Discrete scale-invariant boson-fermion duality in one dimension

Satoshi Ohya (Nihon University)

Phys. Rev. A105 (2022) 033312, arXiv:2110.09723 [quant-ph]

25th European Conference on Few-Body Problems in Physics
July 30–August 4, 2023, Mainz

Introduction

- Today I am going to talk about discrete scale invariance in N-body problems of identical particles in one dimension.
- Before doing this, I will first discuss the impact of scale invariance in quantum many-body problems.

Discrete scale-invariant S-matrix theory: A toy example

- Consider a 1×1 S-matrix $S(E) \in U(1)$, where E stands for energy.
- The most general scaling law consistent with unitarity $|S(E)| = 1$ is

$$S(E) = S(\mathrm{e}^{t} E) \quad (t : \text{real parameter}) \tag{1}$$

- Depending on the range of t, there exist two types of solutions to eq. (1):
- Case $t \in \mathbb{R} = (-\infty, \infty)$: Continuous scale invariance. In this case, there is only a constant solution:

$$S(E) = \text{const} \tag{2}$$

  Hence, in continuous scale-invariant theory, S-matrix must be trivial.
- Case $t \in t_* \mathbb{Z} = \{0, \pm t_*, \pm 2t_* \cdots\}$: Discrete scale invariance. In this case, S-matrix can be nontrivial. In fact, as we will see next, we can prove the following:
  - log-periodicity of $S(E)$
  - emergence of geometric sequence of bound states

- Log-periodicity of S-matrix. First, observe that the scale transformation for E is equivalent to the constant shift for $\log E$:

$$E \mapsto \mathrm{e}^{nt_*} E \quad \Leftrightarrow \quad \log E \mapsto \log E + nt_* \tag{3}$$

The general solution to the scaling law $S(E) = S(\mathrm{e}^{nt_*} E)$ is therefore

$$S(E) = f(\log E) \tag{4}$$

where $f(x) = f(x + t_*)$ is a periodic function with the period t_*. Hence, in discrete scale-invariant theory, S-matrix must be a periodic function of $\log E$.

- Geometric sequence of bound states. Suppose that $S(E)$ has a simple pole (bound-state pole) at $E = -E_*$:

$$S(E) = \frac{N_*}{E + E_*} + O(1) \quad \text{as} \quad E \to -E_* \tag{5}$$

Then, the scaling law implies there exist infinitely many poles of the form:

$$S(E) = S(\mathrm{e}^{nt_*} E) = \frac{N_* \, \mathrm{e}^{-nt_*}}{E + E_* \, \mathrm{e}^{-nt_*}} + O(1) \tag{6}$$

This implies the existence of infinitely many bound states with the binding energies $E_n = -E_* \, \mathrm{e}^{-nt_*}$, which satisfy $E_{n+1} = E_n \, \mathrm{e}^{-t_*}$. Hence, in discrete scale-invariant theory, bound states must form a geometric sequence.

- As we have seen, typical predictions of discrete scale-invariant theory are:
  - periodic oscillation of S-matrix as a function of $\log E$
  - geometric sequence of bound states
- And a typical example realizing these features is the Efimov effect for three identical bosons with two-body short-range interactions **[Efimov '70]**, where the three-body bound-state energies satisfy

$$E_{n+1} = E_n \, \mathrm{e}^{-t_*}, \quad \mathrm{e}^{-t_*} \approx (22.7)^{-2} \tag{7}$$

  This result is independent of the details of interactions and hence universal.
- Note, however, that the appearance of the Efimov effect highly depends on spatial geometry and particle statistics.
- For example, for systems of identical bosons with two-body contact interactions, it was shown that the Efimov effect appears only if the spatial dimension d is in the range $2.3 < d < 3.8$ **[Nielsen-Fedorov-Jensen-Garrido '01]**.
- For systems of identical fermions with two-body contact interactions, the Efimov effect was not realized in lower dimensions.

- In this work, I revisited N-body problems of identical particles in one dimension, where interparticle interactions are only two-body contacts.
- First, I classified all possible two-body contact interactions that respect:
  - unitarity (probability conservation)
  - permutation invariance (indistinguishability of identical particles)
  - translation invariance (total momentum conservation)
  - scale invariance

  [Note: I did not impose the cluster-decomposition property, which was (implicitly) assumed in the previous works.]
- Then, I showed that, for both bosonic and fermionic systems, continuous scale invariance can be broken to discrete scale invariance for any $N \geq 3$.
- Further, I derived the exact N-body bound-state spectrum as well as the exact N-body S-matrix elements for any $N \geq 3$.
- In the rest of the talk, I will explain these results briefly.
- The key is the boson-fermion duality.

Boson-fermion duality in one dimension

- In one dimension, any bosonic N-body problem has its fermionic dual.
- Typical examples are the following:
  - **Lieb-Liniger model** (bosonic model) [Lieb-Liniger '63]

$$H = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2}{m}\sum_{1\le j<k\le N}\delta(x_j - x_k; g_{\mathsf{B}}) \tag{8}$$

  - **Cheon-Shigehara model** (fermionic model) [Cheon-Shigehara '98]

$$H = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2}{m}\sum_{1\le j<k\le N}\varepsilon(x_j - x_k; g_{\mathsf{F}}) \tag{9}$$

  Here $\delta(x; g_{\mathsf{B}})$ and $\varepsilon(x; g_{\mathsf{F}})$ are defined by

$$\delta(x; g_{\mathsf{B}}) = g_{\mathsf{B}}\delta(x) \tag{10a}$$

$$\varepsilon(x; g_{\mathsf{F}}) = \lim_{a\to 0}\left(\frac{1}{2g_{\mathsf{F}}} - \frac{1}{2a}\right)(\delta(x+a) + \delta(x-a)) \tag{10b}$$

- When $g_{\mathsf{B}} = 1/g_{\mathsf{F}}$, these models become equivalent and satisfy (i) spectral equivalence, (ii) boson-fermion mapping, and (iii) strong-weak duality.

- In one dimension, we can classify all possible two-body contact interactions that respect unitarity, permutation invariance, and translation invariance. The result is **[Ohya '21]**

$$H_{\mathsf{B/F}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + V_{\mathsf{B/F}}(x_1, \cdots, x_N) \tag{11}$$

where

$$V_{\mathsf{B}} = \frac{\hbar^2}{m}\sum_{j=1}^{N-1}\sum_{\sigma\in A_N}\left[\prod_{k\in\{1,\cdots,N-1\}\setminus\{j\}}\theta(x_{\sigma(k)} - x_{\sigma(k+1)})\right]\delta(x_{\sigma(j)} - x_{\sigma(j+1)}; g_{\mathsf{B}j}) \tag{12a}$$

$$V_{\mathsf{F}} = \frac{\hbar^2}{m}\sum_{j=1}^{N-1}\sum_{\sigma\in A_N}\left[\prod_{k\in\{1,\cdots,N-1\}\setminus\{j\}}\theta(x_{\sigma(k)} - x_{\sigma(k+1)})\right]\varepsilon(x_{\sigma(j)} - x_{\sigma(j+1)}; g_{\mathsf{F}j}) \tag{12b}$$

Here A_N stands for the alternating group of order $N!/2$.

- When $g_{\mathsf{B}j} = 1/g_{\mathsf{F}j}$, these models become dual to each other. Further, if $g_{\mathsf{B/F}j}$ has a certain coordinate dependence, these models become scale invariant. In addition, for sufficiently strong attractive interactions, continuous scale invariance can be broken to discrete scale invariance.

- In the discrete scale-invariant phase, the N-body Schrödinger equation can be reduced to the following one-body problem on the half line with the attractive inverse square potential:

$$\left(-\frac{d^2}{dr^2} + \frac{-\nu^2 - \frac{1}{4}}{r^2}\right)\psi(r) = \frac{2mE}{\hbar^2}\psi(r) \tag{13}$$

where E stands for the energy of N-body relative motion, $\nu > 0$ is a constant determined by coupling strengths, and r is the hyperradius defined by

$$r = \sqrt{\frac{1}{N}\sum_{1\le j<k\le N}(x_j - x_k)^2} \tag{14}$$

By solving eq. (13), we can obtain the exact solutions of N-body problem. The results are as follows:

- **Exact N-body bound-state spectrum**

$$E_n = -E_* \exp\left(-\frac{2n\pi}{\nu}\right) \tag{15}$$

- **Exact N-body S-matrix elements**

$$S(E) = \frac{1}{i}\frac{\sin\left(\frac{\nu}{2}\log\left(\frac{E}{E_*}\right) + \frac{i\nu\pi}{2}\right)}{\sin\left(\frac{\nu}{2}\log\left(\frac{E}{E_*}\right) - \frac{i\nu\pi}{2}\right)} \tag{16}$$

Summary and outlook

Summary

- I classified all possible two-body contact interactions that respect:
  - unitarity
  - permutation invariance
  - translation invariance
  - scale invariance
- By using those two-body contact interactions, I constructed N-boson and N-fermion models that exhibit:
  - boson-fermion duality
  - breakdown of continuous scale invariance to discrete scale invariance
- I derived the exact N-body bound-state spectrum as well as the exact N-body S-matrix elements in the discrete scale-invariant phase.

Outlook

- Generalization to (non)identical particles with internal degrees of freedom.
- Construction of field-theory description.