Wilsonian RG with a multitude of cutoffs applied to halo EFT

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Outline

- Demonstrating the usage of RG in PT
- RG and LS equation;
- Chiral EFT for P-wave halo states and RG;
  - Subtractive renormalization;
  - Wilsonian RG with two cutoffs;
- Summary;

Talk based on
E. Epelbaum, J. Gegelia, H. P. Huesmann, U.-G. Meißner and X. L. Ren, Few Body Syst. **62**, no.3, 51 (2021),
B. Mindadze, E. Epelbaum and J. Gegelia, Few Body Syst. **64**, no.2, 35 (2023)

Demonstrating the usage of RG in PT

The main idea of using the RG to our advantage in PT can be shown in a demonstrating example as follows:

Let a physical quantity be given by exact expression

$$f(x) = \frac{x}{1 - \hbar x}\,,$$

where x is a parameter and $\hbar$ controls the quantum corrections.

Suppose for whatever reason we calculate this quantity in PT order-by-order using a power series expansion.

If $|x| < 1$ we can expand $f(x)$ in Taylor series around $x = 0$ and approximate the exact function by the sum of first N terms

$$f(x) \approx S_N = x + \hbar x^2 + \hbar^2 x^3 + \cdots + \hbar^{N-1} x^N\,.$$

For $|x| > 1$ our expansion leads to a divergent series. In this case we can try an alternative way by rewriting $f(x)$ identically and expanding in a different way:

$$\begin{aligned} f(x) &= \frac{x}{1-\hbar x} \equiv \frac{x}{1-\hbar\mu x - \hbar x(1-\mu)} \\ &= \frac{1}{1-\mu}\,\frac{x_\mu}{1-\hbar x_\mu} = \frac{x_\mu}{1-\mu}\left(1+\hbar x_\mu + \hbar^2 x_\mu^2 + \cdots\right), \end{aligned}$$

where $x_\mu = x(1-\mu)/(1-\hbar\mu x)$.

The exact expression of $f(x)$ is μ-independent, however the sum of any finite number of terms depends on μ.

While formally this dependence is of higher order $\sim \hbar^{N+1}$ for the sum of the first N terms, the convergence of the series crucially depends on the choice of μ.

For example, for $x = 2$ this series converges only if $\mu > 3/4$, the convergence being best close to $\mu = 1$.

RG and LS equation

Consider an integral equation:

$$T(p',p) = V(p',p) + \int_0^\infty dk V(p',k) G(k) T(k,p)\,. \tag{1}$$

Realization of RG *a la* Wilson: Modify the equation and the potential without changing $T(p',p)$:

$$T(p',p) = \tilde{V}(p',p,\Lambda) + \int_0^\infty dk\, \tilde{V}(p',k,\Lambda) G(k) f(\Lambda,k) T(k,p)\,. \tag{2}$$

The solutions of the original and modified equations are identical for any choice of Λ provided that $\tilde{V}(p',p,\Lambda)$ satisfies RG equation.

In PT the sum of any finite number of terms depends on Λ.

One chooses such values of the scale parameter which lead to optimal convergence of perturbative series.

If perturbative expressions were (approximately) RG invariant, then the RG in this context would be useless.

Chiral EFT for P-wave halo states and RG

C.A. Bertulani, H.-W. Hammer, U. Van Kolck, Nucl.Phys. A712, 37-58 (2002) .

Consider two non-relativistic particles with range of interaction $R \sim 1/M_{\rm hi}$.

ERE for the orbital angular momentum l:

$$T(k) \propto \frac{1}{k\cot\delta - ik} \simeq \frac{k^{2l}}{\left(-1/a + 1/2\, r\, k^2 + v_2 k^4 + \ldots\right) - ik^{2l+1}}\,,$$

where a, r and v_i are the scattering length, effective range and the shape parameters.

If $k^{2l+1}\cot\delta$ does not have poles for small k, the coefficients in the ERE starting from r are expected to be natural
$r \sim M_{\rm hi}^{2l-1}$, $v_2 \sim M_{\rm hi}^{2l-3}$, ... etc.,
while the scattering length a can take any value.

We consider the EFT for P-waves for momenta $k \sim M_{\rm lo} \ll M_{\rm hi}$.

We are interested in fine-tuned systems, for which the scattering amplitude has poles located within the validity range of the EFT.

Assume that the first two terms in the ERE are fine tuned as

$$1/a \sim M_{\rm lo}^3\,, \qquad r \sim M_{\rm lo}\,, \qquad v_n \sim M_{\rm hi}^{3-2n}\,. \tag{3}$$

In low-energy EFT with contact interactions only the two lowest-order contact interactions in the effective potential

$$V = C_2\, p' p + C_4\, p' p \left(p'^2 + p^2\right) + \dots\,, \tag{4}$$

need to be iterated in the LS equation to all orders.
$p \equiv |\vec{p}\,|$ and $p' \equiv |\vec{p}\,'|$ refer to the initial and final momenta of the particles in the center-of-mass system,

We solve

$$T(p',p) = V(p',p) + m\int_0^\Lambda \frac{l^2 dl}{2\pi^2}\,\frac{V(p,l)\;T(l,p')}{k^2 - l^2 + i\,\epsilon}\,, \tag{5}$$

and obtain for the on-shell amplitude $T(k) \equiv T(k,k)$:

$$\frac{k^2}{T(k)} = -I(k)\,k^2 - I_3 + \frac{(C_4 I_5 - 1)^{\,2}}{C_4\left(k^2\,(2 - C_4 I_5) + C_4 I_7\right) + C_2}\,. \tag{6}$$

where the integrals I_n and $I(k)$ are defined via

$$\begin{aligned} I_n &= -m\int_0^\Lambda \frac{l^2 dl}{2\pi^2}\, l^{n-3}, \quad n = 1,3,5,\ldots\,, \\ I(k) &= \int_0^\Lambda \frac{l^2 dl}{2\pi^2}\,\frac{m}{k^2 - l^2 + i\epsilon}\,. \end{aligned} \tag{7}$$

Renormalization and RG can be implemented *a la* Gell-Mann and Low or *a la* Wilson.

Subtractive renormalization

We renormalize the amplitude by applying BPHZ procedure, i.e. subtracting *all* UV divergences prior to taking the limit $\Lambda \to \infty$.

Specifically, we first separate out power-like UV divergences in the appearing integrals in the most general way via

$$\begin{aligned} I_n &= -m \int_0^{\mu_n} \frac{l^2 dl}{2\pi^2} l^{n-3} - m \int_{\mu_n}^{\Lambda} \frac{l^2 dl}{2\pi^2} l^{n-3} \equiv I_n^R(\mu_n) + \Delta_n(\mu_n)\,, \\ & \quad \text{with} \quad n = 1, 3, 5, \ldots\,, \\ I(k) &\equiv I^R(k, \mu_1) - \Delta_1(\mu_1)\,, \end{aligned}$$

where μ_n denote the corresponding subtraction scales.

We renormalize the amplitude by replacing the integrals I_n and $I(k)$ with $I_n^R(\mu_n)$ and $I^R(k, \mu_1)$ and the bare couplings C_2 and C_4 by the corresponding μ_n-dependent renormalized couplings C_2^R and C_4^R.

Since the renormalized amplitude depends only on UV-convergent integrals, we can now safely take the limit $\Lambda \to \infty$.

Fixing the renormalized LECs by the requirement to reproduce a and r leads to our final result:

$$k^3 \cot\delta = -\frac{1}{a} + \frac{1}{2} r k^2 - \frac{3k^4}{2\pi} \frac{(4\mu_1 + \pi r)^2}{6\pi a^{-1} - 4\mu_3^3 + 3k^2(4\mu_1 + \pi r)} \,.$$

The renormalized scattering amplitude depends on μ_1 and μ_3.

The choice of μ_i plays a key role in setting up a self-consistent power counting.

For the resonant P -wave scattering the choice of renormalization conditions is rather delicate due to the strong fine tuning.

Indeed, one *must* choose $\mu_3 \sim M_{\rm hi}$ since setting $\mu_3 \sim M_{\rm lo}$ would lead to poles in the effective range function located at $k \sim M_{\rm lo}$, thereby resulting in enhanced values of the coefficients in the ERE.

Consequently, no KSW-like scheme is possible for resonant P-wave systems under consideration.

A self-consistent Weinberg-like scheme with manifest power counting for renormalized loop diagrams and all LECs scaling according to NDA emerges if we set $\mu_5 \sim \mu_7 \sim \mu_9 \sim \ldots \sim M_{\rm lo}$. The scale μ_1 can be chosen either as $\mu_1 \sim M_{\rm hi}$ or $\mu_1 \sim M_{\rm lo}$.

Wilsonian RG with two cutoffs

Using the approach of
E. Epelbaum, J. Gegelia and U. G. Meißner, Commun. Theor. Phys. **69**, no.3, 303 (2018)
we rewrite the potential as

$$V = (C_2 + 2C_4k^2)pp' + C_4pp'(p^2 - k^2 + p'^2 - k^2)\,,$$

and introduce two cutoffs via

$$\begin{aligned} V &= (C_2 + 2C_4k^2)pp'\theta(\Lambda_1 - p)\theta(\Lambda_1 - p') \\ &+ C_4pp'\theta(\Lambda_1 - p)\theta(\Lambda_1 - p') \\ &\times \left[(p^2 - k^2)\theta(\Lambda_2 - p) + (p'^2 - k^2)\theta(\Lambda_2 - p')\right]\,, \end{aligned}$$

where it is implied that $\Lambda_1 \geq \Lambda_2$.

This potential can be represented in a separable form:

$$
\begin{aligned}
V &= \left(\, p'\theta(\Lambda_1 - p'), \quad p'(p'^2 - k^2)\theta(\Lambda_2 - p') \, \right) \\
&\times \left(\begin{array}{cc} C_2 + 2C_4k^2, & C_4 \\ C_4, & 0 \end{array} \right) \left(\begin{array}{c} p\theta(\Lambda_1 - p) \\ p(p^2 - k^2)\theta(\Lambda_2 - p) \end{array} \right),
\end{aligned}
$$

and therefore the corresponding LS equation for the scattering amplitude can be straightforwardly solved analytically.

Matching the solution to the ERE we fix the LECs C_2 und C_4:

$$
\begin{aligned}
C_2 &= \frac{1}{350 m\pi^2 (3\pi - 2a\Lambda_1^3)} \Big\{ 75 C_4^2 m^2 \pi \Lambda_2^7 + a\big[4200\pi^4 \\
&\quad + 840 C_4 m \pi^2 \Lambda_2^5 + 2 C_4^2 m^2 \Lambda_2^7 (21\Lambda_2^3 - 25\Lambda_1^3)\big]\,, \\
C_4 &= \frac{10\sqrt{5}\pi^2 (3\pi - 2a\Lambda_1^3)^2}{m\Lambda_2^5 \sqrt{(3\pi - 2a\Lambda_1^3)^2\, \alpha(\Lambda_1, \Lambda_2)}} - \frac{10\pi^2}{m\Lambda_2^5} \Big\}\,.
\end{aligned}
$$

The LECs C_2 and C_4 must be real, therefore the argument of the square root has to be non-negative.

This leads to the condition

$$
\alpha(\Lambda_1, \Lambda_2) \equiv 45\pi^2 + 4a^2\Lambda_1 (5\Lambda_1^5 - 9\Lambda_2^5) - 3a\pi(20\Lambda_1^3 + 3ar\Lambda_2^5) \geq 0\,.
$$

For two independent cutoffs Λ_1 and Λ_2, the condition that $\alpha(\Lambda_1, \Lambda_2)$ has to be non-negative can be satisfied for any values of a and r.

To check the convergence of the ERE we subtract $-\frac{1}{a} + \frac{1}{2} r k^2$ from the calculated expression of $k \cot \delta$ and obtain the remainder:

$$S_{Rest} = \frac{k^3}{2\pi}\left(-\frac{3ak(\pi r + 4\Lambda_1)^2}{3\pi(2 + ak^2 r) - 4a\Lambda_1(\Lambda_1^2 - 3k^2)} - 2\ln\frac{\Lambda_1 - k}{\Lambda_1 + k}\right).$$

The second term in the bracket has a convergent expansion in k^2 for $\Lambda_1 \gg k$ and the expansion of the first term converges if

$$-1 < \frac{3\,(\pi\, r + 4\,\Lambda_1)}{6\,\pi/a - 4\,\Lambda_1^3}\, k^2 < 1\,.$$

By taking sufficiently large Λ_1 this condition can always be fulfilled. For considered system this amounts to taking $\Lambda_1 \sim M_{\rm hi}$ or larger.

By taking $\Lambda_1 \sim M_{\rm hi}$ and $\Lambda_2 \sim M_{\rm lo}$, we find that $C_2 \sim 1/(mM_{\rm hi}^3)$ and $C_4 \sim 1/(mM_{\rm hi}^5)$, i.e. both are of natural size.

For $\Lambda_1 = \Lambda_2 = \Lambda$ we have

$$\alpha(\Lambda, \Lambda) = 45\pi^2 - 16a^2\Lambda^6 - 9\pi a^2 r\Lambda^5 - 60\pi a\Lambda^3 \,,$$

which turns negative for sufficiently large Λ values and, therefore, the LECs C_2 and C_4 become complex.

For our system the cutoff Λ cannot be taken larger than $\sim M_{\rm lo}$. This observation is in line with the causality bound $r \leq -2/R\,(1 + \mathcal{O}(R^3/a))$ obtained in

H. W. Hammer and D. Lee, Annals Phys. **325**, 2212-2233 (2010).

if the range of the interaction R is identified with $1/\Lambda$.

Renormalization versus "peratization":

E. Epelbaum and J. Gegelia, “Regularization, renormalization and ’peratization’ in effective field theory for two nucleons,” Eur. Phys. J. A **41**, 341-354 (2009)

Summary

- The presence of shallow P-wave states demands resummation of two contact interactions $C_2 p' p$ and $C_4 p' p(p'^2 + p^2)$ in halo EFTs.
- Expressing the bare LECs $C_2(\Lambda)$ and $C_4(\Lambda)$ in terms of a and r, we found no real LECs if $\Lambda \sim M_{\rm hi}$ or larger.
- We have renormalized the amplitude using BPHZ scheme. The residual dependence of the amplitude on renormalization scales μ_i appears beyond the actual order of the calculation.
- Wilsonian approach with multiple cutoffs leads to the results equivalent to the ones of the subtractive renormalization.