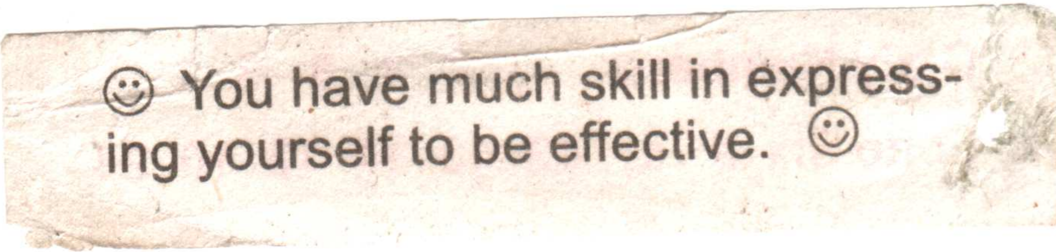
☺ You have much skill in express-
ing yourself to be effective. ☺

Universality in Resummed-Range EFT:
Three Identical Bosons with Large, Negative Effective Range

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Universality Classes, Efimov Towers and More in
Contact EFT with Anomalous Effective Range
Without LO Three Body Interaction

hg/U. van Kolck: 2308.01394

building on 2B: Habashi/Sen/Fleming/van Kolck
arXiv:2007.07360, arXiv:2012.14995, arXiv:2209.08432

1. Resummed-Range EFT for Identical Bosons

(a) Two Identical Bosons in Resummed-Range EFT

follow Habashi/Sen/Fleming/van Kolck
arXiv:2007.07360, arXiv:2012.14995, arXiv:2209.08432

If effective range $|r_0| \ll |a| \to \infty$ scattering length $\Longrightarrow$ r_0 perturbative: $\mathcal{A}_{\mathrm{LO}} = \dfrac{1}{-\frac{1}{a} - \mathrm{i}\,K}$ **"Short-Range" EFT**

If r_0 also large in magnitude, both must be non-perturbative at LO: ***"Resummed-Range EFT"***

Let $|r_0|$ set scale. $\Longrightarrow$ Momenta K in $|r_0^{-1}|$ etc. $\Longrightarrow$ Universal, i.e. depend only on dimensionless $\xi := \dfrac{2r_0}{a}$.

Wigner bound, causality: momentum-dependent BB contact interactions **only renormalisable for** $r_0 \leq 0$.

Fewster hep-th/9412050, Phillips/Cohen nucl-th/9607048, Phillips/Beane/Cohen hep-th/9706070, nucl-th/9709062,. . .

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Physical Systems?: Usually $r_0 \sim$ interaction range > 0, but fine-tuning exists.

Scattering with resonance just above $K = 0$: level repulsion $\Longrightarrow$ $k \cot \delta$ has large negative curvature

Possibly in $D_s^*(2317) \equiv (DK - D_s\eta)$ & ΞN, $\Xi\Xi$ Matuschek/Baru/Guo/Hanhart 2007:05329 Haidenbauer/Meißner/Petschauer 1511.05859

Tell Me If You Know One!

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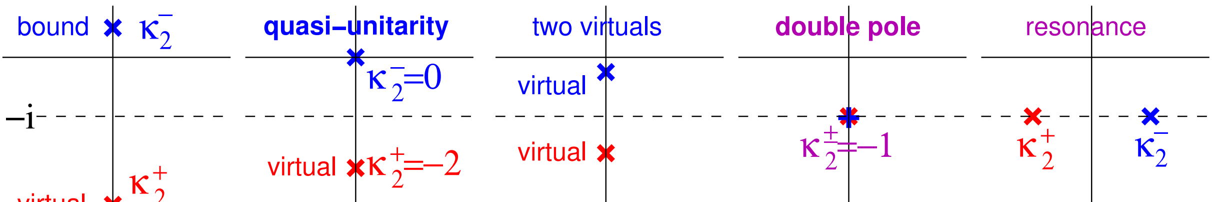
Alternative: Second-simplest $S = \dfrac{K + \mathrm{i}\,\kappa_2^-}{K - \mathrm{i}\,\kappa_2^-}\,\dfrac{K + \mathrm{i}\,\kappa_2^+}{K - \mathrm{i}\,\kappa_2^+}$: 2 poles $\mathrm{i}\,\kappa_2^\pm$ in complex-momentum plane ($B = \frac{(\kappa_2^\pm)^2}{M|r_0|^2}$).

Poles of BB propagator $\dfrac{2}{\xi - K^2 - 2\mathrm{i}K}$ from $r_0 \leq 0$, a: $\kappa_2^\pm = -\left[1 \pm \sqrt{1 - \xi(a, r_0)}\right] \xrightarrow[\text{un-scale}]{r_0 \ll a} \left(\dfrac{1}{a}, -\dfrac{2}{|r_0|}\right)$

$\xi < 0$	$\xi = 0$	$0 < \xi < 1$	$\xi = 1$	$\xi > 1$
$(r_0 < 0 < a)$	$(r_0 = 0 \vee \lvert a\rvert = \infty)$	$(a < 2r_0 < 0)$	$(a = 2r_0 < 0)$	$(2r_0 < a < 0)$

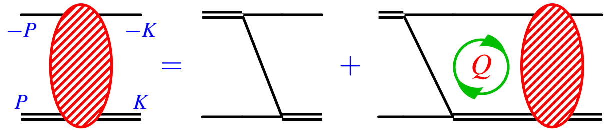


We Will Deal With These Three Cases

(b) Three Identical Bosons in Resummed-Range EFT

hg/van Kolck arXiv:2308.01394

Faddeev integral equation for half off-shell S-wave amplitude $T(\mu^2;K,Q)$, total cm energy μ^2, rescaled by $M^m|r_0^{-1}|^n$:



with $V_{S\text{-ex}}(\mu^2;P,Q) = \frac{1}{PQ}\ln\frac{P^2+Q^2+PQ-\mu^2}{Q^2+Q^2-PQ-\mu^2}$

$$T(\mu^2;K,P) = 4\pi V_{S\text{-ex}}(\mu^2;P,K) + \frac{4}{\pi}\int\limits_0^\infty \mathrm{d}Q\;Q^2\;\overbrace{\frac{V_{S\text{-ex}}(\mu^2;P,Q)}{\xi+\frac{3Q^2}{4}-\mu^2+2\sqrt{\frac{3Q^2}{4}-\mu^2}}}^{\text{kernel }\mathcal{K}(\mu^2;P,Q)}\;T(\mu^2;K,Q)$$

Scattering: B with relative momentum K on (BB) system with binding momentum $\mathrm{i}\,\kappa_2^-$: $\mu^2 = \frac{3K^2}{4}-(\kappa_2^-)^2$.

Bound State: $\det[1-\mathcal{K}(\mu=-\kappa_3^2;P,Q)]=0$ at 3B binding momentum $\mathrm{i}\,\kappa_3$.

Expectations from (BB) propagator $\dfrac{1}{\xi+\underbrace{\frac{3Q^2}{4}+\kappa_3^2}_{\text{eff. range}}+2\sqrt{\frac{3Q^2}{4}+\kappa_3^2}}$:

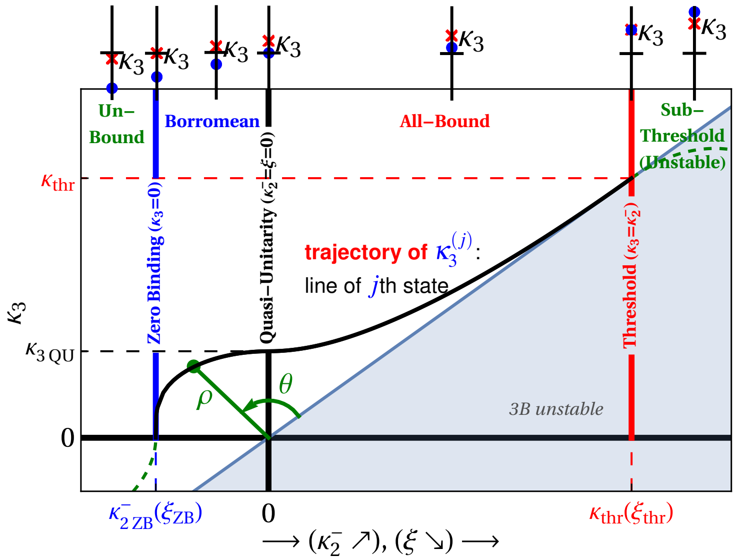
$Q \gg \kappa_3, \xi \qquad \Longrightarrow \to \frac{1}{Q^2}$ tames UV, no divergence $\Longrightarrow$ no 3BI $\Longrightarrow$ no limit cycle.

$\frac{3Q^2}{4}+\kappa_3^2 \gg \sqrt{3Q^2+4\kappa_3^2} \gtrsim 4 \qquad \Longrightarrow$ Quick convergence – can bound state be supported?

$\frac{3Q^2}{4}+\kappa_3^2 \gtrsim 4 \gtrsim \sqrt{3Q^2+4\kappa_3^2} \qquad \Longrightarrow$ All of similar size. $\Longrightarrow$ $\kappa_3 \lesssim 2$ likely, effective-range effects large.

$\frac{3Q^2}{4}+\kappa_3^2 \ll \sqrt{3Q^2+4\kappa_3^2} \lesssim 4 \qquad \Longrightarrow \to \dfrac{1}{\xi+2\sqrt{\frac{3Q^2}{4}+\kappa_3^2}}$: Effective range perturbative. $\Longrightarrow$ Efimov'ish.

(a) Expectations and Nomenclature



3B pole vs. shallow 2B pole κ_2^- in complex momentum plane

expect ground state $\kappa_3 \lesssim 2$

expect Efimov-like for $\kappa_3 \to 0$

expect Efimov-like for $\kappa_2^- \to 0 \; (\xi \to 0)$

gray: 3B unstable against B(2B)

Threshold: 3B bound state becomes stable, emerges from 2B continuum (fate before not yet clear).

Quasi-Unitarity: $\kappa_2^- = 0 = \xi = \dfrac{2r_0}{a}$, but still virtual state at $\kappa_2^+ = -2\,(|r_0^{-1}|)$

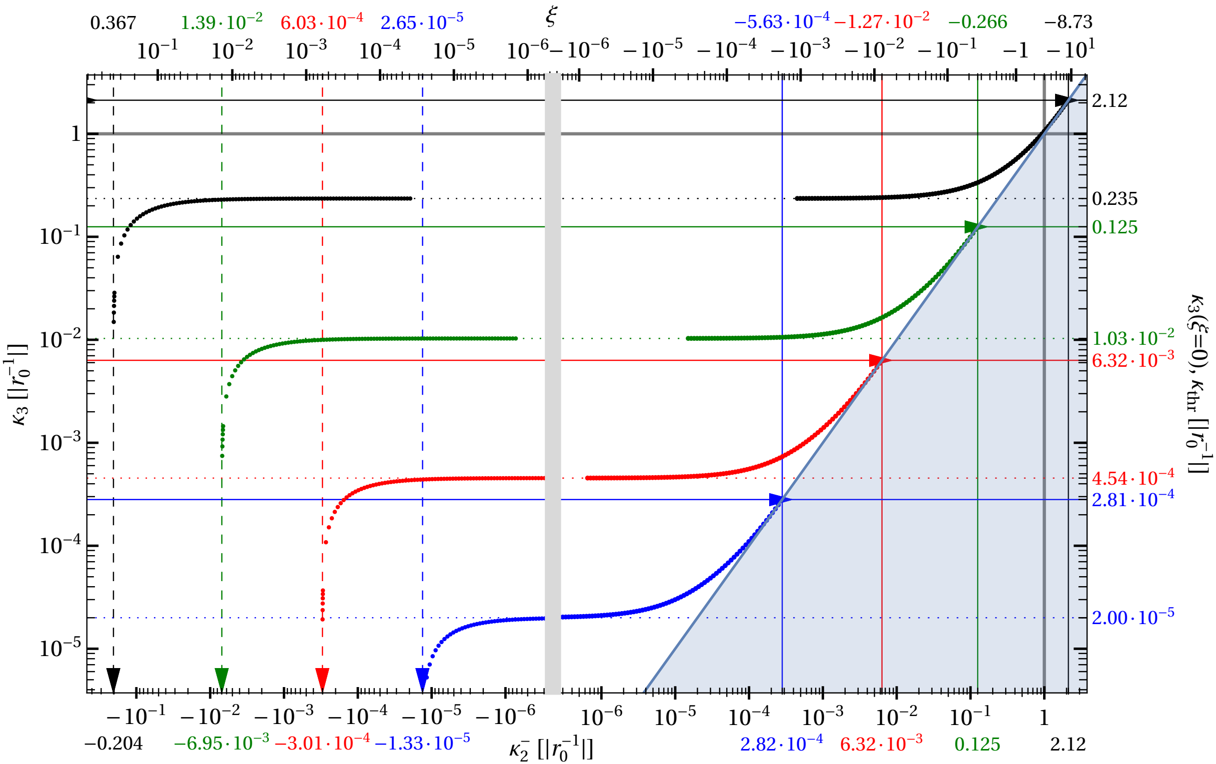
$\Longrightarrow$ 2B scale *continues to exist*, sets scale for 3B and radius of convergence.

Zero Binding: 3B becomes unbound (fate beyond not yet clear).

(b) 3B Bound States in Resummed Range EFT

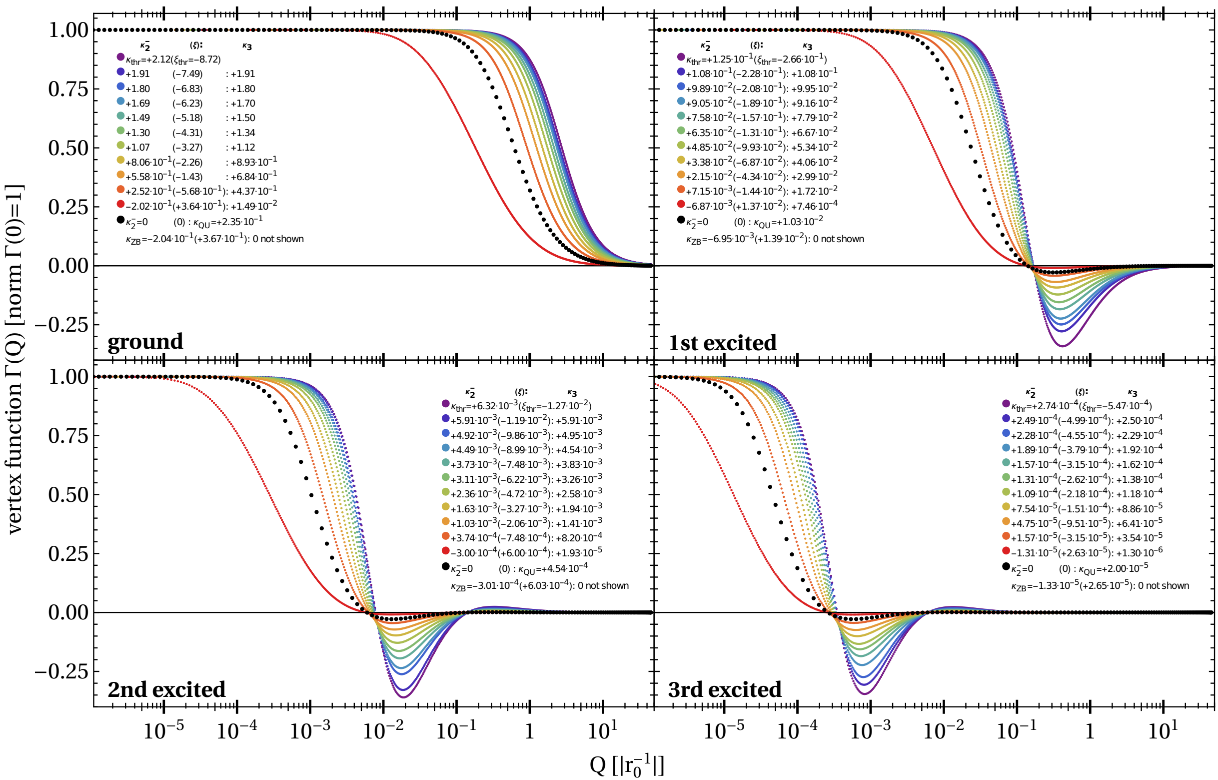
Renormalisable at LO without 3B Interaction. $\Longrightarrow$ Stable ground state, no (new) 3B parameter.

Meets expectations: $\kappa_3^{(0)} \leq 2.1 \leftrightarrow \lesssim 2$; no state for large $|\kappa_2^-|$; Efimov's Discrete Scale Invariance approximate.



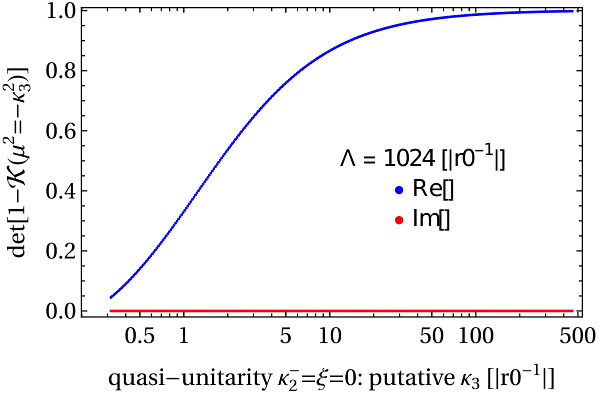
Are You Sure There Is No Deeper State? – No State Outside The Window? – No 3BI?

No node in 3B vertex (wave) function

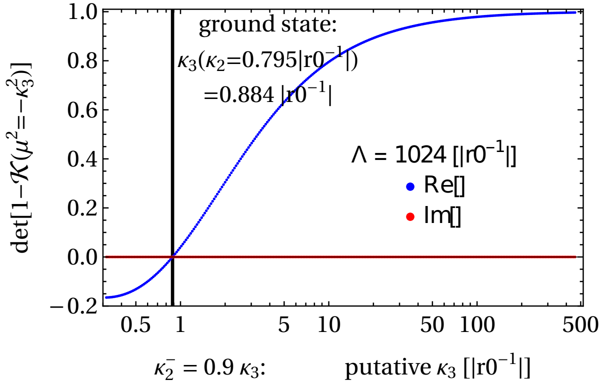


Are You Sure There Is No Deeper State? – No State Outside The Window? – No 3BI?

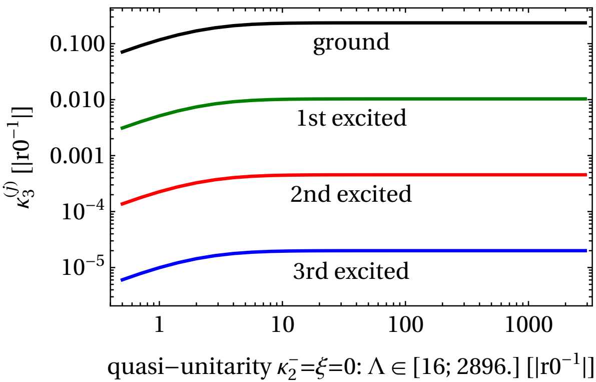
No zero in $\det[1-\text{kernel}]$ at fixed $\kappa_2^- = \xi = 0$



No zero in $\det[1-\text{kernel}]$ at floating $\kappa_2^-(\xi) = 0.9\kappa_3$



No visible effect of cutoff variation by $\times 3000 \approx (22.7)^{2.5}$



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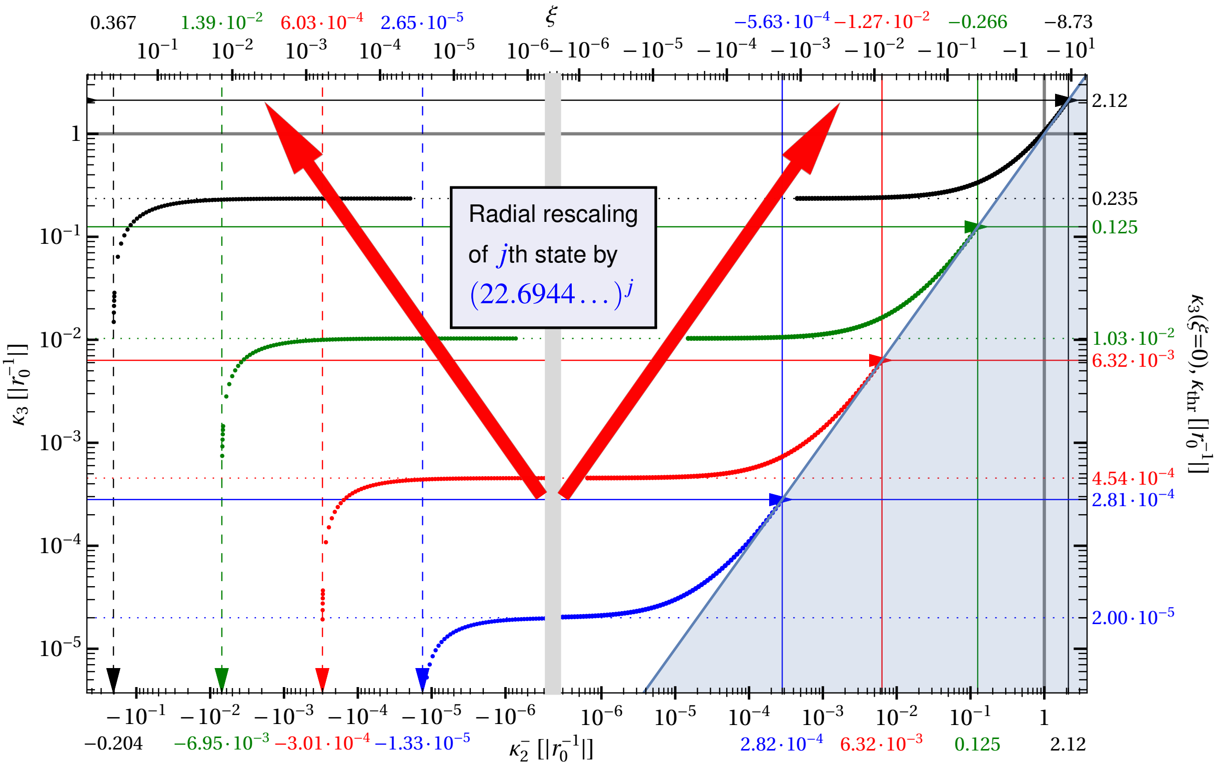
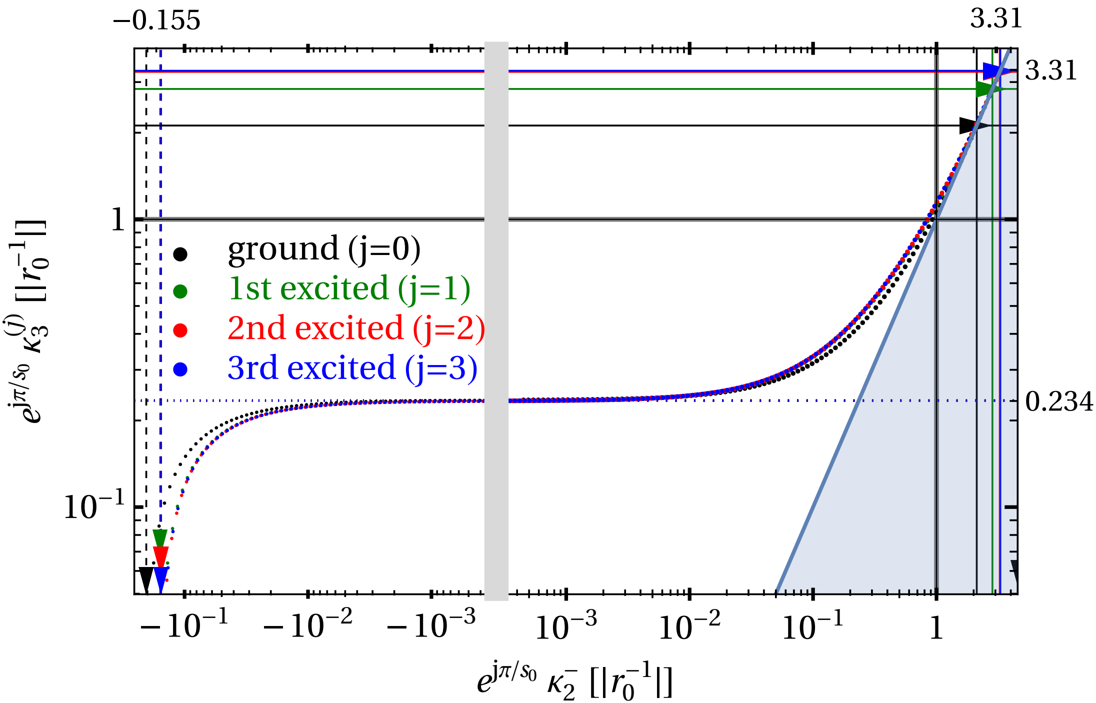


(c) Are Trajectories Self-Similar?: Lay On Top Of Each Other!

Radially Rescale jth state by Efimov's Discrete Scale Invariance factor $e^{j\pi/s_0} = (22.6944\dots)^j$, $s_0 = 1.0062\dots$.



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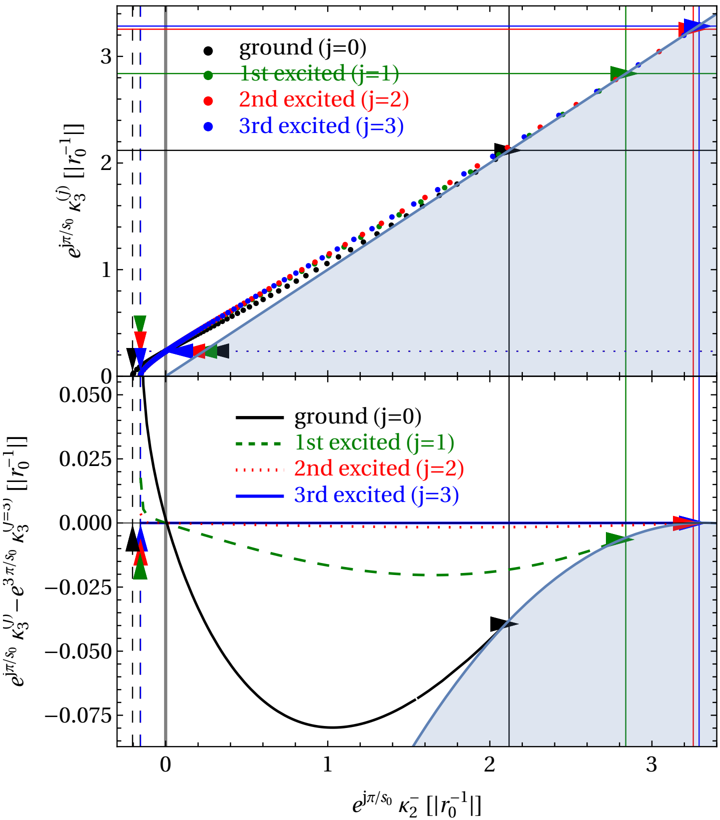


In-In plot hides size of differences

$\Longrightarrow$ lin-lin plots

Points Of Interest relative to Efimov:

threshold: ground / 1st excited state 30% / 15% *smaller*

quasi-unitarity: no visible difference since $\kappa_2^- = \xi = r_0 = 0$.

zero binding: ground / 1st excited state 30% / 10% *bigger*

Meets expectations:

$r_0 \neq 0$ has biggest effect on lowest states.

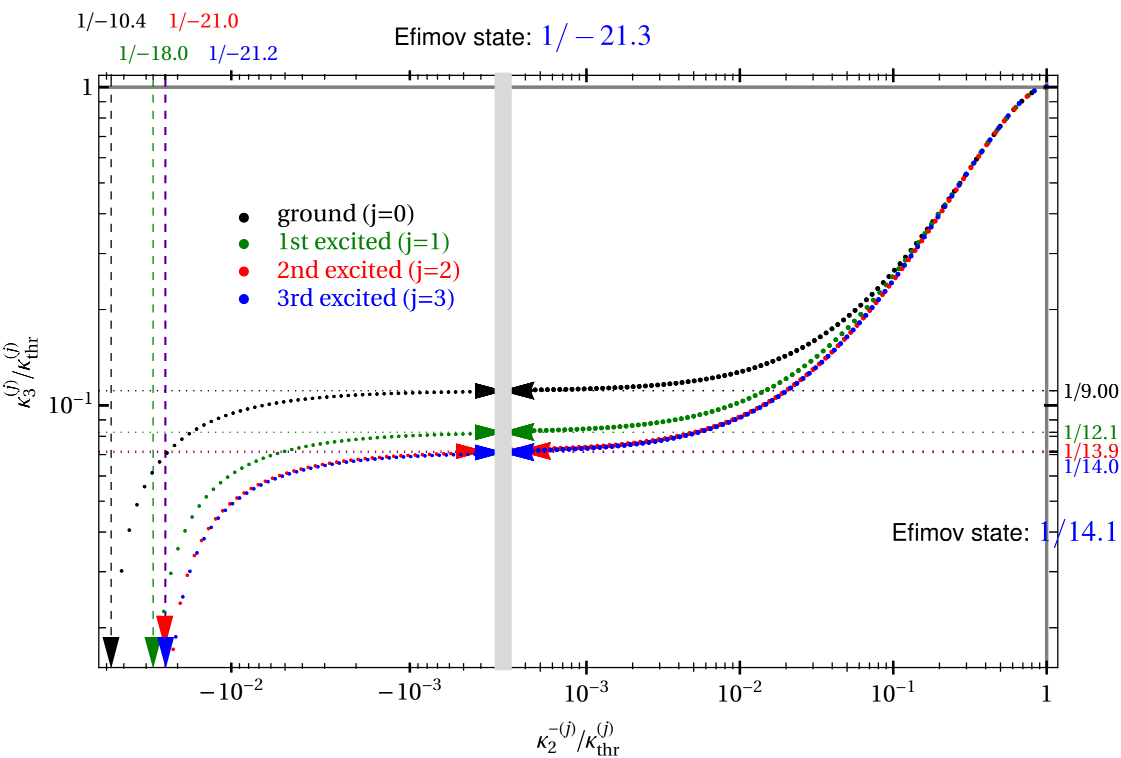
Consistent with N^{2}LO perturbation in r_0.

Platter/Ji/Phillips 0808.1230

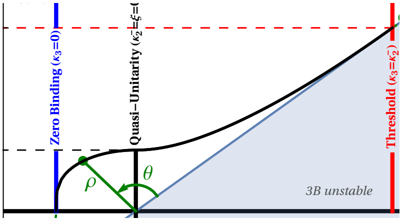
(d) Differences Between Trajectories: Normalised to Threshold

3rd excitation ($|\xi = \frac{2r_0}{a}| < 10^{-3}$) identical to Efimov ($r_0 = 0$) within numerical errors $\ll$ line thickness.

Efimov ratios from Braaten/Hammer cond-mat/0410417, Gogolin/Mora/Egger 0802.0549, Deltuva 1201.2326



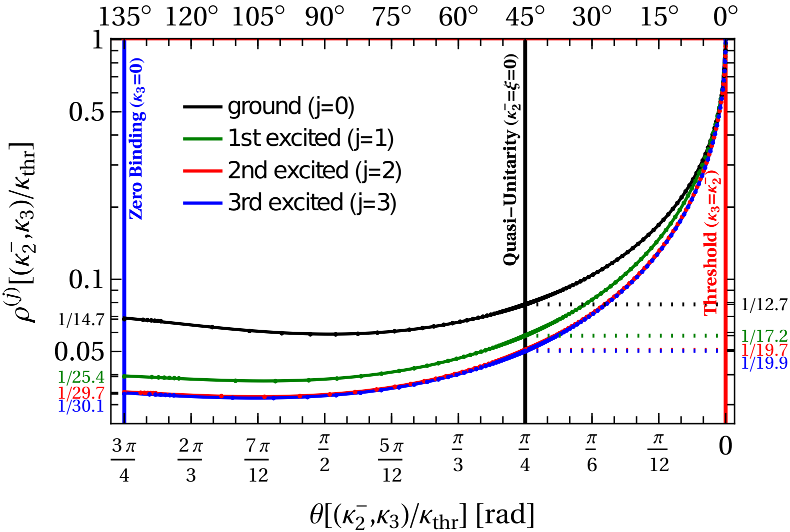
(e) Parametrising Rescaled Trajectories in Polar Angles



7-parameter fit inspired by
Gattobigio/Göbel/Hammer/Kievsky 1903.05493

$$\rho^{(j)}_{\mathrm{param}}(\theta) = \exp\left(\sum_{n=1}^{7} c_n^{(j)}\ \theta^{n/2}\right)$$

with norm $\rho(\mathrm{threshold}) = 1$

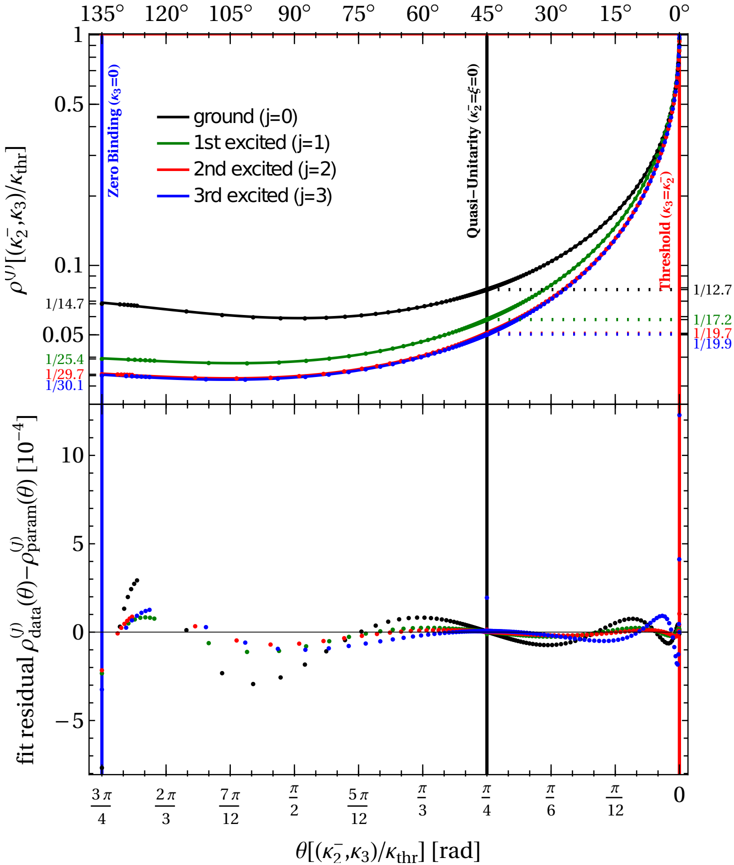


Truncated to significant figures, 68% prediction confidence interval $[\rho^{(j)} \pm w^{(j)}_{\mathrm{pred}}]$.

Correlation matrix entries $|(c_n, c_m)| > 0.6 \implies$ Find more efficient parametrisation (information compression)?

state	$c_1^{(j)}$	$c_2^{(j)}$	$c_3^{(j)}$	$c_4^{(j)}$	$c_5^{(j)}$	$c_6^{(j)}$	$c_7^{(j)}$	$w^{(j)}_{\mathrm{pred}}$
ground ($j = 0$)	-4.7545	4.3710	-7.4404	11.714	-11.0015	5.54726	-1.12595	0.00009
1st excited ($j = 1$)	-4.2764	1.0734	-1.6005	5.4777	-6.58104	3.64577	-0.768214	0.00003
2nd excited ($j = 2$)	-4.6823	1.4352	-1.2221	4.1199	-5.08001	2.87318	-0.613720	0.00003
3rd excited ($j = 3$)	-4.7003	1.1952	0.014514	1.5657	-2.44535	1.53708	-0.348975	0.00011

Parametrisation Residuals and Quality/Confidence



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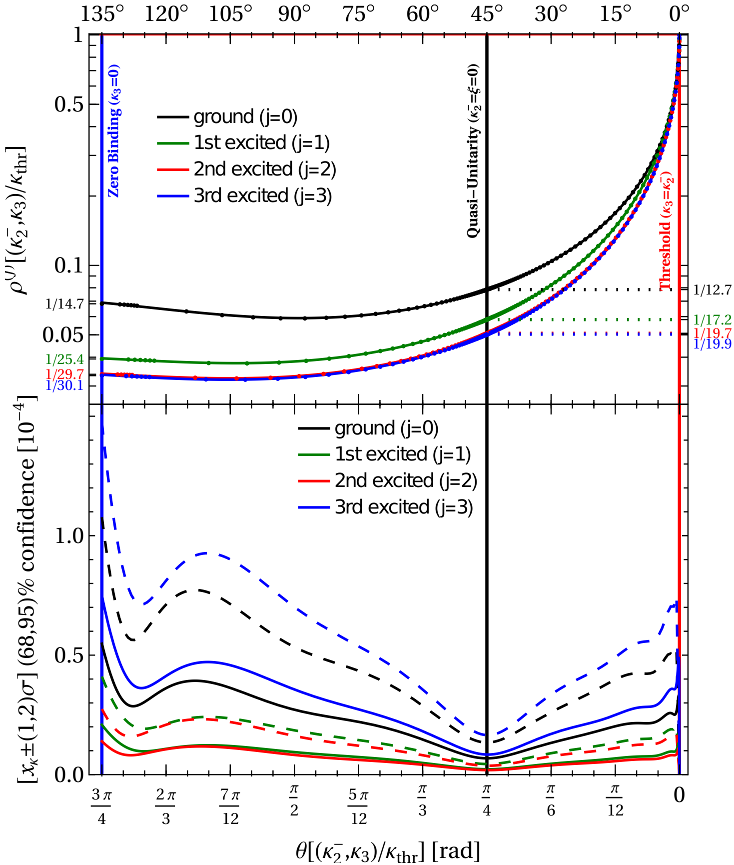


(f) Short-Range EFT as Low-Energy ReΣRangeEFT: Fixing Efimov's Tower

From 2B propagator in 3B kernel to Efimov:

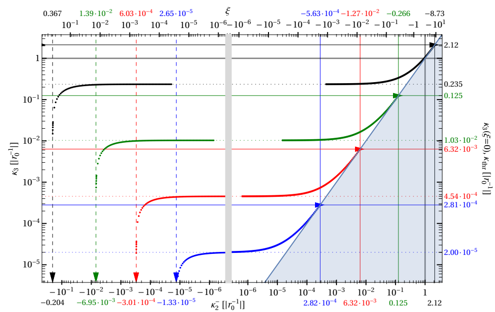
$$\frac{1}{\xi + \underbrace{(3Q^2/4+\kappa_3^2)}_{\text{eff. range}} + \sqrt{3Q^2+4\kappa_3^2}} \xrightarrow{Q\sim\kappa_3\to 0} \underbrace{\frac{1}{\xi+\sqrt{3Q^2+4\kappa_3^2}}}_{\text{Short-Range EFT}} \xrightarrow{\xi\to 0} \underbrace{\frac{1}{\sqrt{3}\,Q}}_{\text{Efimov's Discrete Scale Inv.}}$$

$\kappa_2^+(\xi=0) = -2|r_0^{-1}| \neq 0$ still sets scale of 2B. $\Longrightarrow$ Quasi-unitarity: No 2B scale invariance, except around 0.

$\Longrightarrow$ **"Near-Discrete Scale Invariance"** in 3B, Efimov tower fixed in limit $\xi = \dfrac{2r_0}{a} \to 0$ at $r_0 < 0$.

$\Longrightarrow$ (Over-accurately) predict binding energies in physical systems with large $|a|$ and $r_0 < 0$?!

state	zero binding $\kappa_2^-(\kappa_3=0)$ $[\lvert r_0^{-1}\rvert]$	quasi-unitarity $\kappa_3(\kappa_2^-=0)$ $[\lvert r_0^{-1}\rvert]$	threshold $\kappa_3 \overset{!}{=} \kappa_2^-$ $[\lvert r_0^{-1}\rvert]$
ground	$-2.04318(6)\cdot 10^{-1}$	$2.35412(3)\cdot 10^{-1}$	$2.11862(2)$
1st exc.	$-6.9517(5)\cdot 10^{-3}$	$1.03030(5)\cdot 10^{-2}$	$1.25108(1)\cdot 10^{-1}$
2nd exc.	$-3.0144(1)\cdot 10^{-4}$	$4.53987(1)\cdot 10^{-4}$	$6.320(1)\cdot 10^{-3}$
3rd exc.	$-1.3269(2)\cdot 10^{-5}$	$2.00039(5)\cdot 10^{-5}$	$2.810(3)\cdot 10^{-4}$
$j\to\infty$	$-0.1551(1)\,\mathrm{e}^{-j\frac{\pi}{s_0}}$	$0.23381(8)\,\mathrm{e}^{-j\frac{\pi}{s_0}}$	$3.31(2)\,\mathrm{e}^{-j\frac{\pi}{s_0}}$



Match Efimov scale Λ_* of 3BI in "hard cutoff regularisation"

$$H(\Lambda) \simeq -A\,\frac{\sin[s_0\ln\frac{\Lambda}{\Lambda_*} - \operatorname{arccot} s_0]}{\sin[s_0\ln\frac{\Lambda}{\Lambda_*} + \operatorname{arccot} s_0]}$$

Bedaque/Hammer/van Kolck nucl-th/9809025

to ReΣRangeEFT at same 2B binding $\kappa_2^-(\xi=0)$ per state.

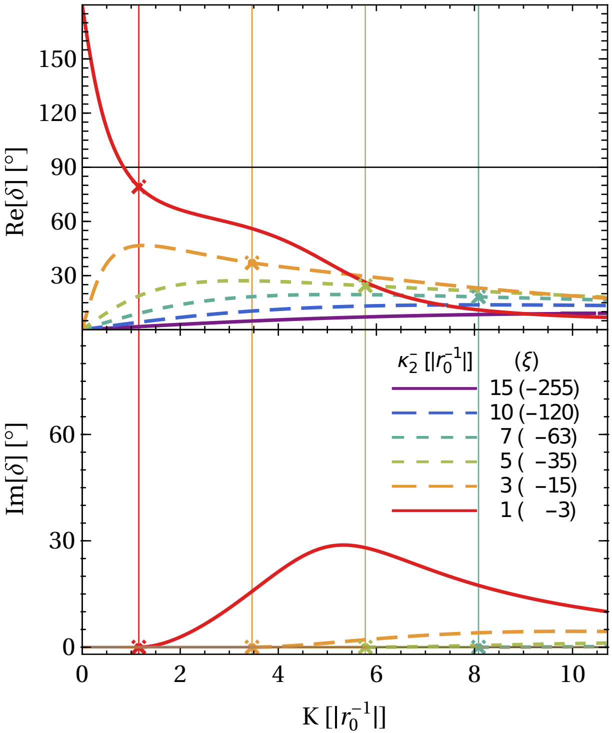
state	Efimov's Λ_* $[\lvert r_0^{-1}\rvert]$	amplitude A
ground	$0.614379(2)$	$0.87866(2)$
1st excited	$0.610223(1)$	$0.87866(1)$
2nd excited	$0.610206(1)$	$0.87866(1)$
3rd excited	$0.610206(1)$	$0.87866(1)$
$j\to\infty$	$0.610206(1)$	$0.87866(1)$

A: cf. Braaten/Kang/Platten1101.2854: 3 SigFig, no uncertainy

3. Quick Look At Scattering in Resummed-Range EFT

(a) Scattering B on Bound (BB) at cm Momentum K, Energy $\frac{3K^2}{4} - (\kappa_2^-)^2$

3B breakup threshold at $\sqrt{\frac{4}{3}}\,\kappa_2^- > 0$ (vertical lines) $\Longrightarrow$ Hetherington-Schick contour deformation.



3B ground state $\kappa_{\text{thr}} = 2.12\ldots$

$$K\cot\delta(K\to 0) = -\frac{1}{a_3} + \frac{r_3}{2}K^2 + \mathcal{O}(K^4)$$

$\kappa_2^- > \kappa_{3\text{thr}} = 2.12\ldots$: $a_3 < 0$ & $\delta(0) = 0$

$\kappa_2^- < \kappa_{3\text{thr}} = 2.12\ldots$: $a_3 > 0$ & $\delta(0) = 180^\circ$

more indication that ground state indeed ground

B(BB) eff. range $r_3 > 0$ always, while BB $r_0 < 0$.

$\kappa_2^- \searrow 1$: plateau-like structure around $K \sim 3$

– not present in Efimov version.

Appreciable inelasticities only for $\kappa_2^- \lesssim 1$.

(a) Scattering B on Bound (BB) at cm Momentum K, Energy $\frac{3K^2}{4} - (\kappa_2^-)^2$

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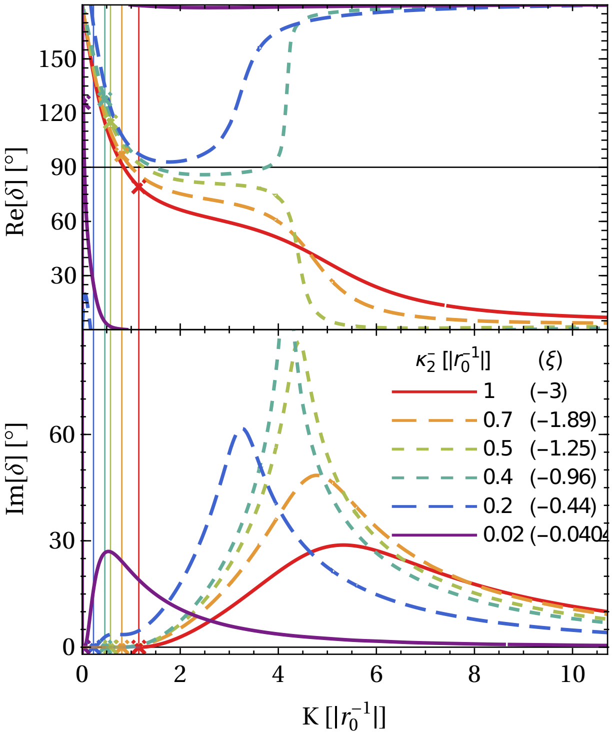


$\kappa_2^- = 1$ (red) same as previous.

3B binding thresholds $\kappa_{\text{thr}}^{(0)} = 2.12\ldots$, $\kappa_{\text{thr}}^{(1)} = 0.12\ldots$

Plateau continues to flatten & grow to $K \in [1;4]$,

turns into trough at $\kappa_2^- \approx 0.43$ ($\approx$ln-mean of $\kappa_{\text{thr}}^{(0,1)}$).

$\delta(K \in [1;4]) \approx 80^\circ$ smells unitary, but $\kappa_2 \neq 0$ bound.

"Near-Unitarity Window"

Cliff in Re correlates to huge inelasticity (peak in Im).

$\Longrightarrow$ Resonance?? (not BW+const. background)

hg/UvK investigating

$K \searrow \kappa_{\text{thr}}^{(2)}$: trough disappears, $\delta \approx 0$ except for $K \to 0$.

$\delta(K \to \infty)$ flips $0 \to 180^\circ$, more rapid with $K \searrow 0$.

$\Longrightarrow$ Perturbative, BB propagator $\sim \frac{1}{Q^2}$ Coulomb-like.

B(BB) eff. range $r_3 > 0$ always, while BB $r_0 < 0$.

4. Concluding Questions

Rescaled-Range EFT: $|a|$ & $r_0 < 0$ large in magnitude. $\Longrightarrow$ **Resum both at LO: universal in $\xi := \dfrac{2r_0}{a}$.**

– **3B bound in narrow range**, and only when 2B has bound+virtual or two virtual states.

– **Renormalisable without 3BI.** $\Longrightarrow$ Different universality class from perturbative $r_0 > 0$? $\exists$??RREFT$(r_0 > 0)$

– **2B predicts 3B:** Even at quasi-unitarity $\kappa_2^- = 0$, second pole $\kappa_2^+ \to -2|r_0^{-1}|$ sets scale in 2B & 3B.

$\Longrightarrow$ No 2B Scale Invariance, but 3B Discrete Scale Invariance approached as $\kappa_3 \to 0$ or $\xi \to 0$.

Efimov-like "Near-Discrete Scale Invariance" without 3BI: 3B ground state with 30% differences to Efimov.

$\Longrightarrow$ Efimov-like Tower fixed, 3B predicted, unique/stable ground state, universal relations.

– **Scattering:** Interesting structures; **"Near-Unitarity Window" ended by resonance??**

Shopping List

– **Fate of 3B pole** in un-bound & un-stable regions; close to BB double-pole $\kappa_2^\pm(\xi = 1) = -|r_0^{-1}|$;...

– Additional virtual/resonance 3B poles?

– **Scattering:** more details, e.g. 3B resonance signal; (BB) unbound; near-unitary plateau/trough;...

– Identical fermions; different masses.

– **Beyond LO:** shape-parameter etc in perturbation.

– **4B, 5B,... "Rinse And Repeat":** Two 4B states for each Efimov-like 3B state? with L. Contessi, J. Kirscher

– **Compare to physically realised systems in AMO & Nuclear/Particle!** $\Longrightarrow$ **HELP!**


END OF
IMPROVEMENT
THANK YOU
FOR YOUR PATIENCE

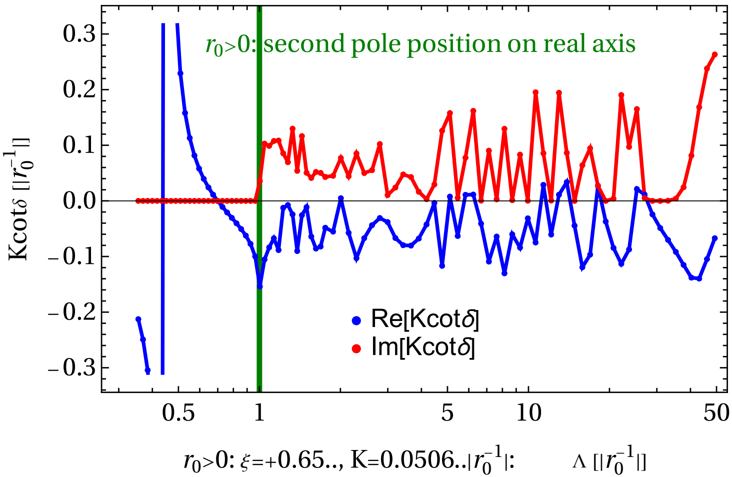
(a) Is There A Renormalisable Resummed-Range EFT With $r_0 > 0$?

$$\xrightarrow{r_0<0\to r_0\lessgtr 0} \frac{1}{\xi - K^2 + 2\mathrm{i}K\,\mathrm{sgn}r_0} \Longrightarrow$$ For $a, r_0 > 0$, $\kappa_2^\pm > 0$ are two **bound** states – κ_2^+ with residue < 0??

Example: Cutoff-dependence at fixed, low K, using

$\kappa_2^- = 0.23\ldots|r_0^{-1}|$, but B(BB) scattering has *second pole* on real axis at $1\ |r_0^{-1}|$ from

$\kappa_2^+ = 0.90\ldots|r_0^{-1}|$ (circumnavigated by Hertherington-Schick complex contour).



Not a numerical problem, but a cutoff-independent result appears elusive. . .

Stick to $r_0 \leq 0$ for now. . .

(b) The Fate of Some Theory Discussions

curse of statistics, that it can never prove things, only disprove them! At best, you can substantiate a hypothesis by ruling out, statistically, a whole long list of competing hypotheses, every one that has ever been proposed. After a while your adversaries and competitors will give up trying to think of alternative hypotheses, or else they will grow old and die, and *then your hypothesis will become accepted*. Sounds crazy, we know, but that's how science works!* Numerical Recipes chap. 14.0

* Science advances one funeral at a time.

[“Eine neue wissenschaftliche Wahrheit pflegt sich nicht in der Weise durchzusetzen, daß ihre Gegner überzeugt werden und sich als belehrt erklären, sondern vielmehr dadurch, daß ihre Gegner allmählich aussterben und daß die heranwachsende Generation von vornherein mit der Wahrheit vertraut gemacht ist.”] Max Planck: Scientific Autobiography