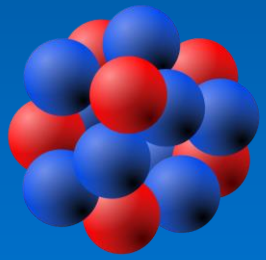
Exploring non-implausible domain of low-energy constants in delta-full chiral effective field theory with history matching



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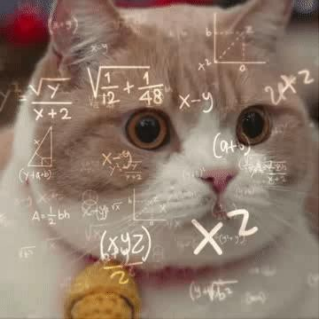
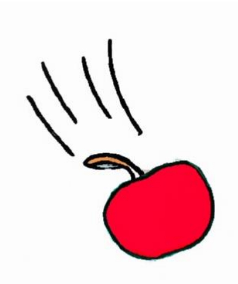
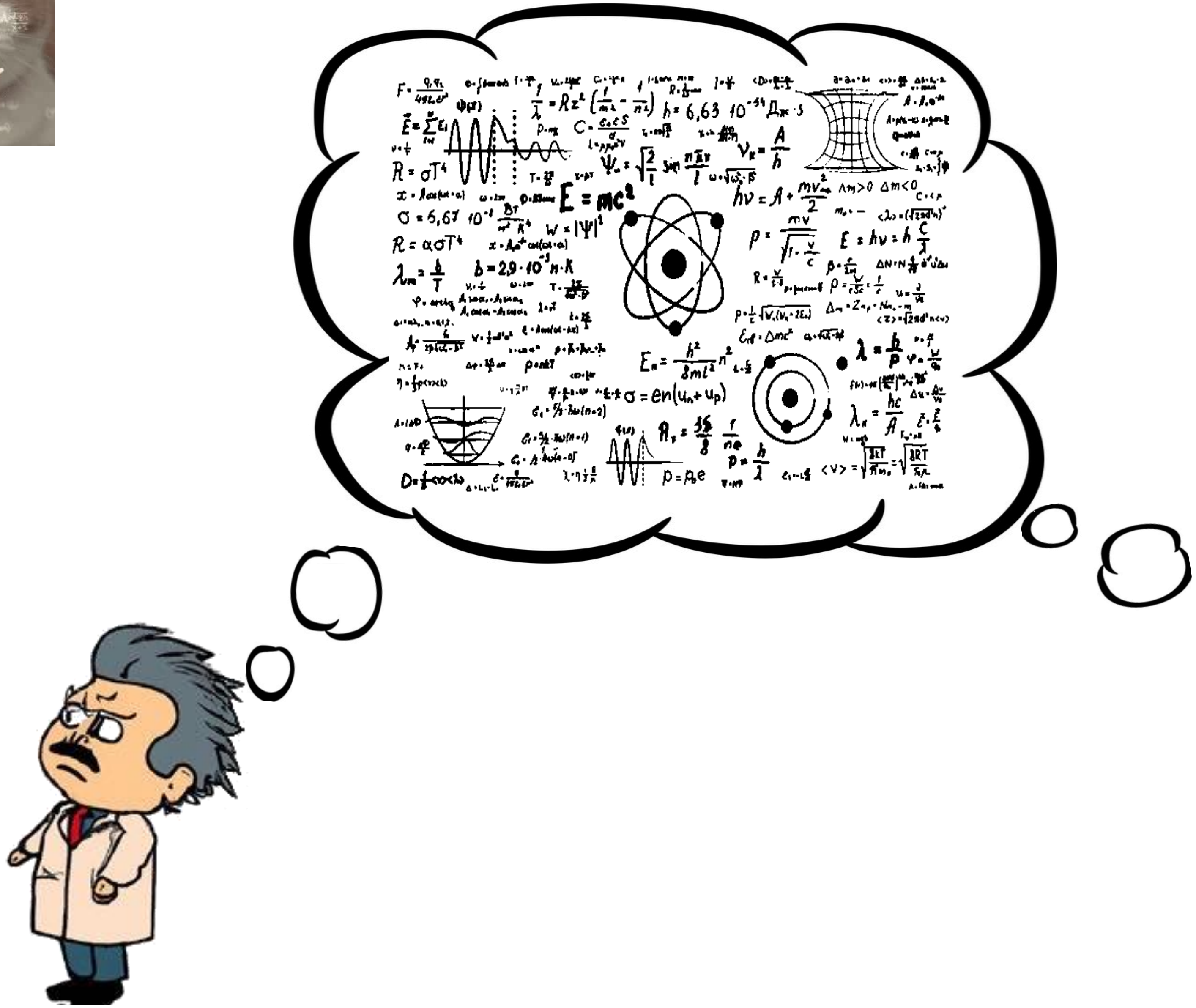
EFB-25, Mainz, July 31



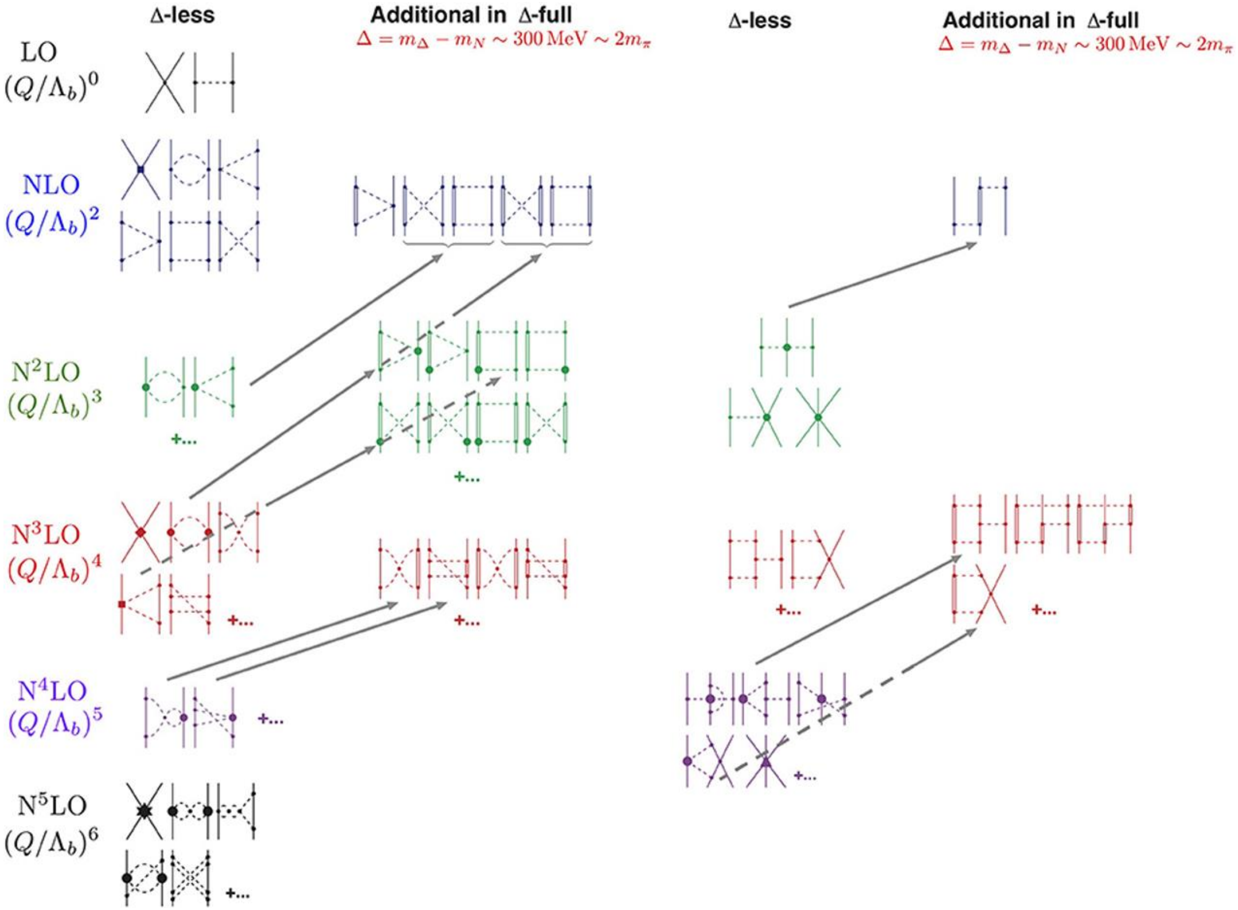




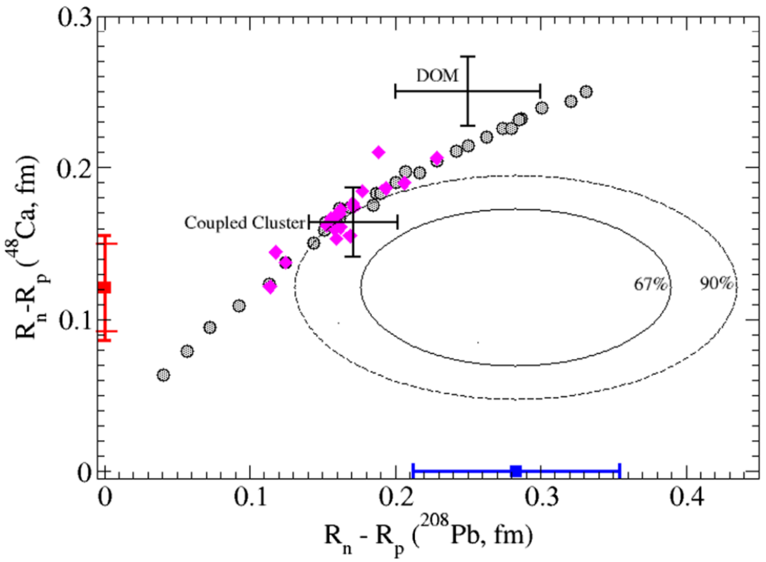
$$
\begin{gathered}
\mathcal{L}_{SM} = -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \\
igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - \\
W^-_\nu \partial_\nu W^+_\mu)) - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - \\
Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \tfrac{1}{2}\partial_\mu H \partial_\mu H - 2M^2\alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\beta_h\left(\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \tfrac{2M^4}{g^2}\alpha_h - \\
g\alpha_h M\,(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-) - \\
\tfrac{1}{8}g^2\alpha_h\,(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2) - \\
gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \\
\tfrac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) + \\
\tfrac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + \\
M\,(\tfrac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+) - ig\tfrac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - \\
W^-_\mu\phi^+) - ig\tfrac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \\
\tfrac{1}{4}g^2W^+_\mu W^-_\mu\,(H^2 + (\phi^0)^2 + 2\phi^+\phi^-) - \tfrac{1}{8}g^2\tfrac{1}{c_w^2}Z^0_\mu Z^0_\mu\,(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-) - \\
\tfrac{1}{2}g^2\tfrac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \tfrac{1}{2}ig^2\tfrac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- - \\
g^2 s_w^2 A_\mu A_\mu\phi^+\phi^- + \tfrac{1}{2}ig_s\,\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda(\gamma\partial + m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + \\
m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_w A_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right) + \\
\tfrac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{d}^\lambda_j\gamma^\mu(\tfrac{4}{3}s_w^2 - 1 - \gamma^5)d^\lambda_j) + \\
(\bar{u}^\lambda_j\gamma^\mu(1 - \tfrac{8}{3}s_w^2 + \gamma^5)u^\lambda_j)\} + \tfrac{ig}{2\sqrt{2}}W^+_\mu\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5){U^{lep}}_{\lambda\kappa}e^\kappa) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right) + \\
\tfrac{ig}{2\sqrt{2}}W^-_\mu\left((\bar{e}^\kappa {U^{lep}}^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_e(\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m^\lambda_\nu(\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa}(1+\gamma^5)e^\kappa\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_e(\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa}(1+\gamma^5)\nu^\kappa) - m^\kappa_\nu(\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa}(1-\gamma^5)\nu^\kappa\right) - \tfrac{g}{2}\tfrac{m^\lambda_\nu}{M}H(\bar{\nu}^\lambda\nu^\lambda) - \\
\tfrac{g}{2}\tfrac{m^\lambda_e}{M}H(\bar{e}^\lambda e^\lambda) + \tfrac{ig}{2}\tfrac{m^\lambda_\nu}{M}\phi^0(\bar{\nu}^\lambda\gamma^5\nu^\lambda) - \tfrac{ig}{2}\tfrac{m^\lambda_e}{M}\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda) - \tfrac{1}{4}\,\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa - \\
\tfrac{1}{4}\,\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa + \tfrac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_d(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_d(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j\right) - \tfrac{g}{2}\tfrac{m^\lambda_u}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \\
\tfrac{g}{2}\tfrac{m^\lambda_d}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j) + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu\bar{G}^a G^b g^c_\mu + \\
\bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 - \tfrac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_w W^+_\mu(\partial_\mu\bar{X}^0 X^- - \\
\partial_\mu\bar{X}^+ X^0) + igs_w W^+_\mu(\partial_\mu\bar{Y}X^- - \partial_\mu\bar{X}^+ Y) + igc_w W^-_\mu(\partial_\mu\bar{X}^- X^0 - \\
\partial_\mu\bar{X}^0 X^+) + igs_w W^-_\mu(\partial_\mu\bar{X}^- Y - \partial_\mu\bar{Y}X^+) + igc_w Z^0_\mu(\partial_\mu\bar{X}^+ X^+ - \\
\partial_\mu\bar{X}^- X^-) + igs_w A_\mu(\partial_\mu\bar{X}^+ X^+ - \\
\partial_\mu\bar{X}^- X^-) - \tfrac{1}{2}gM\left(\bar{X}^+X^+H + \bar{X}^-X^-H + \tfrac{1}{c_w^2}\bar{X}^0X^0H\right) + \tfrac{1-2c_w^2}{2c_w}igM\left(\bar{X}^+X^0\phi^+ - \bar{X}^-X^0\phi^-\right) + \\
\tfrac{1}{2c_w}igM\left(\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right) + igMs_w\left(\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right) + \\
\tfrac{1}{2}igM\left(\bar{X}^+X^+\phi^0 - \bar{X}^-X^-\phi^0\right).
\end{gathered}
$$

Introduction

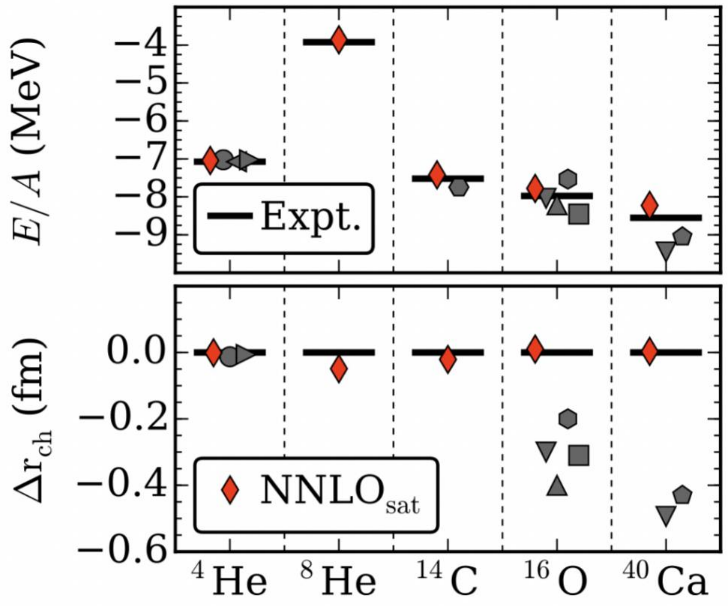
$$H(\vec{\alpha}) = h_0 + \sum_{i=1}^{N_{\mathrm{LECs}}} \alpha_i h_i$$



Nuclear interaction based on chiral effective field theory (EFT), parametrized in terms of low energy constants (LECs)

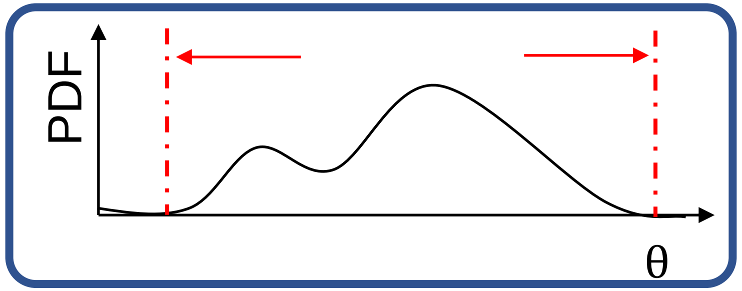


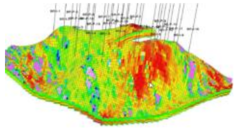
Uncertainty of the nuclear Hamiltonian (nuclear interaction)



History matching - Linking models to reality

- History matching is a statistical method for calibrating complex models (high-dimension)
- Scan the whole model space using random interaction as probes and iteratively remove the implausible parameter domains
- Enabling technology: fast emulators for predicting many-body observables



- oil reservoirs 
- galaxy formation 

different parameterization of interaction samples → **implausibility measure** → **non-implausible samples** / implausible samples

reality | theoretical predictions | experimental errors | theoretical errors

$$z = \widetilde{M}(\theta) + \varepsilon_{\mathrm{exp}} + \varepsilon_{\mathrm{em}} + \varepsilon_{\mathrm{method}} + \varepsilon_{\mathrm{model}}$$

*Vernon, I., Goldstein, M. , Bower, R. Statist. Sci. 29, 81 (2014).
Edwards, T.L., Brandon, M.A., Durand, G. *et al.* *Nature* **566,** 58–64 (2019).

Implausibility measure

individual implausibility measures:

$$I_i^2(\alpha) = \frac{|M_i(\alpha) - z_i|^2}{\mathrm{Var}\left(M_i(\alpha) - z_i\right)}$$

reality; theoretical predictions; experimental errors; theoretical errors

$$z = \widetilde{M}(\theta) + \varepsilon_{\mathrm{exp}} + \varepsilon_{\mathrm{em}} + \varepsilon_{\mathrm{method}} + \varepsilon_{\mathrm{model}}$$

- Includes the squared difference between the model prediction $M_i(\alpha)$ and the observation z_i for observable i.

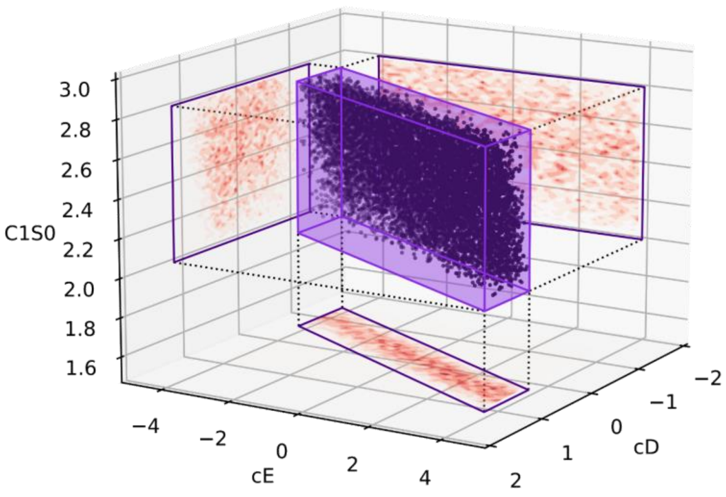
- The total variance in the denominator assumes independent errors and is therefore a sum of variances that in our case includes experimental, model, method, and emulator errors.

- Unless differently specified we use the maximum of the individual implausibility measures to define the constraint. Default choice is $c_I = 3$ inspired by Pukelheim's three-sigma rule

$$I_M(\alpha) \equiv \max_{z_i \in \mathcal{Z}} I_i(\alpha) \leq c_I$$

Iterative history matching procedure

- After a wave of scanning, “good” interactions are preserved and one can observe how they cluster in the high-dimensional space
- Put all these “good” interactions in a box and remove all the other parameter domains
- Repeat the scanning multiple times using different observables as constraints. Stop when the box size can no longer be reduced.

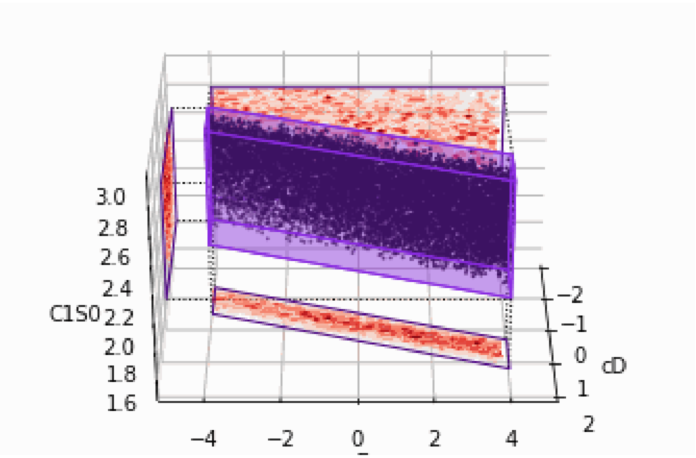


Iterative History matching

I. Use a space-filling design such as Latin Hypercube Sampling to generate well-spaced interaction samples in the input parameter domain.

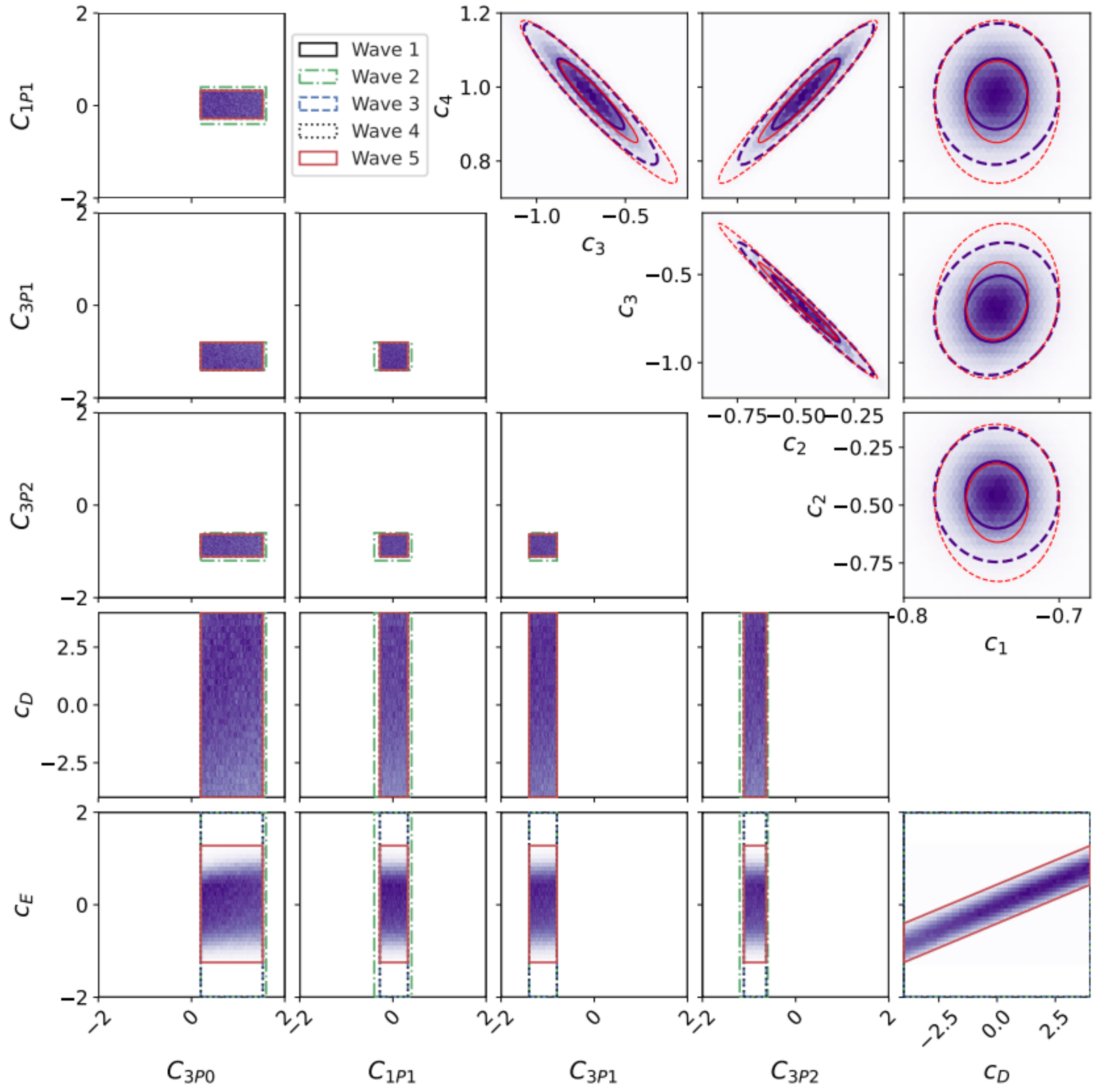
II. Use fast modeling or emulation to compute the implausibility measures for each samples and apply the maximum implausibility constraint.

III. The remaining non-implausible interaction samples are kept and define the non-implausible region for the next wave.



2D Non-implausible region

Applying history matching on delta-full chiral interaction (NNLO with 17 parameters).

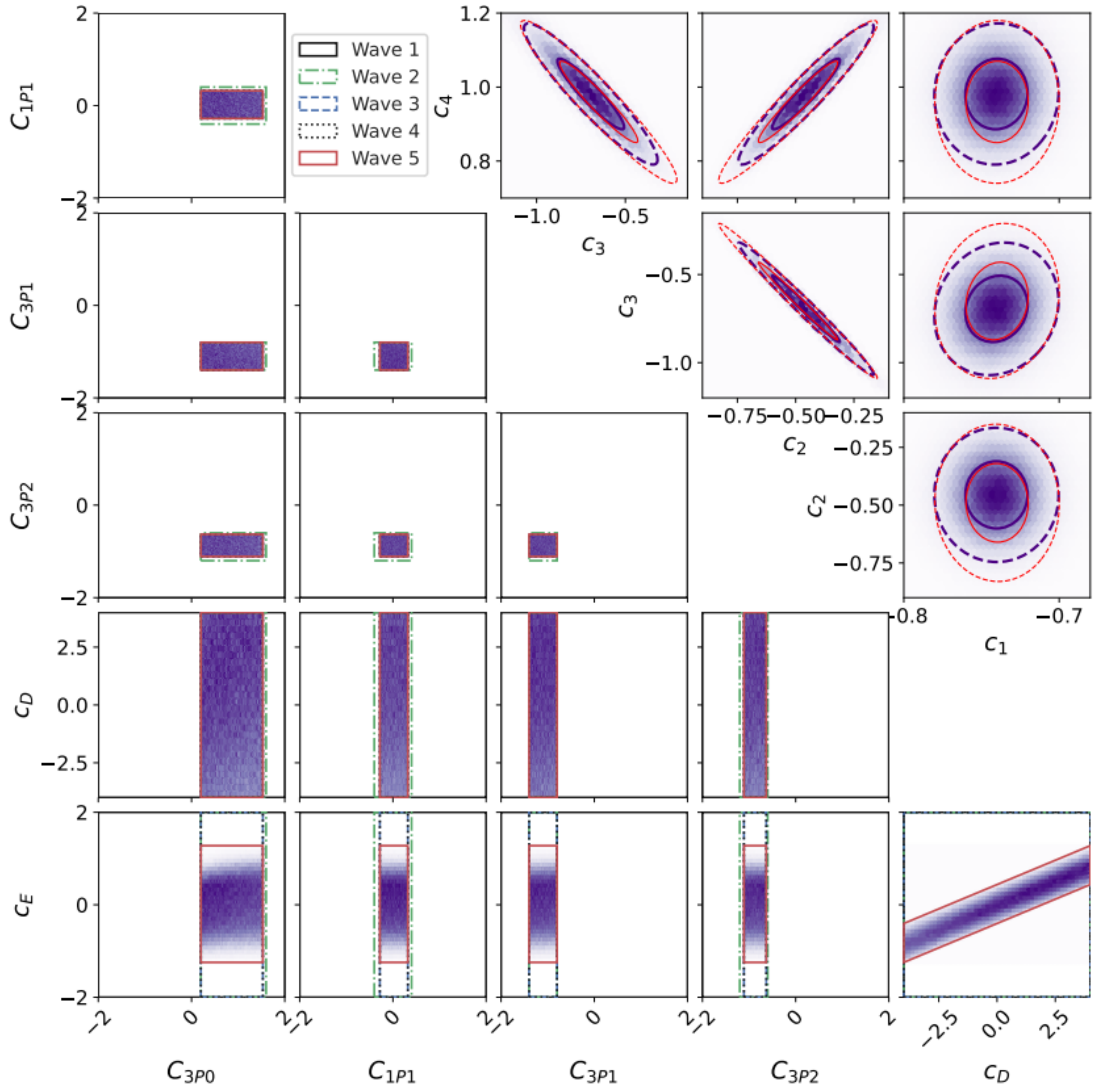


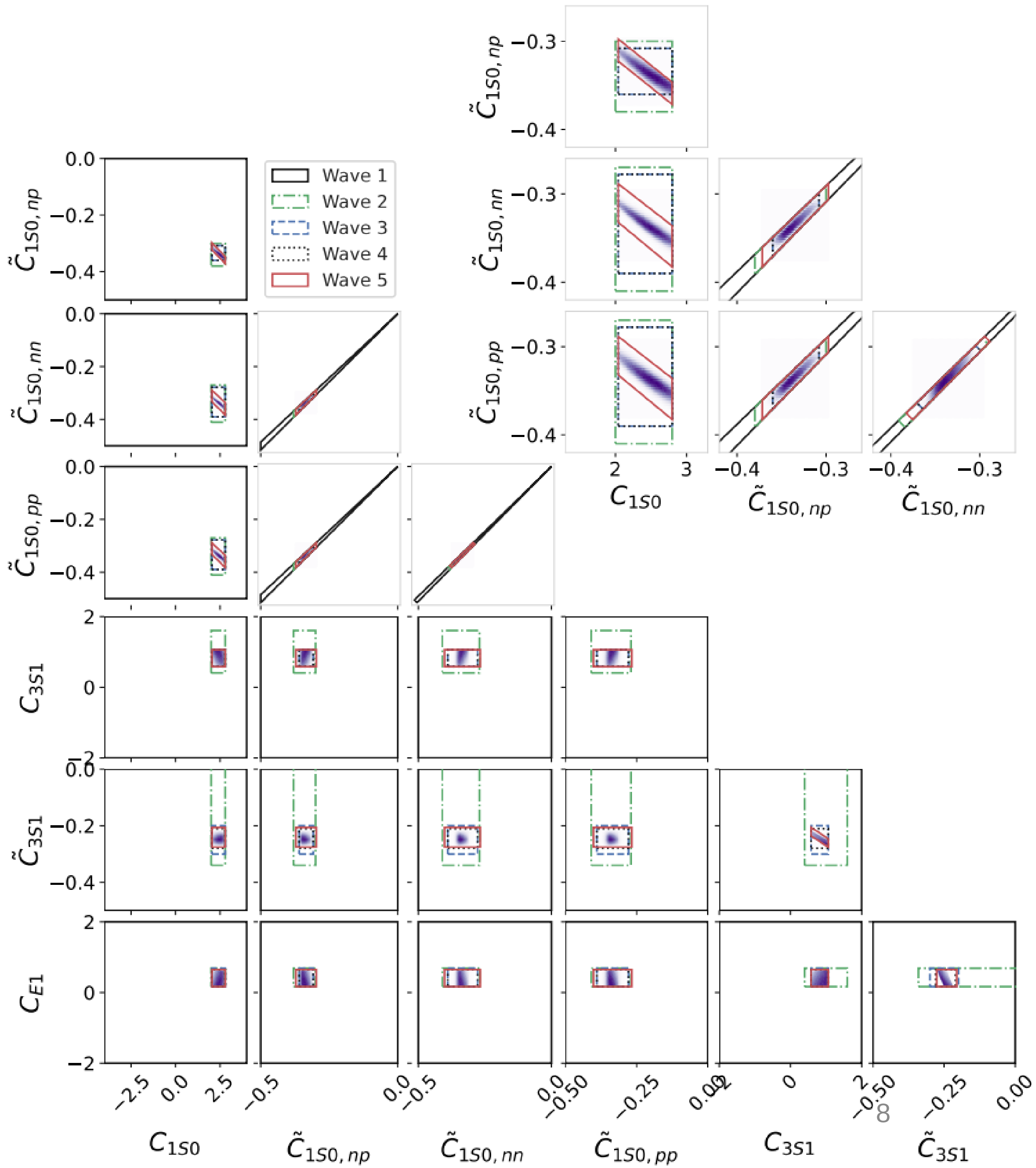
Wave	Target set $\mathcal{Z}$ outputs	Target set $\mathcal{Z}$ systems	Active inputs	Input samples	Non-implausible fraction	Proportion space non-implausible
1	6×6	*np* scattering	5–7	10^6–$2.7 \cdot 10^8$	10^{-1}–10^{-4}	$1.5 \cdot 10^{-6}$
2	6×6	*np* scattering	5–7	10^6–$2.7 \cdot 10^8$	10^{-1}–10^{-4}	$3.7 \cdot 10^{-8}$
3	3	$A = 2$	7	$2.7 \cdot 10^8$	$7 \cdot 10^{-3}$	$2.4 \cdot 10^{-8}$
4	6	$A = 2$–4	13	10^8	$1.3 \cdot 10^{-4}$	$1.0 \cdot 10^{-9}$
5	6	$A = 2$–4	17	10^9	$1.7 \cdot 10^{-3}$	same

- maximum number of interaction probes is limited to 10^8~10^9
- *np* scattering: 5-7 active LECs for a given partial wave
- Deuteron : 7 active LECs

- This domain (the red one) can not be further reduced by more HM iteration.
- We found no disconnected regions outside in this case.

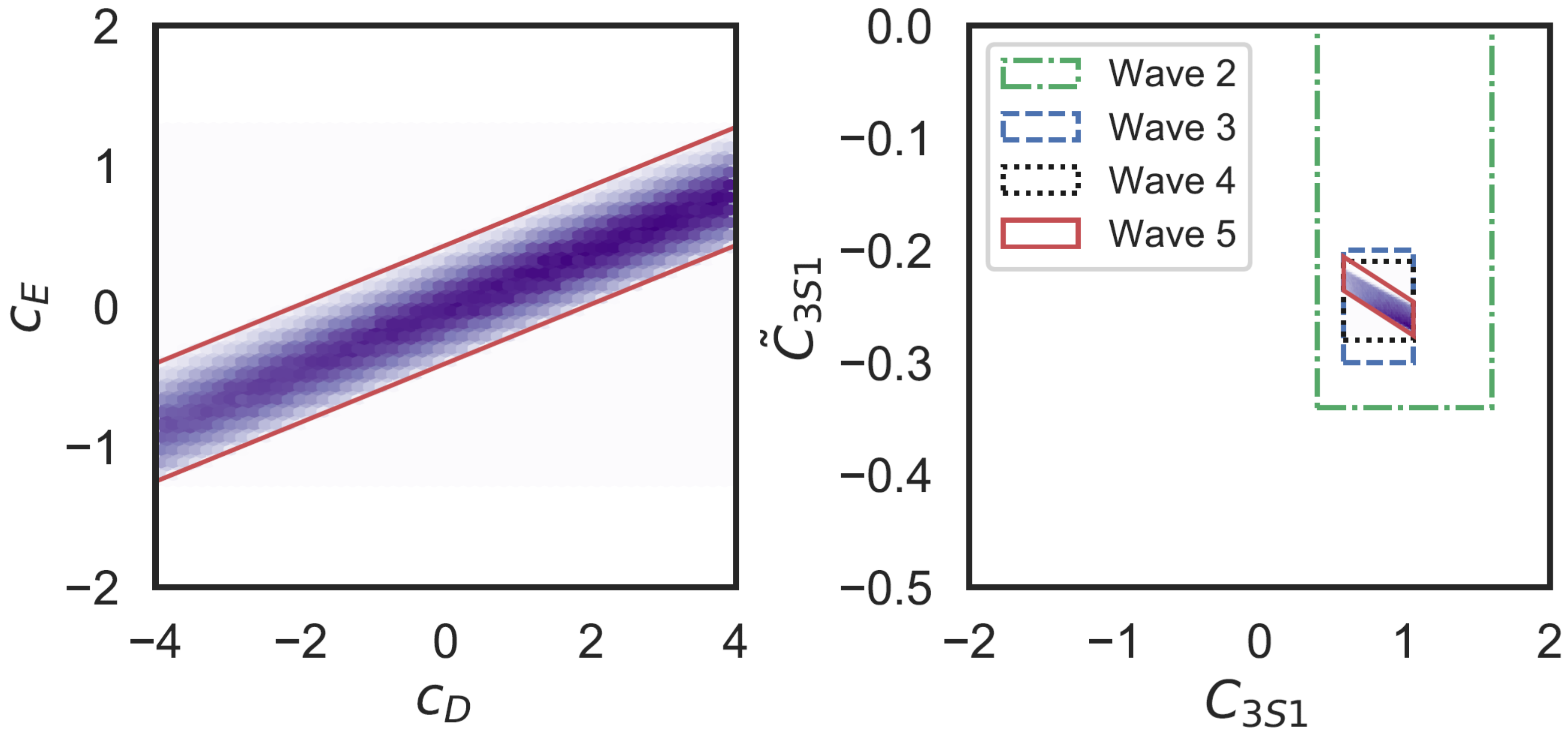
2D Non-implausible region



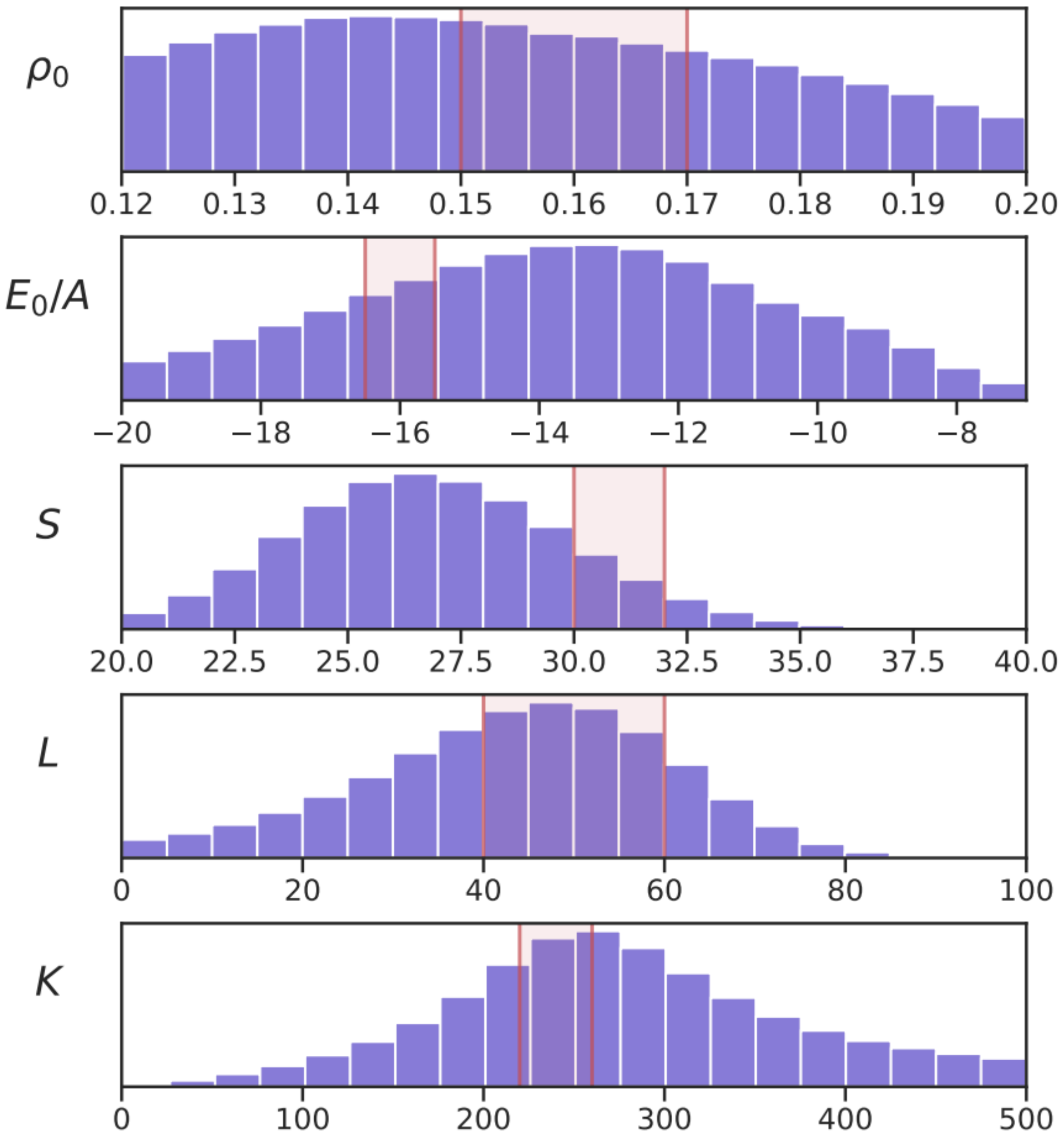


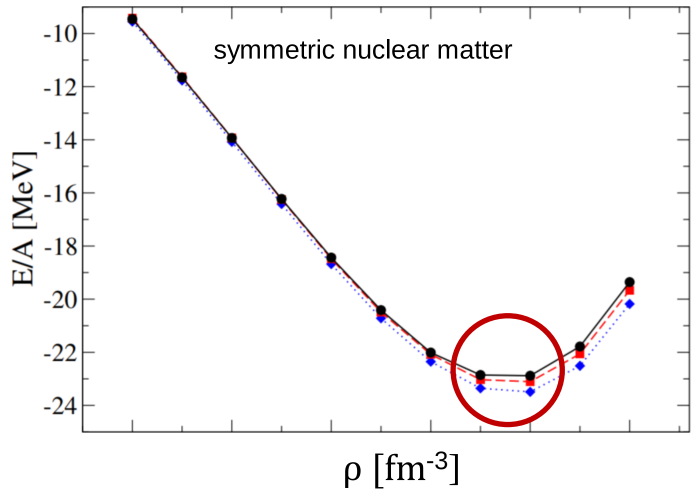
Constraining model parameters

- parameter domain reduced by a factor of 10^7
- strongly correlated LEC pairs
- Only linear combination of contact 3NFs LECs c_D and c_E are constrained by ^{3}H, ^{4}He binding energies and radii



Nuclear matter prediction with non-implausible samples





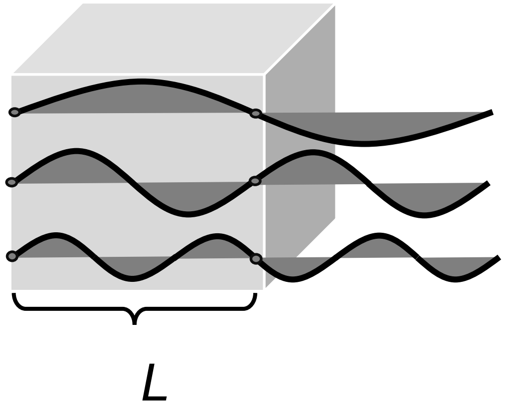
saturation properties

- Target nuclear matter saturation properties:
- Saturation density ρ_0
- Saturation energy E_0/A
- Symmetry energy S
- Others: Slope L, Incompressibility K

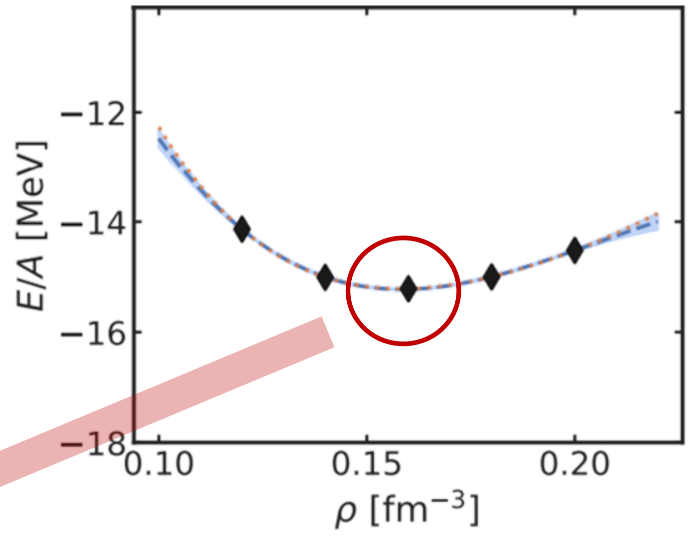
Emulating *ab initio* computations of infinite nucleonic matter

Coupled cluster nuclear matter calculation in momentum space with periodic boundary condition

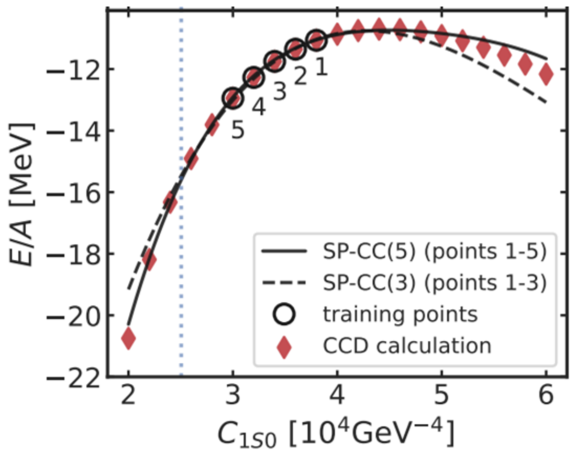
$$H_N \, e^T |\Phi_0\rangle = E \, e^T |\Phi_0\rangle$$

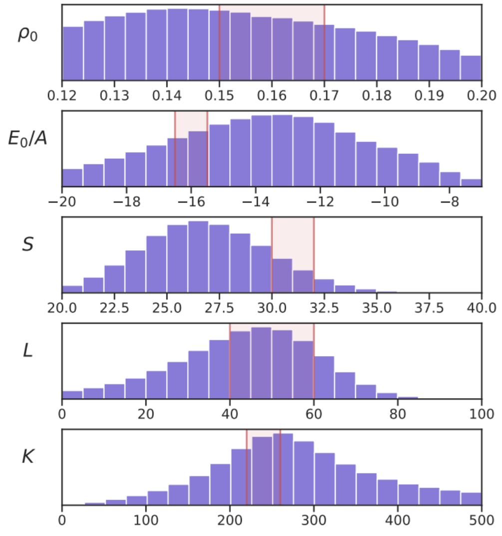


Nuclear matter equation of state for arbitrary interactions



Nuclear matter emulator based on Subspace projected coupled cluster

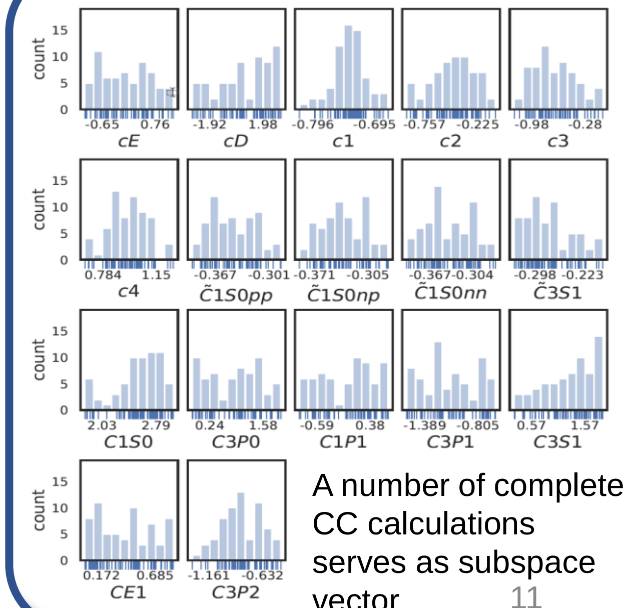




Different saturation properties

Emulator enables 10^6 times acceleration in this case

eg: for SNM (ccd ~200 CPU-hour) vs (emulator ~2ms)

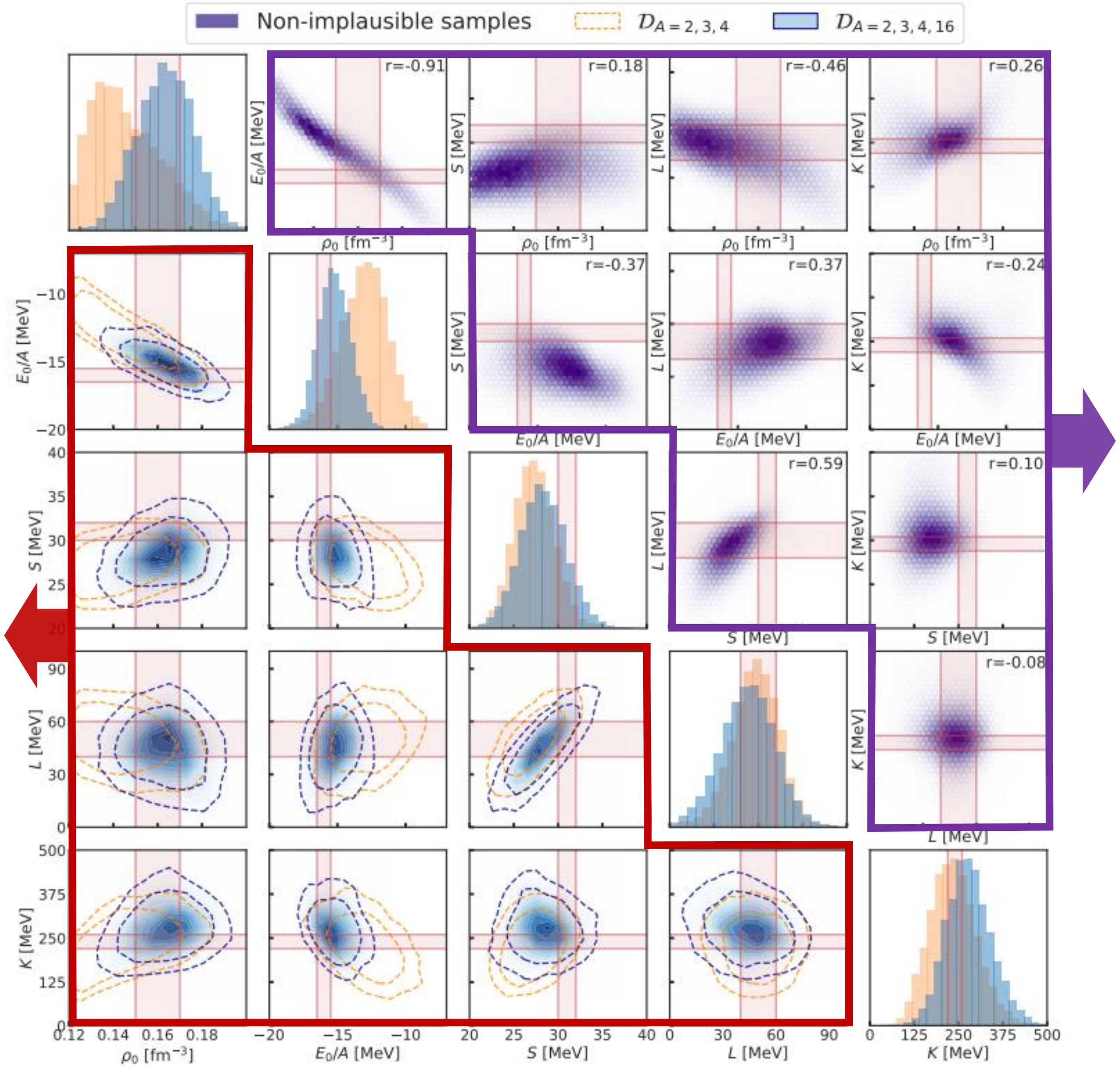


A number of complete CC calculations serves as subspace vector

Bayesian studies build upon history-matching results

Two PPDs with different PDFs:
$D_{A=2,3,4}$, $D_{A=2,3,4,16}$

Note that the same interaction samples are used for different importance resampling stages.

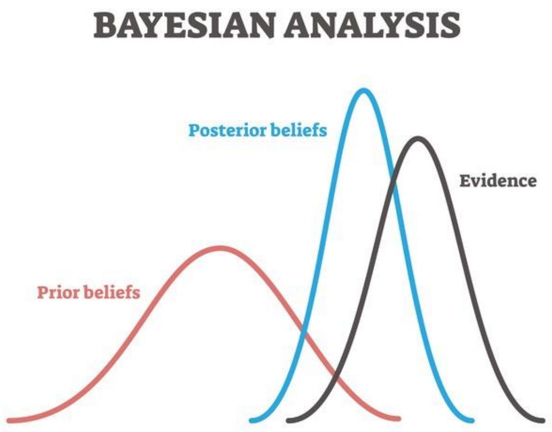


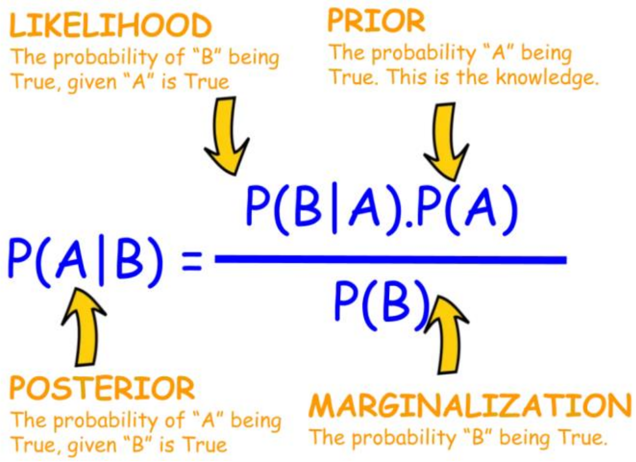
~10^6 Non-implausible interaction samples form history matching

Summary

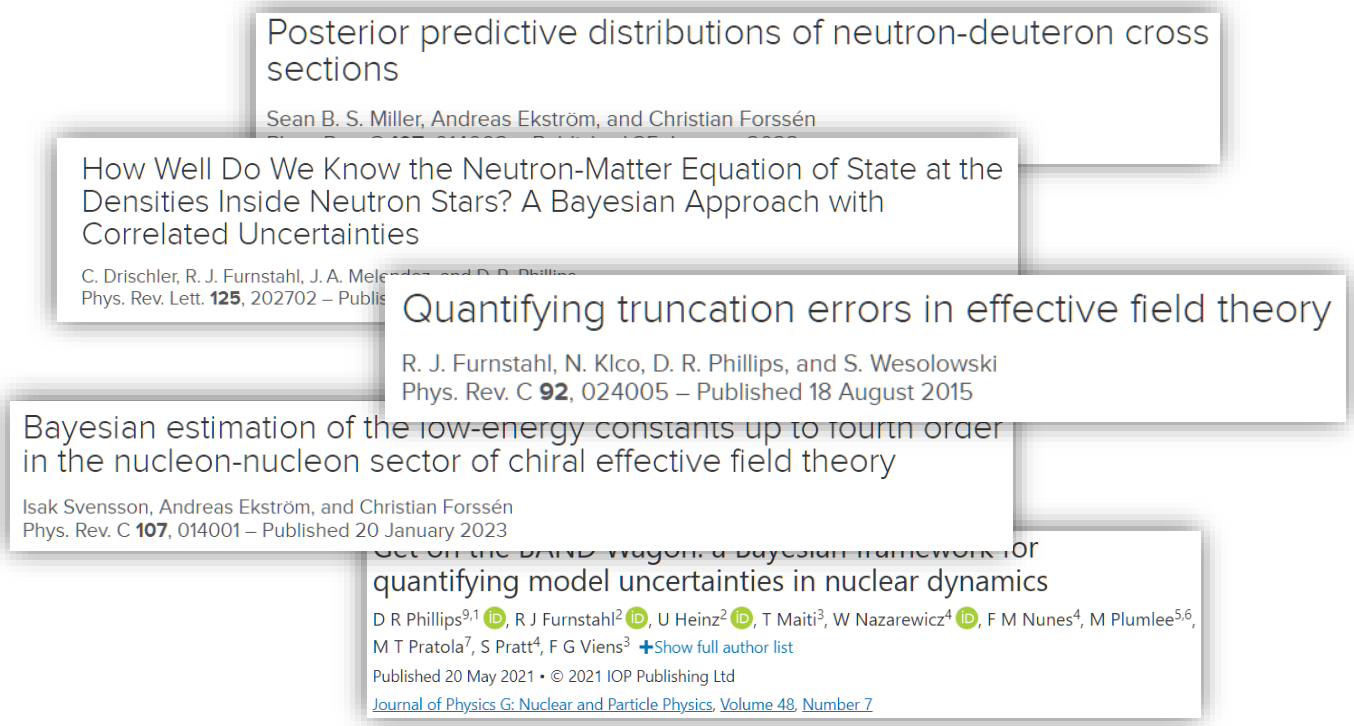
- Use history matching approach to acquire non-implausible domain of low energy constant.
- Reveal correlation structure between different LEC pairs.
- One will likely obtain unsatisfactory predictions of nuclear matter when relying on interaction models constrained by few-body data only.
- Observables that can constrain 3NF effectively are called for.

Backup





Bayesian inference is an appealing approach for dealing with theoretical uncertainties and has been applied in different nuclear physics studies



Posterior predictive distributions of neutron-deuteron cross sections

Sean B. S. Miller, Andreas Ekström, and Christian Forssén

How Well Do We Know the Neutron-Matter Equation of State at the Densities Inside Neutron Stars? A Bayesian Approach with Correlated Uncertainties

C. Drischler, R. J. Furnstahl, J. A. Mele

Phys. Rev. Lett. **125**, 202702 – Publis

Quantifying truncation errors in effective field theory

R. J. Furnstahl, N. Klco, D. R. Phillips, and S. Wesolowski

Phys. Rev. C **92**, 024005 – Published 18 August 2015

Bayesian estimation of the low-energy constants up to fourth order in the nucleon-nucleon sector of chiral effective field theory

Isak Svensson, Andreas Ekström, and Christian Forssén

Phys. Rev. C **107**, 014001 – Published 20 January 2023

quantifying model uncertainties in nuclear dynamics

D R Phillips[9,1], R J Furnstahl[2], U Heinz[2], T Maiti[3], W Nazarewicz[4], F M Nunes[4], M Plumlee[5,6], M T Pratola[7], S Pratt[4], F G Viens[3] +Show full author list

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Journal of Physics G: Nuclear and Particle Physics, Volume 48, Number 7

Bayesian Probability and Sampling/Importance Resampling

Bayesian inference is an excellent framework to incorporate different sources of uncertainty and propagate errors to the model predictions.

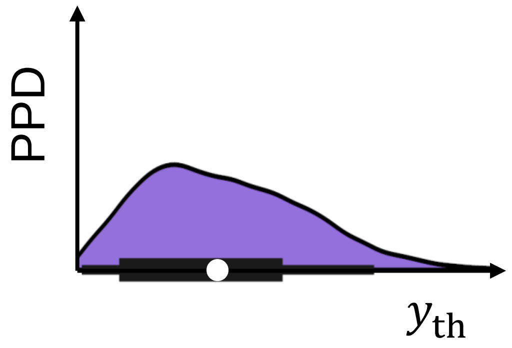
- Posterior probability density function (PDF) in Bayes' theorem :

$$\mathrm{pr}(\theta|\mathcal{D}) \propto \underline{\mathcal{L}(\mathcal{D}|\theta)}\ \underline{\mathrm{pr}(\theta)}$$

Likelihood function (usually not analytical) — Prior

- Posterior predictive distribution (PDD):

$$\mathrm{PDD} = \{y_{\mathrm{th}}(\theta): \theta \sim \mathrm{pr}(\theta|\mathcal{D})\}$$



Prior:
a priori hypothesis of parameterization θ (e.g. LECs under uniform distribution in a certain range)

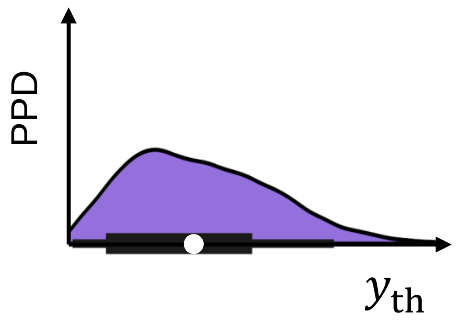
Likelihood:
different sources of uncertainty (EFT truncation error, the many-body method error, experimental error…) go in here

Predicting new observables

- e.g. The expectation value of certain observable $y(\theta)$:

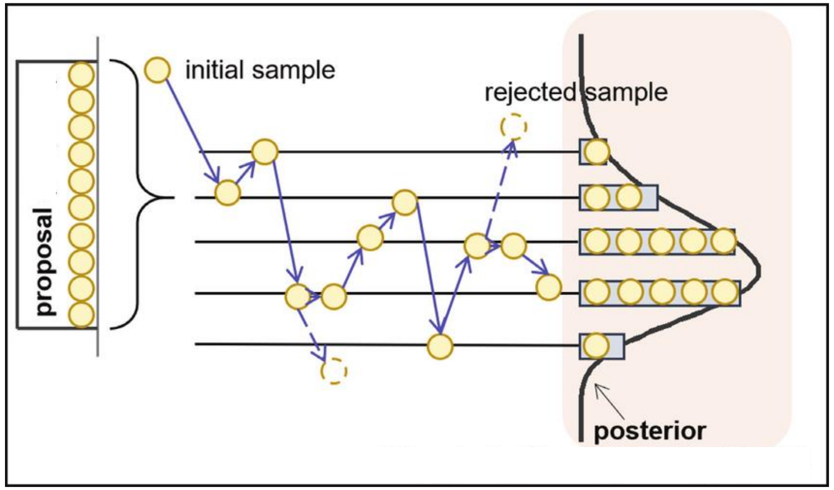
$$\int d\theta\, y(\theta)\, \mathrm{pr}(\theta|\mathcal{D}) \rightarrow \mathrm{mean}(y(\theta_i)) \quad \text{with } \{\theta_i \sim \mathrm{pr}(\theta|\mathcal{D})\}$$

Sampling

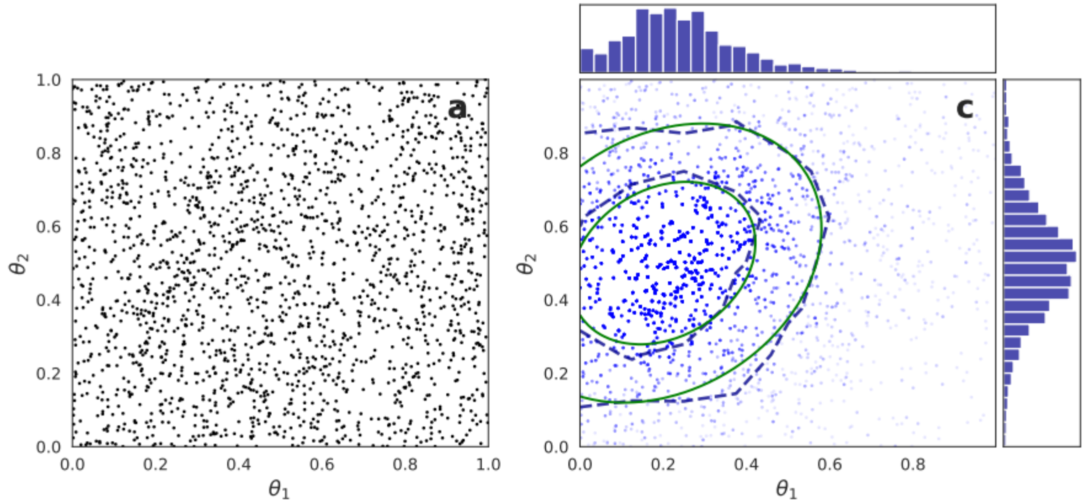


- Sampling method:

  Markov chain Monte Carlo (MCMC), Sampling/Importance Resampling(SIR)…



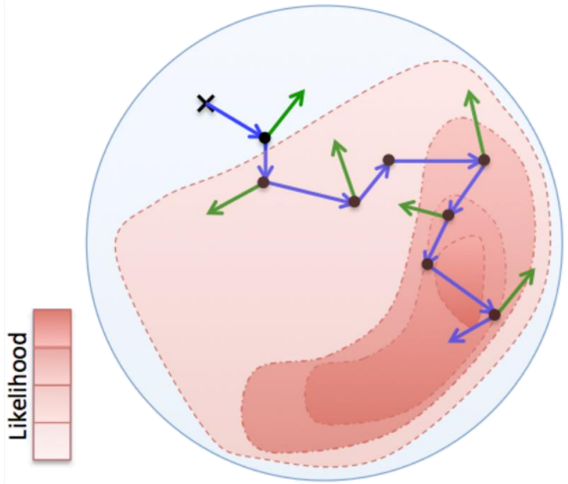
MCMC



SIR

Applications in Nuclear Theory

MCMC sampling typically requires many likelihood evaluations, which is often a costly operation in nuclear theory



MCMC stochastic processes of "walkers"

There are certain situations where MCMC sampling is not ideal or even becomes infeasible:

1) When the posterior is conditioned on some calibration data for which our model evaluations are very costly. Then we might only afford a limited number of full likelihood evaluations.

2) Bayesian posterior updates in which calibration data is added in several different stages. Or in model checking where we want to explore the sensitivity to prior assignments. This typically requires that the MCMC sampling must be carried out repeatedly from scratch.

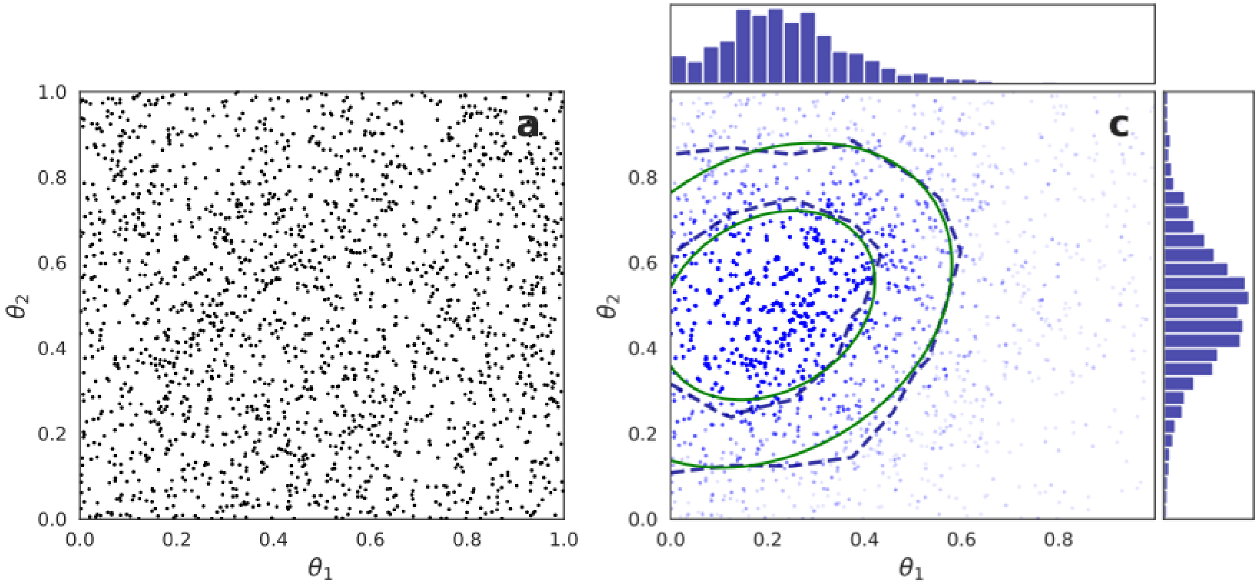
3) Even after we get the pdf using MCMC, the prediction of target observables for which our model evaluations could be very costly and the handling of a large number of MCMC samples becomes infeasible.

Methodology: Sampling/Importance Resampling

The basic idea of SIR is to utilize the inherent duality between samples and the density (probability distribution) from which they were generated

This duality offers an opportunity to indirectly recreate a density (that might be hard to compute) from samples that are easy to obtain.

$$\{\theta_i \sim \mathrm{pr}(\theta|\mathcal{D})\}$$



Bayesian Statistics without Tears: A Sampling-Resampling Perspective
Author(s): A. F. M. Smith and A. E. Gelfand
Source: *The American Statistician*, May, 1992, Vol. 46, No. 2 (May, 1992), pp. 84-88

Methodology: Sampling/Importance Resampling

weighted bootstrap

Assuming we are interested in the target density *h(**θ**)* = *f(**θ**) / ∫ f(**θ**) d**θ***, the procedure of resampling via weighted bootstrap can be summarized as follows:

1) Generate the set $\{\boldsymbol{\theta}_i\}_{i=1}^{n}$ of samples from a sampling density *g(**θ**)*.

2) Calculate $\omega_i = f(\boldsymbol{\theta}_i) / g(\boldsymbol{\theta}_i)$ for the n samples and define importance weights as: $q_i = \omega_i / \sum_{j=1}^{n} \omega_j$.

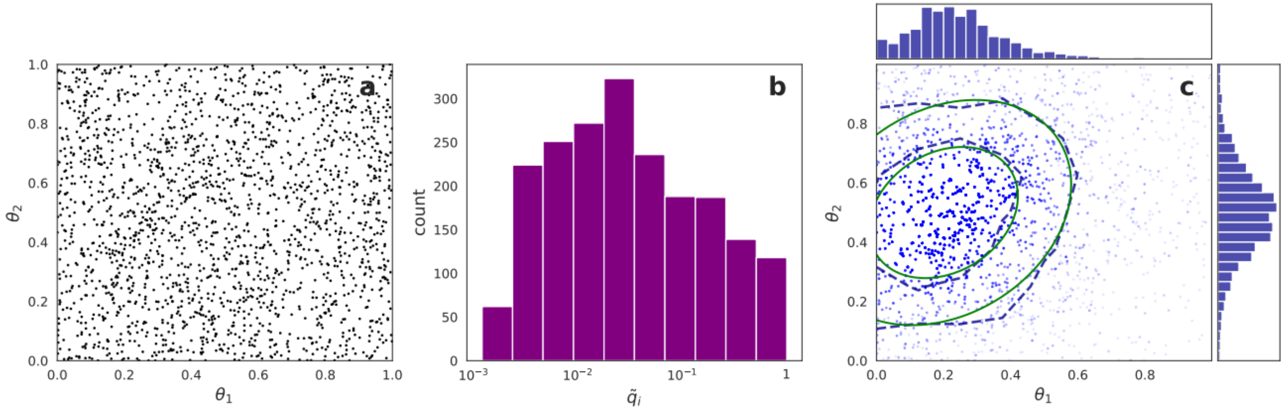
3) Draw *N* new samples $\{\boldsymbol{\theta}_i^*\}_{i=1}^{N}$ from the discrete distribution $\{\boldsymbol{\theta}_i\}_{i=1}^{n}$ with probability mass q_i on $\boldsymbol{\theta}_i$.

4) The set of samples $\{\boldsymbol{\theta}_i^*\}_{i=1}^{N}$ will then be approximately distributed according to the target density *h(**θ**)*.

Intuitively, the distribution of ***θ**** should be good approximation of *h*(***θ***) when *n* is large enough. Here we justify this claim *via* the cumulative distribution function of *θ** (for the one-dimensional case)

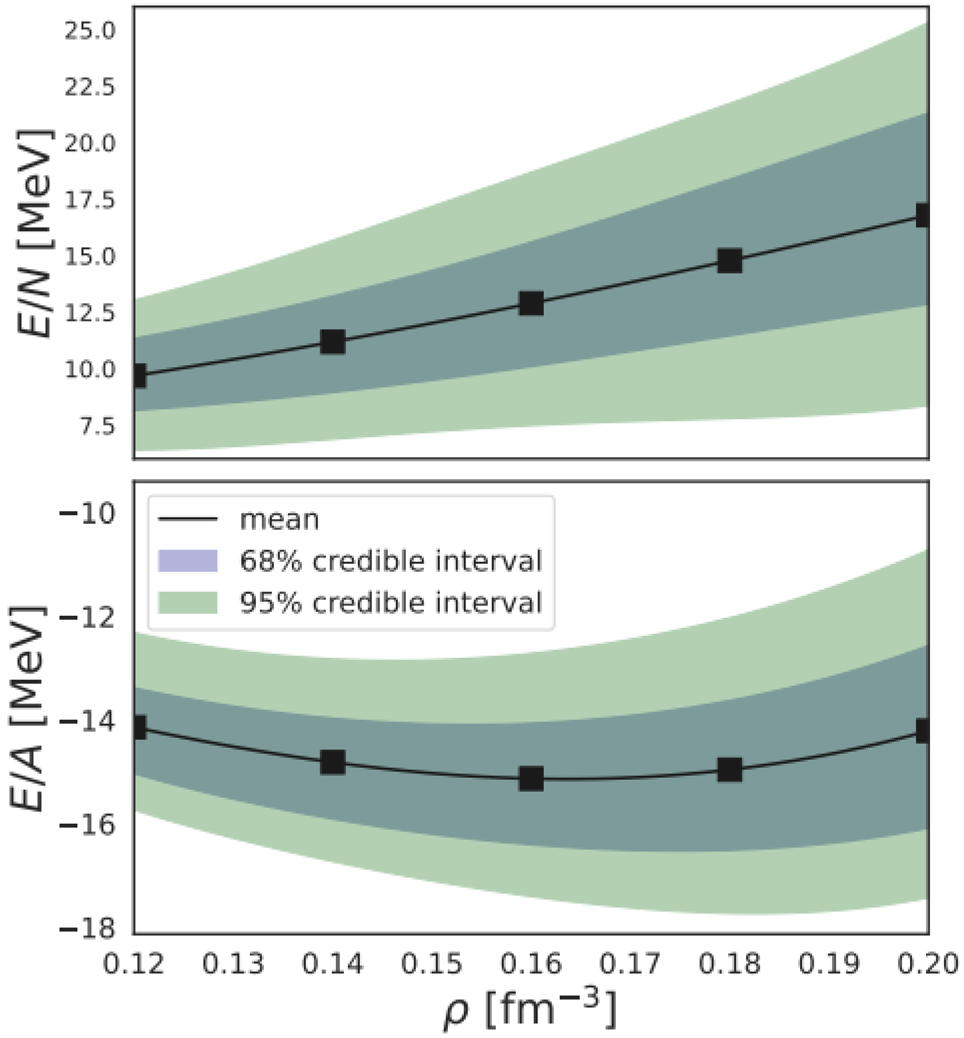
$$\mathrm{pr}\left(\theta^* \leq \mathrm{a}\right) = \sum_{i=1}^{n} q_i \cdot H\left(a - \theta_i\right) = \frac{\frac{1}{n}\sum_{i=1}^{n} \omega_i \cdot H(a-\theta_i)}{\frac{1}{n}\sum_{i=1}^{n} \omega_i}$$

$$\xrightarrow{n \to \infty} \frac{\mathbb{E}_g\left[\frac{f(\theta)}{g(\theta)} \cdot H(a-\theta)\right]}{\mathbb{E}_g\left[\frac{f(\theta)}{g(\theta)}\right]} = \frac{\int_{-\infty}^{a} f(\theta)\, d\theta}{\int_{-\infty}^{\infty} f(\theta)\, d\theta} = \int_{-\infty}^{a} h\left(\theta\right) d\theta,$$



Sampling (from "simple" distribution) → Importance weights → Importance resampling

Application – nuclear matter



The PPD for the EOS around saturation density

Convergence test