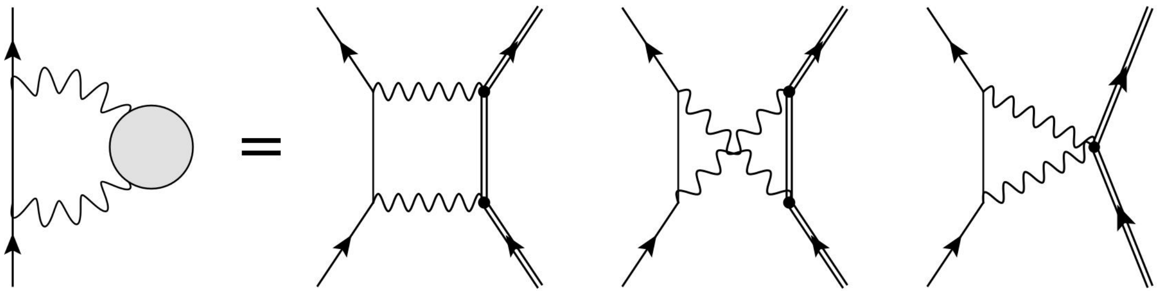


$$\Delta E_{nl} = -8\alpha^2 m\, \phi_{nL}^2 \Bigg\{ \sum_{N\neq 0} \int d^3x\, d^3y\;\; \langle 0|\, \rho^\dagger_{\rm ch}(\mathbf{y})\,|N\rangle\, \langle N|\, \rho_{\rm ch}(\mathbf{x})\,|0\rangle\;\; \mathrm{I_N}(z)$$

$$+ \sum_{N\neq 0} \int d^3x\, d^3y\, \langle 0|\, J_i^\dagger(\mathbf{y})\,|N\rangle\, \langle N|\, J_j(\mathbf{x})\,|0\rangle \left[\delta_{ij} J_N(z) + z^i z^j \bar{J}_N(z)\right]$$

$$+ \int d^3x\, d^3y\; B^{ij}(\mathbf{x},\mathbf{y})\; \frac{1}{2}\left[\delta_{ij} K(z) + z_i z_j \bar{K}(z)\right] \Bigg\}\, .$$

Evaluation of the NS effects



$$\Delta E_{nl} = -8\alpha^2 m\, \phi_{nL}^2 \Bigg\{ \sum_{N\neq 0} \int d^3x\, d^3y\, \langle 0|\, \rho_{\rm ch}^\dagger(\mathbf{y})\,|N\rangle \langle N|\, \rho_{\rm ch}(\mathbf{x})\,|0\rangle\; \mathrm{I_N}(z)$$

$$+ \sum_{N\neq 0} \int d^3x\, d^3y\, \langle 0|\, J_i^\dagger(\mathbf{y})\,|N\rangle \langle N|\, J_j(\mathbf{x})\,|0\rangle \left[\delta_{ij} J_N(z) + z^i z^j \bar{J}_N(z)\right]$$

$$+ \int d^3x\, d^3y\, B^{ij}(\mathbf{x},\mathbf{y})\, \frac{1}{2}\left[\delta_{ij} K(z) + z_i z_j \bar{K}(z)\right] \Bigg\} .$$

Ab-initio strategy

Our strategy is to build models for the operators from first principles

$$H_N$$

$$\{\mathbf{D}, \mathbf{Q}, \ldots\}$$

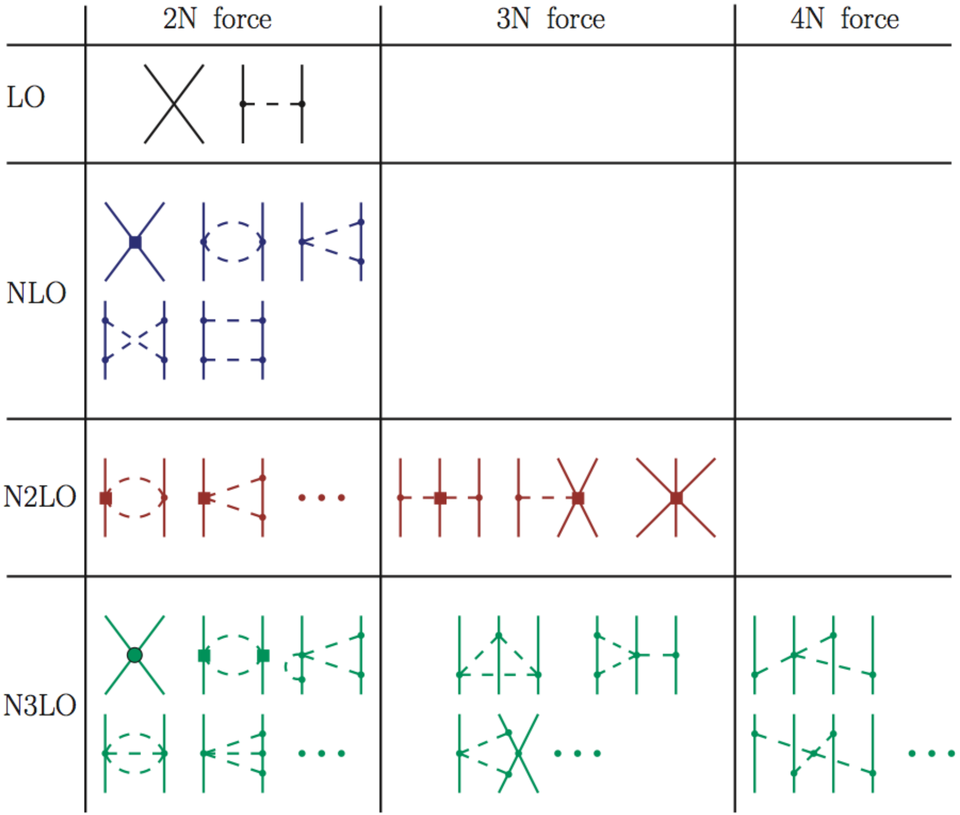
Solve the time-independent Schrödinger equation for the nuclear states

$$\widehat{H}_{\mathrm{N}} \left|\mathrm{N}\right\rangle = E_{\mathrm{N}} \left|N\right\rangle$$

And calculate the relevant matrix elements with controlled approximations.

$$\left\langle N\right| \mathbf{D} \left|0\right\rangle$$

Nuclear Hamiltonians



- Epelbaum E, Krebs H, Meißner UG. PRL 115, 122301 (2015)
- Ekström A, et al. PRL 110, 192502 (2013)
  Ekström A, et al. PRC 91, 05130(R) (2015)
- Entem DR, Machleidt R, Nosyk Y, PRC 96, 024004 (2017)
- Piarulli M, et al. PRC 91, 024003 (2015)
- Gezerlis A, et al. PRC 90 , 054323 (2014)
  Lynn JE, et. al. PRC 96, 054007 (2017)

Bayesian uncertainty quantification

Chiral perturbation theory is an expansion in $Q = \frac{m_\pi}{\Lambda_\chi} \sim 0.3$

It is then reasonable to assume that a similar expansion holds also for the observable calculated with the theory

$$X = \sum_{n=0}^{k} D_n + \sum_{n=k+1}^{\infty} D_n$$

$$= X_{\text{ref}} \left[\sum_{n=0}^{k} c_n \, Q^n + \sum_{n=k+1}^{\infty} c_n \, Q^n \right]$$

One assumes that the expansion coefficients follow the same underlying distribution and uses the calculated coefficients to learn about the distribution.

Outline

Results

NS corrections in muonic Helium

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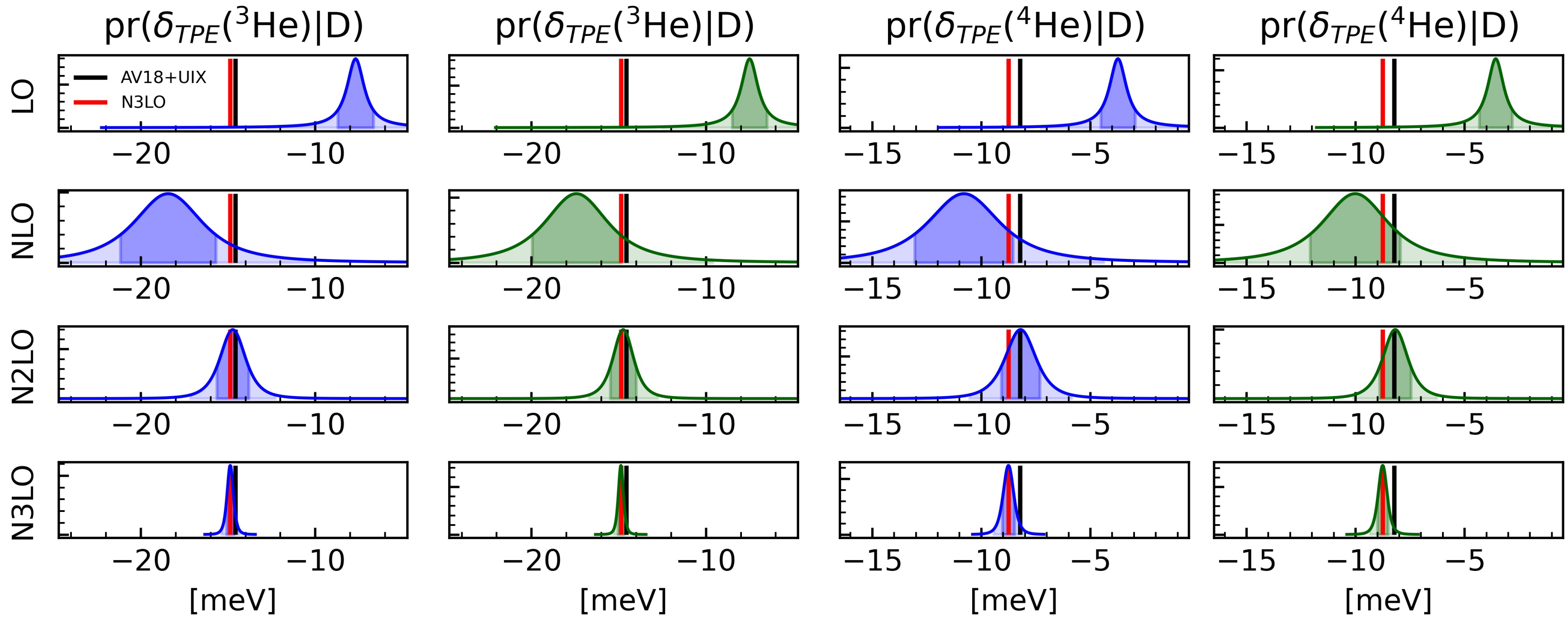
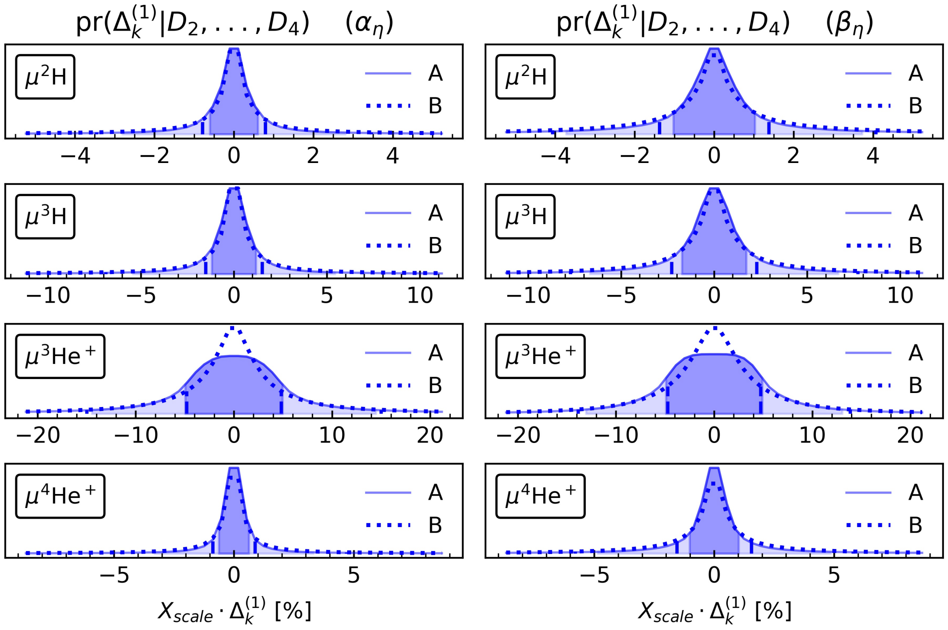


C. Ji, et al. J. Phys. G: Nucl. Part. Phys. 45 (2018)

Structure functions uncertainty

$$\Delta E_{nl}^{\rm NR} = -8\alpha^2\ \phi_{nl}^2 \sum_{N\neq 0} \int d^3x\ d^3y\ \langle 0|\,\rho_{\rm ch}^\dagger(\mathbf{y})\,|N\rangle\,\langle N|\,\rho_{\rm ch}(\mathbf{x})\,|0\rangle\ \ {\rm I}_{\rm NR}(z)$$
$$= -8\alpha^2\ \phi_{nl}^2 \sum_{N\neq 0} \int d^3x\ d^3y\ \langle 0|\,\rho_{\rm ch}^\dagger(\mathbf{y})\,|N\rangle\,\langle N|\,\rho_{\rm ch}(\mathbf{x})\,|0\rangle\ \left({\rm I}_{\rm NR}^{(2)}(z) + {\rm I}_{\rm NR}^{(3)}(z) + {\rm I}_{\rm NR}^{(4)}(z) + ..\right)$$



S.S.LM, et al. 2022 J. Phys. G: Nucl. Part. Phys. 49 105101

	μ^2H	μ^3H	μ^3He$^+$	μ^4He$^+$
[1]	0.4%	1.3%	1.1%	0.8%
This work	0.8%	1.5%	4.8%	0.9%

[1] C. Ji, et al. J. Phys. G: Nucl. Part. Phys. 45 (2018)

Uncertainty budget in TPE

	μ^3He^+			μ^4He^+		
	δ_{pol} [%]	δ_{Zem} [%]	δ_{TPE} [%]	δ_{pol} [%]	δ_{Zem} [%]	δ_{TPE} [%]
Numerical **[1]**	0.4	0.1	0.1	0.4	0.3	0.4
Numerical	0.1	0.2	0.1	0.4	0.3	0.2
Nuclear model **[1]**	0.7	1.8	1.5	3.9	4.6	4.4
Nuclear model (N2LO)	4.8	6.9	6.2	14.5	9.4	11
Nuclear model (N3LO)	1.6	1.6	1.4	4.1	2.8	3
η-expansion **[1]**	1.1	–	0.3	0.8	–	0.2
η-expansion	4.8	–	1.4	0.9	–	0.2
$Z\alpha$ **[1]**	1.5	0.0	0.4	1.5	0.0	0.4
$Z\alpha$	–	–	–	–	–	–
ISB **[1]**	1.8	0.2	0.5	2.2	0.5	0.5
Nucleon-size **[1]**	1.2	1.3	0.9	2.7	2.0	1.2
Relativistic **[1]**	0.4	–	0.1	0.1	–	0.0
Coulomb **[1]**	3.0	–	0.9	0.4	–	0.1
Total **[1]**	4.2	2.2	2.1	5.5	5.1	4.6
Total (N2LO)	7.6	6.9	6.5	10.6	9.6	10.9
Total (N3LO)	6.3	2.1	2.4	5.3	3.5	3.5

S.S.LM, et al. In preparation for 2023

[1] C. Ji, et al. J. Phys. G: Nucl. Part. Phys. 45 (2018)

Helion charge radius

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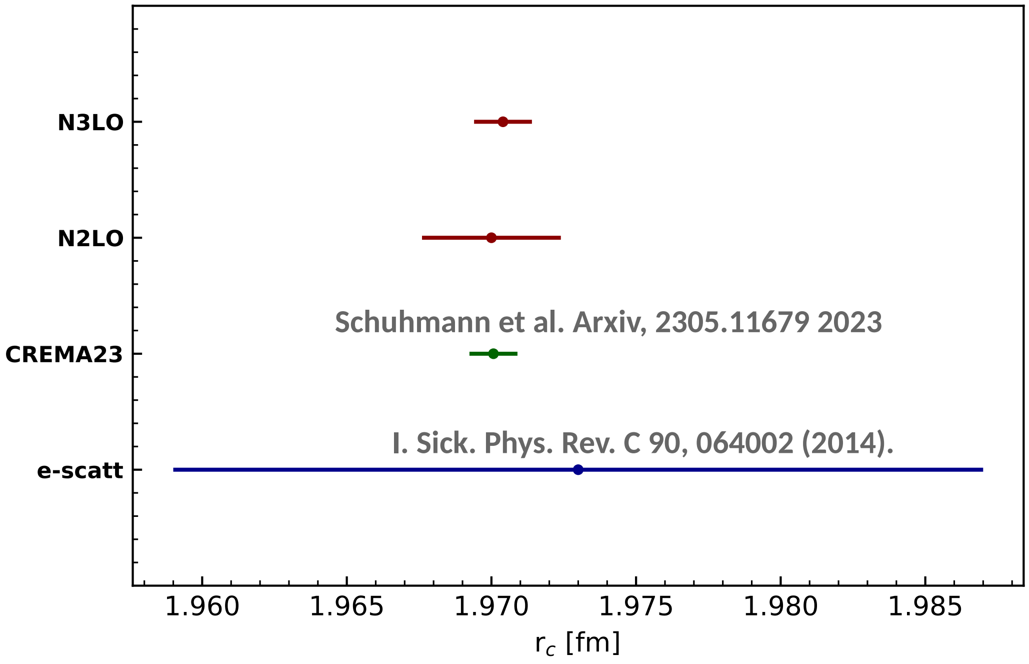

Alpha particle charge radius

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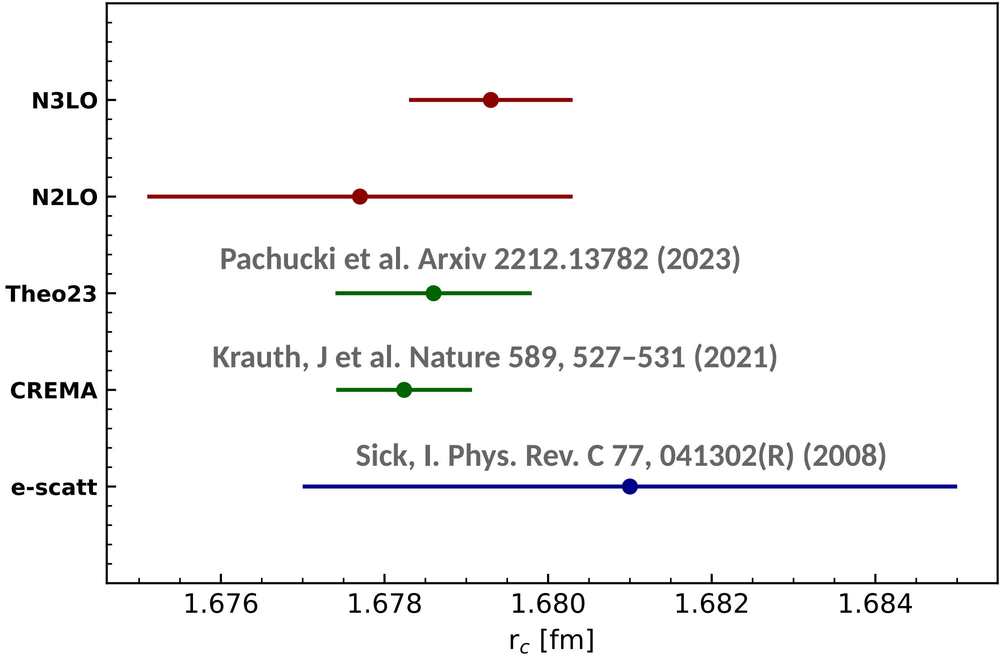

Conclusions

- The nuclear model uncertainty of the two-photon-exchange correction to the Lamb-shift in muonic Helium ions is larger at N2LO and at a similar level at N3LO compared to the uncertainty quoted in the past.

- Our analysis of the η -expansion uncertainty suggests that in muonic He-3 the error band might be larger than previously expected.

- The calculation of the 3PE will improve the precision of the nuclear charge radii of the Helium isotopes by ~10%.

- These techniques can be applied to the study of other observable, like the Hyperfine splitting in muonic Deuterium and muonic He-3.

Backup

Uncertainties in muonic Helium

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	μ^3He^+			μ^4He^+		
	δ_{pol} [%]	δ_{Zem} [%]	δ_{TPE} [%]	δ_{pol} [%]	δ_{Zem} [%]	δ_{TPE} [%]
Numerical (2019)	0.4	0.1	0.1	0.4	0.3	0.4
Numerical	0.1	0.2	0.1	0.4	0.3	0.2
Nuclear model (2019)	0.7	1.8	1.5	3.9	4.6	4.4
Nuclear model (N2LO)	4.8	6.9	6.2	14.5	9.4	11
Nuclear model (N3LO)	1.6	1.6	1.4	4.1	2.8	3
η-expansion (2019)	1.1		0.3	0.8		0.2
η-expansion	4.8		1.4	0.9		0.2

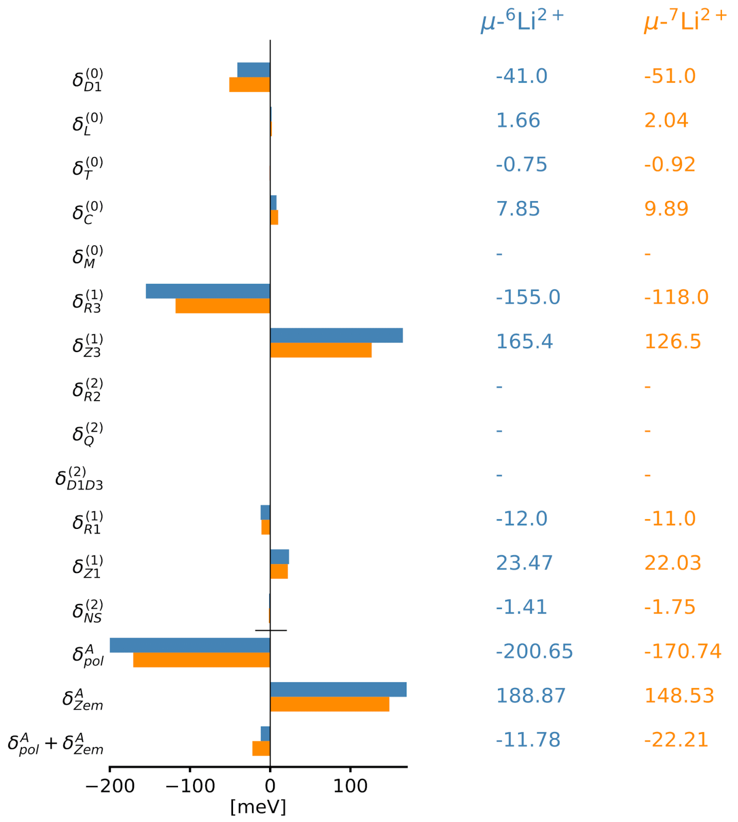
Preliminary works on Li atoms

$$\delta_{\rm TPE} = \delta^{(0)}_{\rm D1} + \delta^{(0)}_{\rm C} + \delta^{(1)}_{\rm Z1} + \delta^{(1)}_{\rm Z3} + \delta^{(2)}_{\rm NS} + \delta^{(2)}_{\rm Q} + \dots$$

$\delta_{\rm TPE}$	Ref. **(1)** **(meV)**	Ref. **(2)** **(meV)**
μ-^{6}Li^{2+}	-11.8(3)	-15(4)
μ-^{7}Li^{2+}	-22.2(4)	-21(4)

(1) S.L., A.Poggialini, S.Bacca, SciPost Phys. Proc. 3, 028 (2020)

(2) Drake et al, Phys. Rev. A **32**, 713 (1985)



Priors

$$O = O_{ref}\left[c_0 + c_1 Q + c_2 Q^2 + ...\right]$$

Priors	$\mathrm{pr}(\eta)$
α_η	$\frac{1}{\lambda}\,\exp(\frac{\eta}{\lambda})$
β_η	$\beta(a,b)$

Priors	$\mathrm{pr}(c_i\|\bar{c})$	$\mathrm{pr}(\bar{c})$
A	$\frac{1}{2\bar{c}}\theta(\bar{c} - \|c_i\|)$	$\frac{1}{\ln(\bar{c}_>/\bar{c}_<)\bar{c}}\theta(\bar{c}-\bar{c}_<)\theta(\bar{c}_> - \bar{c})$
B	$\frac{1}{\sqrt{2\pi}\bar{c}}\exp\left(-\frac{c_i^2}{2\bar{c}^2}\right)$	$\frac{1}{\ln(\bar{c}_>/\bar{c}_<)\bar{c}}\theta(\bar{c}-\bar{c}_<)\theta(\bar{c}_> - \bar{c})$

Evaluation of the NS effects

$$\delta_{\rm pol}^{(5)} = C \sum_{N\neq N_0} \left(\frac{1}{\omega_N}\right)^{\frac{3}{2}} \int d^3R\, d^3R'\; \rho_N^p(\mathbf{R}) \left[\eta^2 - \frac{1}{4}\eta^3 + \frac{1}{20}\eta^4 + \ldots\right] \rho_N^p(\mathbf{R}')$$

LO NLO N2LO

$$\eta = \sqrt{2m_r\omega_N}|\mathbf{R}-\mathbf{R}'| \approx \sqrt{\frac{m_r}{m_N}} \approx 0.33$$

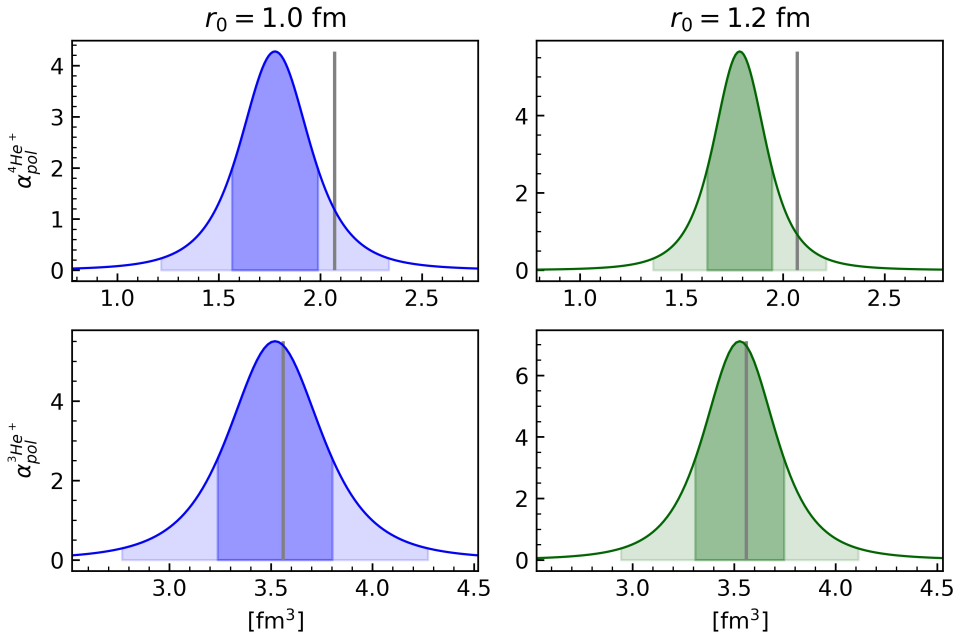
. . . Some more algebra . . .

$$\delta_{\rm pol}^{(5)} = \sum_i \left[\, C_i(Z\alpha, m_r)\; \int_0^\infty \mathcal{F}_i(\omega/m_r)\; S_{O_i}(\omega)\; d\omega \,\right]$$

Nuclear response function

NS effects in 3He+ and 4He+

$$\delta^{(5)}_{\rm pol} = -m\alpha^4 \langle \sum_a \delta^3(r_a) \rangle \, (m^3 \alpha_{\rm pol})$$



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1S-2S	$^3He^+$	$^4He^+$
[1]	48(5)kHz	28(3)kHz
This work	48(6)kHz	24(4)kHz

— [1] K. Pachucki, A.M. Moro Phys.Rev.A 75,032521(2007)

Uncertainty

Uncertainty sources

- **Numerical**
- **Nuclear model**
- **Nucleon model**
- **Truncation of EM multipoles**
- **η-expansion**
- **Expansion in (Zα)**

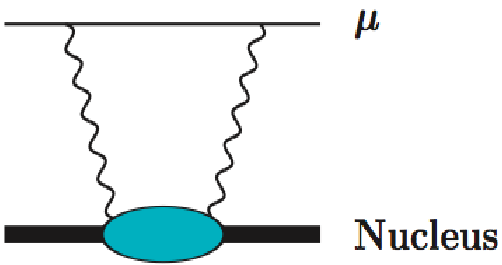
Evaluation of the TPE amplitude

- **Coulomb Distortion**

Contribution coming from intermediate interactions between lepton and nucleus during the TPE process.

We include only the correction of order $(Z\alpha)^6 log(Z\alpha)$

Since we work in the leading dipole approximation, the correction is related to the electric dipole response



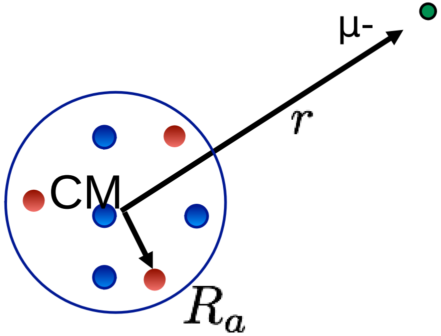
- **Relativistic effects**

They are smaller by factors $\frac{\omega_{th}}{m_r}$

Includes also first effects from electromagnetic currents. Are evaluated in the leading dipole approximation

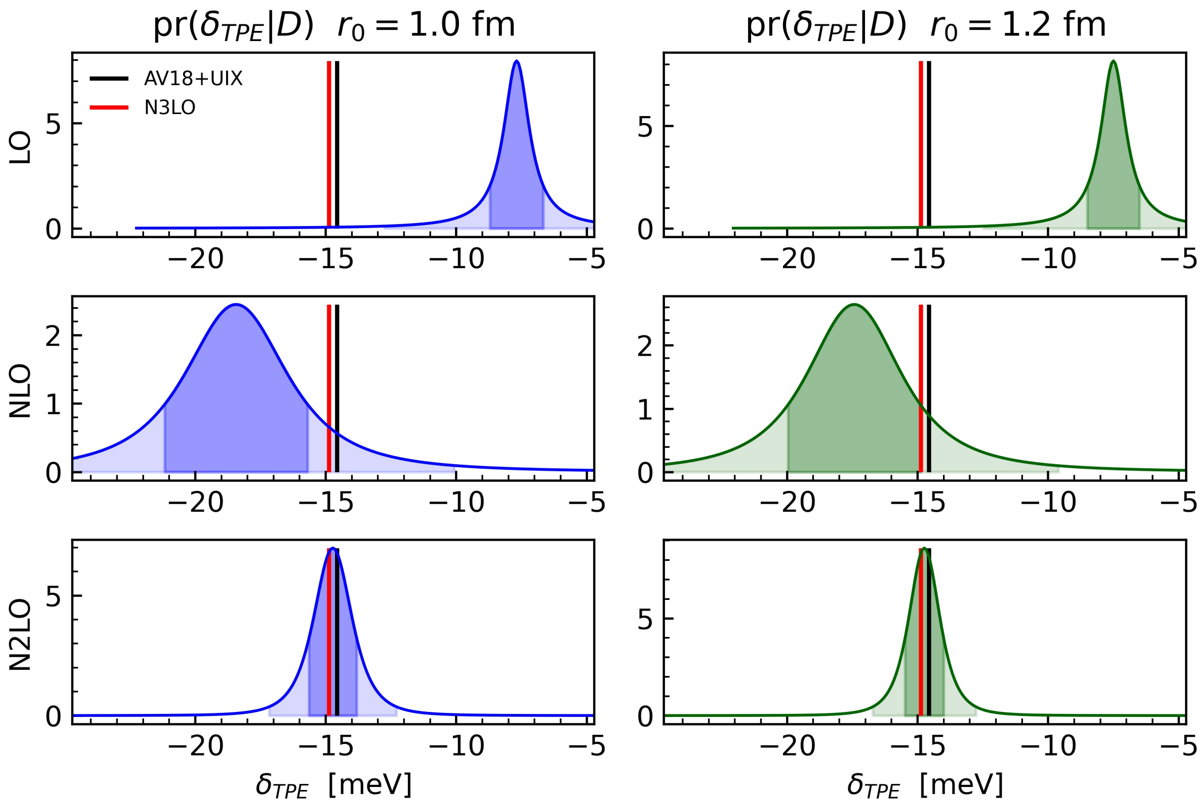
$$\delta^{(0)}_{L(T)} = \frac{2m_r^3}{9}(Z\alpha)^5 \int_{\omega_{th}}^{\infty} K_{L(T)}\left(\frac{\omega}{m_r}\right) S_{D1}(\omega)$$

- **Nucleon size effects**



TPE in He-3

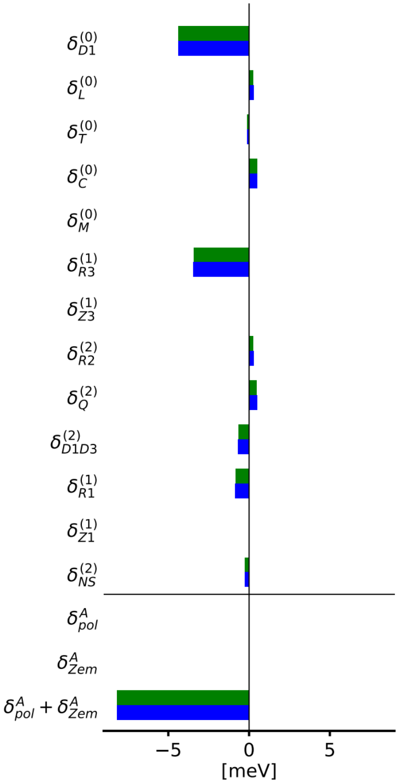
S.S. LM, et al. In preparation



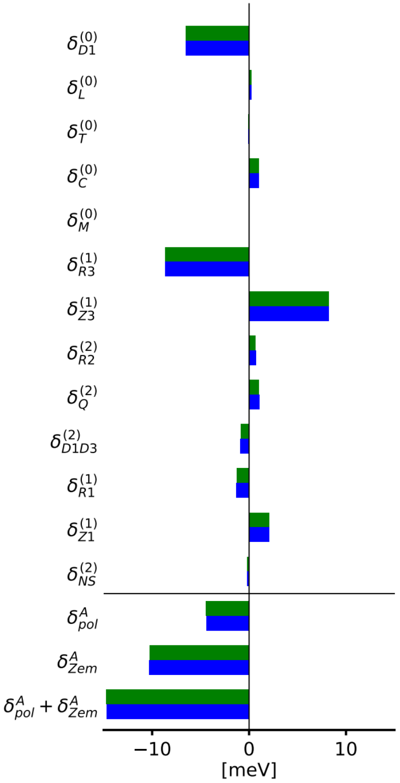
C. Ji, et al. J. Phys. G: Nucl. Part. Phys. 45 (2018)

NS corrections in μ4He+

S.S.LM, et al. In preparation for 2022



	$r_0 = 1.2$	$r_0 = 1.0$
$\delta_{D1}^{(0)}$	-4.386	-4.373
$\delta_{L}^{(0)}$	0.272	0.287
$\delta_{T}^{(0)}$	-0.124	-0.125
$\delta_{C}^{(0)}$	0.517	0.514
$\delta_{M}^{(0)}$	0.011	0.011
$\delta_{R3}^{(1)}$	-3.422	-3.477
$\delta_{Z3}^{(1)}$	-	-
$\delta_{R2}^{(2)}$	0.267	0.285
$\delta_{Q}^{(2)}$	0.484	0.505
$\delta_{D1D3}^{(2)}$	-0.668	-0.69
$\delta_{R1}^{(1)}$	-0.846	-0.856
$\delta_{Z1}^{(1)}$	-	-
$\delta_{NS}^{(2)}$	-0.272	-0.277
δ_{pol}^{A}	-	-
δ_{Zem}^{A}	-	-
$\delta_{pol}^{A} + \delta_{Zem}^{A}$	-8.174	-8.196



	$r_0 = 1.2$	$r_0 = 1.0$
$\delta_{D1}^{(0)}$	-6.562	-6.552
$\delta_{L}^{(0)}$	0.232	0.232
$\delta_{T}^{(0)}$	-0.104	-0.104
$\delta_{C}^{(0)}$	1.018	1.015
$\delta_{M}^{(0)}$	0.008	0.008
$\delta_{R3}^{(1)}$	-8.674	-8.691
$\delta_{Z3}^{(1)}$	8.223	8.255
$\delta_{R2}^{(2)}$	0.67	0.696
$\delta_{Q}^{(2)}$	1.043	1.062
$\delta_{D1D3}^{(2)}$	-0.877	-0.899
$\delta_{R1}^{(1)}$	-1.297	-1.302
$\delta_{Z1}^{(1)}$	2.054	2.057
$\delta_{NS}^{(2)}$	-0.191	-0.193
δ_{pol}^{A}	-4.458	-4.411
δ_{Zem}^{A}	-10.277	-10.312
$\delta_{pol}^{A} + \delta_{Zem}^{A}$	-14.735	-14.723