Investigation of two-body system by considering Dunkl operator

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Tow particles in an infinite well: Preamble

Kind of Particles	Statistical	Wavefunction of two particles in a one dimensional box	Eigenvalues and degeneracy $\left(K = \frac{\pi^2\hbar^2}{2mL^2}\right)$
Distinguished particles ▸ No need to arrange the wavefunction	Maxwell-Boltzmann	$\psi_d(x_1, x_2) = \psi_{n_1}(x_1)\,\psi_{n_2}(x_2)$ $\psi_{n_1}(x_1)\,\psi_{n_2}(x_2) \neq \psi_{n_1}(x_2)\,\psi_{n_2}(x_1)$	$gs: n_1, n_2 = 1 \rightarrow E_{11} = 2k$ 1 $fes: (1{,}2), (2{,}1) \rightarrow E_{12} = 5k$ 2 $ses: (2{,}2) \rightarrow E_{22} = 8k$ 1
Bosons (with integer spin)	Bose-Einstein	$\psi_b = \frac{1}{\sqrt{2}}\left\{\psi_{n_1}(x_1)\,\psi_{n_2}(x_2) + \psi_{n_1}(x_2)\,\psi_{n_2}(x_1)\right\}$	$n_1, n_2 = 1 \rightarrow E_{11} = 2k$ 1 $(1{,}2), (2{,}1) \rightarrow E_{12} = 5k$ 1 the same function $(2{,}2) \rightarrow E_{22} = 8k$ 1
Fermions (with half-integer spin)	Fermi-Dirac	$\psi_f = \frac{1}{\sqrt{2}}\left\{\psi_{n_1}(x_1)\,\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\,\psi_{n_2}(x_1)\right\}$	$n_1, n_2 = 1 \rightarrow$ ~~$E_{11} = 2k$~~ 1 $(1{,}2), (2{,}1) \rightarrow E_{12} = 5k$ 1 $(2{,}2) \rightarrow$ ~~$E_{22} = 8k$~~ 1 $\psi_{12} \neq \psi_{21} \rightarrow$ phase different $\psi_{11} \neq \psi_{22} = 0$ (PEP)

◉ From the experimental view:

- Time reversal makes the transformation $t \to -t$
  - for the reaction: $a + b \to c + d$ the total cross section is σ_1
  - for the reaction: $c + d \to a + b$ the total cross section is σ_2
  - If the σ_1/σ_2 goes to 1, then this reaction is reversible and the time reversal is valid for it.
- The operation of time reversal changes the sign of momentum p and of the direction of the total angular momentum J :

$$P \xrightarrow{T} P' = -P \qquad J \xrightarrow{T} J' = -J$$

- Charge canjugate
  - Particle-antiparticle $C|a\bar{a}\rangle = (-1)^{L+S}|a\bar{a}\rangle$

◉ From the historical view:

★ Eugene Paul Wigner (1950)

$i[H,x] = \dot{x} \qquad i[H,v] = \dot{v}$

★ Lee Yang (1951)

$\hat{P} = \frac{1}{i} D_x^{\text{Yang}} \qquad D_x^{\text{Yang}} = \partial_x - \frac{v}{x}\hat{R}$

★ Francesco Calogero (1969)

Wigner → (two) particle

★ Shuji Watanabe (1987)

Yang → self adjoint

◉ Dunkl operator

★ Charles F. Dunkl (1989)

$$\frac{d}{dx} \to D_x^v \qquad D_x^v = \frac{d}{dx} + \frac{v}{x}(1 - R_x)$$

- Four Dunkl-momentum operator:

$$p_\mu = \frac{\hbar}{i} D_\mu = \frac{\hbar}{i}\left(\partial_\mu - \frac{v}{x_\mu}\left(1 - R_\mu\right)\right)$$

- The operator R is called a reflection operator

$$R_x f(x) = f(-x)$$

- The Wigner parameter v should be $v > -1/2$
- For the even function: $R_x f_e(x) = f_e(x)$
- For the odd function: $R_x f_o(x) = -f_o(x)$

◉ Dunkl derivative - one dimension

- The square of the Dunkl derivative:

$$(D_x^v)^2 = \frac{d^2}{dx^2} + \frac{2v}{x}\frac{d}{dx} - \frac{v}{x^2}(1 - R_x)$$

- Then the Heisenberg relation is deformed as

$$[x, p] = i\,(1 + 2vR)$$

- For a particle in a box:

$$V(x) = \begin{cases} 0 & (-L < x < L) \\ \infty & \text{elsewhere} \end{cases}$$

- The Dunkl-Schrödinger equation is:

$$-\frac{1}{2m}D_x^2\psi = E\psi \quad \text{or} \quad -\frac{1}{2m}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x - \frac{\nu}{x^2}(1-R)\right)\psi = E\psi$$

- For the even parity solution by ψ_+ we get

$$-\frac{1}{2m}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x\right)\psi_+ = E_+\psi_+$$

- For the odd parity solution by ψ_- we get

$$-\frac{1}{2m}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x - \frac{2\nu}{x^2}\right)\psi_- = E_-\psi_-$$

$$\psi = \psi_+ + \psi_-$$

- For the even parity solution, If we set

$$\psi_+^\lambda = \sum_{n=0}^{\infty} a_n^+ x^{2n} |x|^\lambda$$

- with insert it into the Schrödinger equation, we have the recurrence relation:

$$a_{n+1}^+ = -\frac{2mE_+}{(2n+2+\lambda)(2n+1+\lambda+2\nu)}a_n^+$$

- with a characteristic equation:

$$\lambda(\lambda - 1 + 2\nu) = 0$$

- Thus we obtain the wave function and energy for an even parity solution as:

$$\psi_+ = N_+ x^{\frac{1}{2}-2\nu} J_{\nu-\frac{1}{2}}(\sqrt{2mE_+}\,x) \qquad E_n^+ = \frac{1}{2mL^2}a^2_{\nu-\frac{1}{2},n} \qquad n = 1,2,\ldots$$

- For the odd parity solution, If we set

$$\psi_-^\lambda = \sum_{n=0}^{\infty} a_n^- x^{2n+1} |x|^\lambda$$

- with insert it into the Schrödinger equation, we have the recurrence relation:

$$a_{n+1}^- = -\frac{2mE_-}{(2n+2+\lambda)(2n+3+\lambda+2\nu)}a_n^-$$

- with a characteristic equation:

$$\lambda(\lambda + 1 + 2\nu) = 0$$

- Thus we obtain the wave function and energy for an odd parity solution as:

$$\psi_- = N_- x^{\frac{1}{2}-\nu} J_{\nu+\frac{1}{2}}(\sqrt{2mE_-}\,x) \qquad E_n^- = \frac{1}{2mL^2}a^2_{\nu+\frac{1}{2}n} \qquad n = 1,2,\ldots$$

- For two bosons:
  - the wavefunction:

$$\psi^{s_1,s_2}_{b\ n_1,n_2}(x_1,x_2)=\frac{1}{\sqrt{2}}\left\{\psi^{s_1}_{n_1}(x_1)\,\psi^{s_2}_{n_2}(x_2)+\psi^{s_1}_{n_1}(x_2)\,\psi^{s_2}_{n_2}(x_1)\right\}$$

  - and the energy :

$$E^{s_1,s_2}_{n_1,n_2}=E^{s_1}_{n_1}+E^{s_2}_{n_2}$$

  - for $\nu=\frac{1}{2}$ $\qquad s_i=\pm 1,\quad i=1,2$

$$E^{s_i}_n=k\left(\alpha_{\nu-\frac{s_i}{2},n}\right)^2\Rightarrow\begin{cases}E^+_n=k\left(\alpha_{\nu-\frac{1}{2},n}\right)^2\\ E^-_n=k\left(\alpha_{\nu+\frac{1}{2},n}\right)^2\end{cases}$$

  - also

$$\alpha_{\nu-\frac{1}{2},n}<\alpha_{\nu+\frac{1}{2},n}\quad\Rightarrow\quad E^+_n<E^-_n$$

$$\alpha_{\nu+\frac{1}{2},n}<\alpha_{\nu-\frac{1}{2},n+1}\quad\Rightarrow\quad E^-_n<E^+_{n+1}$$

$$\alpha_{0,1}=2.4\qquad \alpha_{1,1}=3.8\qquad \alpha_{0,2}=5.5\qquad \alpha_{1,2}=7.0$$

  - Therefore

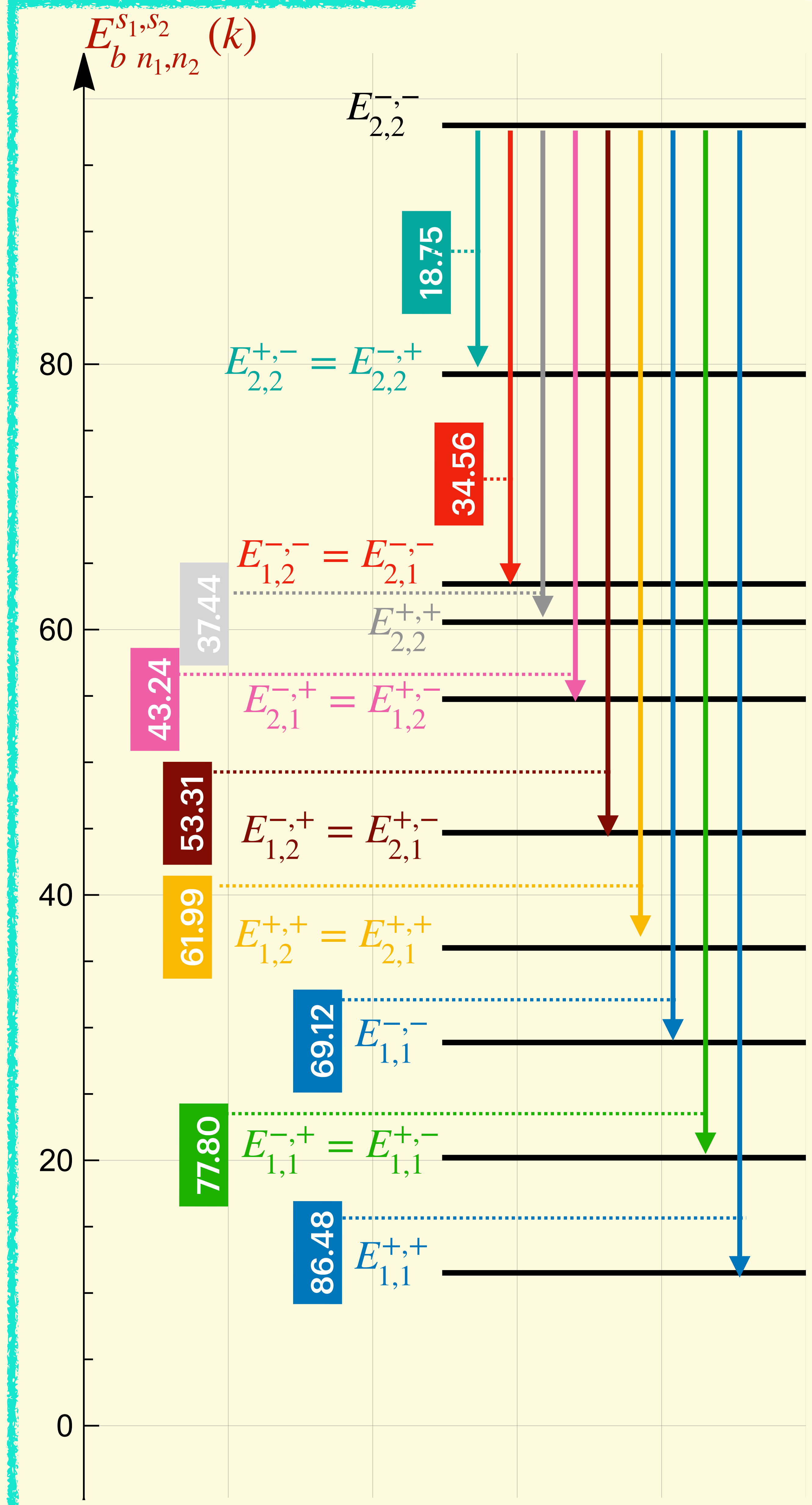
$$\psi^{+,-}_{b\ 1,1}=\frac{1}{\sqrt{2}}\left\{\psi^+_1(x_1)\,\psi^-_1(x_2)+\psi^+_1(x_2)\,\psi^-_1(x_1)\right\}$$

$$\psi^{-,+}_{b\ 1,1}=\frac{1}{\sqrt{2}}\left\{\psi^-_1(x_1)\,\psi^+_1(x_2)+\psi^-_1(x_2)\,\psi^+_1(x_1)\right\}$$

$$\psi^{+,-}_{b\ 1,1}=\psi^{-,+}_{b\ 11}\Rightarrow$$

We don't have degenerate cases. In fact, for two bosons there is not any degenerate case for the composite cases.

$E^{s_1,s_2}_{b\ n_1,n_2}$	Values (k)
$E^{+,+}_{1,1}$	$(2.4)^2+(2.4)^2$
$E^{+,-}_{1,1}$	$(2.4)^2+(3.8)^2$
$E^{-,+}_{1,1}$	$(2.4)^2+(3.8)^2$
$E^{-,-}_{1,1}$	$(3.8)^2+(3.8)^2$
$E^{+,+}_{1,2}$	$(2.4)^2+(5.5)^2$
$E^{+,+}_{2,1}$	$(2.4)^2+(5.5)^2$
$E^{-,+}_{1,2}$	$(5.5)^2+(3.8)^2$
$E^{+,-}_{2,1}$	$(5.5)^2+(3.8)^2$
$E^{-,+}_{2,1}$	$(7.0)^2+(2.4)^2$
$E^{+,-}_{1,2}$	$(2.4)^2+(7.0)^2$
$E^{+,+}_{2,2}$	$(5.5)^2+(5.5)^2$
$E^{-,-}_{1,2}$	$(3.8)^2+(7.0)^2$
$E^{-,-}_{2,1}$	$(3.8)^2+(7.0)^2$
$E^{+,-}_{2,2}$	$(5.5)^2+(7.0)^2$
$E^{-,+}_{2,2}$	$(5.5)^2+(7.0)^2$
$E^{-,-}_{2,2}$	$(7.0)^2+(7.0)^2$



- For two fermions:
  - the wavefunction:

$$\psi^{s_1,s_2}_{f\,n_1,n_2}(x_1,x_2)=\frac{1}{\sqrt{2}}\left\{\psi^{s_1}_{n_1}(x_1)\,\psi^{s_2}_{n_2}(x_2)-\psi^{s_1}_{n_1}(x_2)\,\psi^{s_2}_{n_2}(x_1)\right\}$$

  - and the energy :

$$E^{s_1,s_2}_{n_1,n_2}=E^{s_1}_{n_1}+E^{s_2}_{n_2}=k\left\{\left(\alpha_{v-\frac{s_1}{2},n_1}\right)^2+\left(\alpha_{v-\frac{s_2}{2},n_2}\right)^2\right\}$$

- and the energy :

$$\psi^{s_1,s_2}_{n_1,n_2}=0$$

This is a quite general result and is known as the Pauli exclusion principle.

- also:

$$\psi^{+,-}_{f\;1,1}(x_1,x_2)=\frac{1}{\sqrt{2}}\left\{\psi^{+}_{1}(x_1)\,\psi^{-}_{1}(x_2)-\psi^{+}_{1}(x_2)\,\psi^{-}_{1}(x_1)\right\}$$

$$\psi^{-,+}_{f\;1,1}(x_1,x_2)=\frac{1}{\sqrt{2}}\left\{\psi^{-}_{1}(x_1)\,\psi^{+}_{1}(x_2)-\psi^{-}_{1}(x_2)\,\psi^{+}_{1}(x_1)\right\}$$

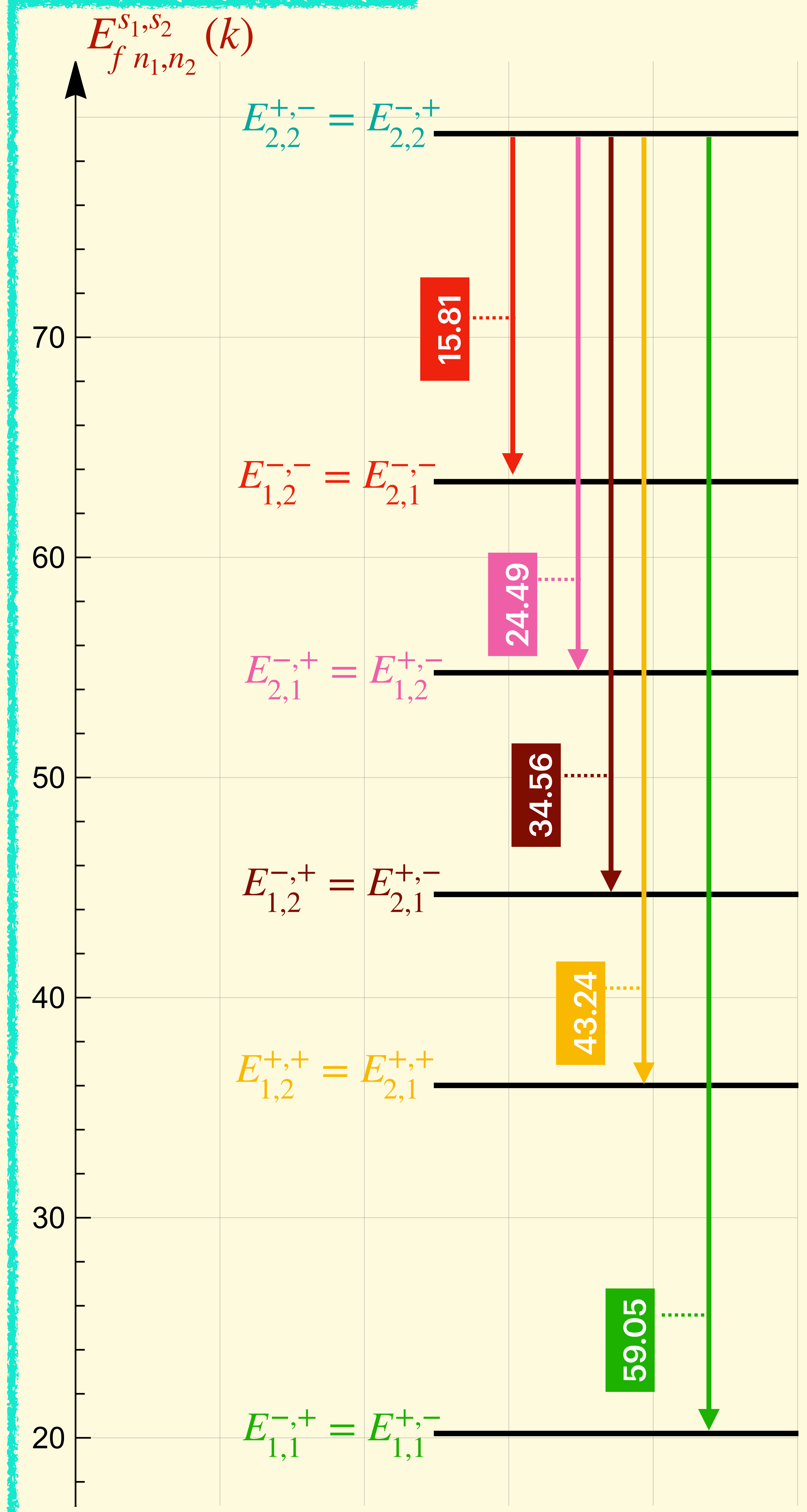
$$\Rightarrow E^{+,-}_{1,1}=E^{-,+}_{1,1}$$

- Although the above states have the same energy, they are not degenerate. In fact

$$\psi^{+,-}_{f\;1,1}=e^{i\pi}\,\psi^{-,+}_{f\;1,1}$$

- such phase factors are very important in the interference phenomenon

$E^{s_1,s_2}_{f\;n_1,n_2}$	Values (k)
$E^{+,+}_{1,1}$	~~$(2.4)^2+(2.4)^2$~~
$E^{+,-}_{1,1}$	$(2.4)^2+(3.8)^2$
$E^{-,+}_{1,1}$	$(2.4)^2+(3.8)^2$
$E^{-,-}_{1,1}$	~~$(3.8)^2+(3.8)^2$~~
$E^{+,+}_{1,2}$	$(2.4)^2+(5.5)^2$
$E^{+,+}_{2,1}$	$(2.4)^2+(5.5)^2$
$E^{-,+}_{1,2}$	$(5.5)^2+(3.8)^2$
$E^{+,-}_{2,1}$	$(5.5)^2+(3.8)^2$
$E^{-,+}_{2,1}$	$(7.0)^2+(2.4)^2$
$E^{+,-}_{1,2}$	$(2.4)^2+(7.0)^2$
$E^{+,+}_{2,2}$	~~$(5.5)^2+(5.5)^2$~~
$E^{-,-}_{1,2}$	$(3.8)^2+(7.0)^2$
$E^{-,-}_{2,1}$	$(3.8)^2+(7.0)^2$
$E^{+,-}_{2,2}$	$(5.5)^2+(7.0)^2$
$E^{-,+}_{2,2}$	$(5.5)^2+(7.0)^2$
$E^{-,-}_{2,2}$	~~$(7.0)^2+(7.0)^2$~~



- Now let us consider the harmonic oscillator problem with reflection symmetry. The Schrödinger equation reads

$$\left(-\frac{1}{2m}D_x^2+\frac{1}{2}m\omega^2x^2\right)\psi=E\psi$$

- If we set $\sqrt{m\omega}\,x=\xi$ we have:

$$-D_\xi^2\psi+\xi^2\psi=\epsilon\psi$$

- where

$$\epsilon=\frac{2E}{\omega}$$

- if we set

$$\psi(\xi)=e^{-\frac{\xi^2}{2}}(\xi)$$

- we have

$$D_\xi^2y-D_\xi(\xi y)-\xi D_\xi y+\epsilon y=0$$

- or

$$D_\xi^2y-2\xi D_\xi y+(\epsilon-1-2\nu P)\,y=0$$

- Even solution
  - For the even solution, we set

$$y=\sum_{k=0}^{\infty}a_n\xi^{2k}$$

- Then

$$a_{n+1}=\left(\frac{2\,[2n]_\nu+1+2\nu-\epsilon_+}{[2n+2]_\nu\,[2n+1]_\nu}\right)a_n$$

- Requiring that the series be terminated, we have

$$(\epsilon_+)_N=2\,[2N]_\nu+1+2\nu$$

$$[N]_\nu=N+\nu\left(1-(-1)^N\right)\qquad N=0,1,2,\ldots$$

- Thus the energy level for the even solution is

$$E_N^+=\frac{\omega}{2}\left(2\,[2N]_\nu+1+2\nu\right)$$

- Then we have the polynomial solution whose recurrence relation is

$$a_{n+1}=\left(\frac{2\left([2n]_\nu-[2N]_\nu\right)}{[2n+2]_\nu\,[2n+1]_\nu}\right)a_n$$

- Let us denote the function y corresponding to N by H_N^+. The first few H_N^+'s are

$$H_0^+(x)=1\qquad H_1^+(x)=1-\frac{2}{[1]_\nu}x^2$$

$$H_2^+(x)=1-\frac{2\,[4]_\nu}{[2]_\nu!}x^2+\frac{2^2\,[4]_\nu\left([4]_\nu-[2]_\nu\right)}{[4]_\nu!}x^4$$

- Even solution
  - For the odd solution, we set

$$y=\sum_{k=0}^{\infty}b_n\xi^{2k+1}$$

- Then

$$b_{n+1}=\left(\frac{2\,[2n+1]_\nu+1-2\nu-\epsilon_-}{[2n+3]_\nu\,[2n+2]_\nu}\right)b_n$$

- Requiring that the series be terminated, we have

$$(\epsilon_-)_N=2\,[2N+1]_\nu+1-2\nu$$

- Thus the energy level for the odd solution is

$$E_N^-=\frac{\omega}{2}\left(2\,[2N+1]_\nu+1-2\nu\right)$$

- Then we have the polynomial solution whose recurrence relation is

$$b_{n+1} = \left(\frac{2\left([2n+1]_\nu - [2N+1]_\nu \right)}{[2n+3]_\nu [2n+2]_\nu} \right) b_n$$

- Let us denote the function y corresponding to N by H_N^-. The first few H_N^- 's are

$$H_0^-(x) = x \qquad H_1^-(x) = x - \frac{2\left([3]_\nu - [1]_\nu\right)}{[3]_\nu !} x^3$$

$$H_2^-(x) = x - \frac{2\left([5]_\nu - [1]_\nu\right)}{[3]_\nu !} x^3 + \frac{2^2\left([5]_\nu - [3]_\nu\right)\left([5]_\nu - [1]_\nu\right)}{[5]_\nu !} x^5$$

$$\psi_{1,0}^{+,-}(x_1,x_2) = \frac{1}{\sqrt{2}} \left\{ \psi_1^+(x_1)\psi_0^-(x_2) - \psi_1^+(x_2)\psi_0^-(x_1) \right\}$$

$$\psi_{0,1}^{-,+}(x_1,x_2) = \frac{1}{\sqrt{2}} \left\{ \psi_0^-(x_1)\psi_1^+(x_2) - \psi_0^-(x_2)\psi_1^+(x_1) \right\}$$

$$\psi_{1,0}^{-,+}(x_1,x_2) = \frac{1}{\sqrt{2}} \left\{ \psi_1^-(x_1)\psi_0^+(x_2) - \psi_1^-(x_2)\psi_0^+(x_1) \right\}$$

$$\psi_{0,1}^{+,-}(x_1,x_2) = \frac{1}{\sqrt{2}} \left\{ \psi_0^+(x_1)\psi_1^-(x_2) - \psi_0^+(x_2)\psi_1^-(x_1) \right\}$$

$E_{b\ n_1,n_2}^{s_1,s_2}$	Values (ω)
$E_{0,0}^{+,+}$	$1 + 2\nu$
$E_{0,0}^{+,-}$	$2 + 2\nu$
$E_{0,0}^{-,+}$	$2 + 2\nu$
$E_{0,0}^{-,-}$	$3 + 2\nu$
$E_{0,1}^{+,+}$	$3 + 2\nu$
$E_{1,0}^{+,+}$	$3 + 2\nu$
$E_{1,0}^{+,-}$	$4 + 2\nu$
$E_{0,1}^{-,+}$	$4 + 2\nu$
$E_{1,0}^{-,+}$	$4 + 2\nu$
$E_{0,1}^{+,-}$	$4 + 2\nu$
$E_{0,1}^{-,-}$	$5 + 2\nu$
$E_{1,0}^{-,-}$	$5 + 2\nu$
$E_{1,1}^{+,+}$	$5 + 2\nu$
$E_{1,1}^{+,-}$	$6 + 2\nu$
$E_{1,1}^{-,+}$	$6 + 2\nu$
$E_{1,1}^{-,-}$	$7 + 2\nu$

$E_{f\ n_1,n_2}^{s_1,s_2}$	Values (ω)
$E_{0,0}^{+,+}$	~~$1 + 2\nu$~~
$E_{0,0}^{+,-}$	$2 + 2\nu$
$E_{0,0}^{-,+}$	$2 + 2\nu$
$E_{0,0}^{-,-}$	~~$3 + 2\nu$~~
$E_{0,1}^{+,+}$	$3 + 2\nu$
$E_{1,0}^{+,+}$	$3 + 2\nu$
$E_{1,0}^{+,-}$	$4 + 2\nu$
$E_{0,1}^{-,+}$	$4 + 2\nu$
$E_{1,0}^{-,+}$	$4 + 2\nu$
$E_{0,1}^{+,-}$	$4 + 2\nu$
$E_{0,1}^{-,-}$	$5 + 2\nu$
$E_{1,0}^{-,-}$	$5 + 2\nu$
$E_{1,1}^{+,+}$	~~$5 + 2\nu$~~
$E_{1,1}^{+,-}$	$6 + 2\nu$
$E_{1,1}^{-,+}$	$6 + 2\nu$
$E_{1,1}^{-,-}$	~~$7 + 2\nu$~~

- How can we test whether our approaches are right or not?
  - The best way is to compare them with experimental results.
- Is there any such transition in cold atoms?
- Are there any such transitions in hadronic states? (where k and ν are adjustable parameters)
- If yes, we should fit the data using the scanning method to obtain other properties of the system.
- For a perturbation potential in the excited state, we should consider a 9 by 9 matrix.
- Expanding the calculations to 2 and 3 dimensions is necessary, especially for spherical and cylindrical coordinates.
- Does this method give us the same results as those obtained by non-commutative methods, minimal length formalism, DSR method, and other methods?
- Are the results in the presence of a magnetic field different for odd and even parities?
- Do you think considering Dunkl operators in your field would make any difference?

References

1. E. P. Wigner, Phys. Rev. 77, 711 (1950).
2. L. Yang, Phys. Rev. 84, 788 (1951).
3. F. Calogero, J. Math. Phys. 10, 2191 (1969).
4. F. Calogero, J. Math. Phys. 12, 419 (1971).
5. M. A. Vasiliev, Int. J. Mod. Phys. A 6, 1115 (1991).
6. R. Chakrabarti and R. Jagannathan, J. Phys. A 27, L2777 (1994).
7. A. Turbiner, Phys. Lett. B 320, 281 (1994).
8. T. Brzezinski, T. Egusquiza and A. J. Macfarlane, Phys. Lett. B 311, 202 (1993)
9. L. Brink, T. H. Hansson and M. A. Vasiliev, Phys. Lett. B 286, 109 (1992).
10. A. P. Polychronakos, Phys. Rev. Lett. 69, 703 (1992).
11. A. J. Macfarlane, J. Math. Phys. 35, 1054 (1994).
12. V. Genest, M. Ismail, L. Vinet and A. Zhedanov, J. Phys. A 46, 145201 (2013)
13. W. S. Chung and H.Hassanabadi, Mod. Phys. Lett. A,Vol. 34, No. 24 (2019) 1950190

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