S matrix of elastic α-^{12}C scattering at low energies in cluster EFT

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Outline of talk

- Introduction
- Formalism: Overview

  -- Cluster EFT for radiative α capture on ^{12}C
- Numerical results:

  -- Elastic α-^{12}C scattering for l =0,1,2,3,4,5,6

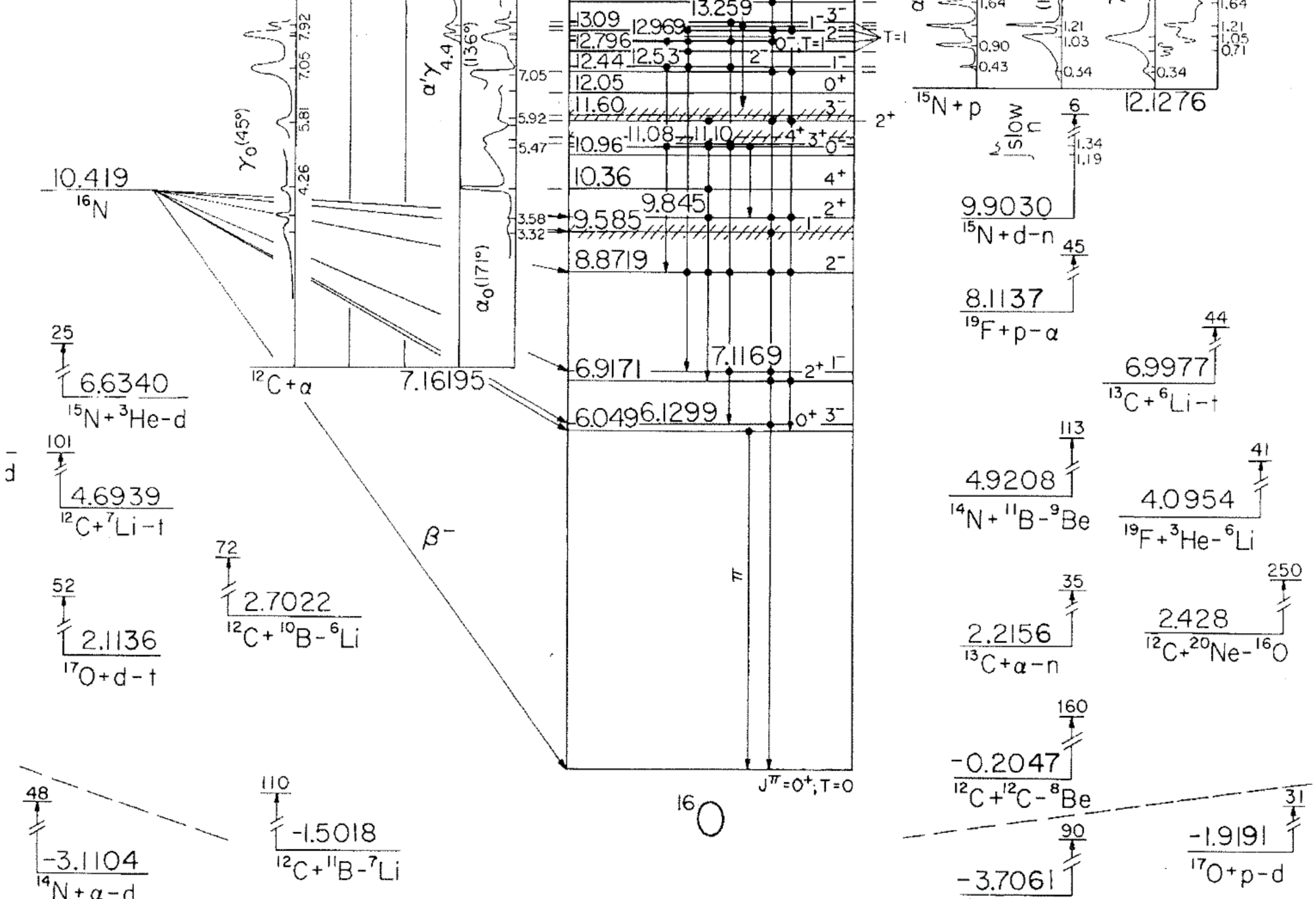
  -- ANCs of 0_1^+, 0_2^+, 1_1^-, 2_1^+, 3_1^- states of ^{16}O for α-^{12}C system
- Results and discussion

Introduction

- Radiative alpha capture on carbon-12, $^{12}C(\alpha,\gamma)^{16}O$, is a fundamental reaction in nuclear-astrophysics
- It determines C/O ratio after helium burning in red giant stars
- The core is exploded as supernova explosion type Ia
- Supernova type Ia is used as a standard candle to measure the size and expansion speed of the universe
- Our universe is expanding and consists of 72% dark energy

Elastic alpha-cabon-12 scattering

- Precise phase shift data for l=0,1,2,3,4,5,6 in $2.6 < E_\alpha < 6.62$ MeV from University of Notre Dame by Tischhauser et al. (2009)
- Construct S matrices from the amplitudes of subthreshold and resonant states of oxygen-16, which are represented in terms of effective range parameters
- The parameters are fixed by using the binding energies, resonant energies, and widths and fit to the phase shift data
- The fitted parameters are used to extrapolate the reaction rate of ${}^{12}C(\alpha,\gamma){}^{16}O$ to very low energy



10.419
^{16}N
$\gamma_0(45°)$
$\alpha'\gamma_{4.4}(136°)$
$\alpha_0(171°)$
^{12}C+α
7.16195
β⁻
π
13.259
13.09
12.969
12.796
12.53
12.44
12.05
11.60
11.08
11.10
10.96
10.36
9.845
9.585
8.8719
7.1169
6.9171
6.049
6.1299
T=1
0⁻; T=1
2⁺
J^π=0⁺; T=0
^{16}O
^{15}N+p
12.1276
slow n
6
9.9030
^{15}N+d−n
45
8.1137
^{19}F+p−α
44
6.9977
^{13}C+^{6}Li−t
113
4.9208
^{14}N+^{11}B−^{9}Be
41
4.0954
^{19}F+^{3}He−^{6}Li
35
2.2156
^{13}C+α−n
250
2.428
^{12}C+^{20}Ne−^{16}O
160
−0.2047
^{12}C+^{12}C−^{8}Be
90
−3.7061
31
−1.9191
^{17}O+p−d
25
6.6340
^{15}N+^{3}He−d
101
4.6939
^{12}C+^{7}Li−t
72
2.7022
^{12}C+^{10}B−^{6}Li
52
2.1136
^{17}O+d−t
48
−3.1104
^{14}N+α−d
110
−1.5018
^{12}C+^{11}B−^{7}Li

Cluster EFT for radiative alpha capture on carbon-12

-- The relevant degrees of freedom:

α and ^{12}C; nonrelativistic point-like scalar fields

-- The separation scale:

p-^{15}N breakup energy; $\Delta\mathrm{E} = 5\mathrm{MeV}$, $\Lambda_H = 160$ MeV

-- Momentum expansion at Gamow-peak energy,

$E_G = 0.3$ MeV; $Q = 40$ MeV, thus $Q/\Lambda_H = 0.25$

-- Typical length scale of the reaction; $1/Q = 5$ fm

-- Coulomb interaction; included non-pertabatively

-- Introducing the composite fields for resonant states

Expanding around unitary limit; effective range expansion

Formalism for the elastic scattering

- S matrix

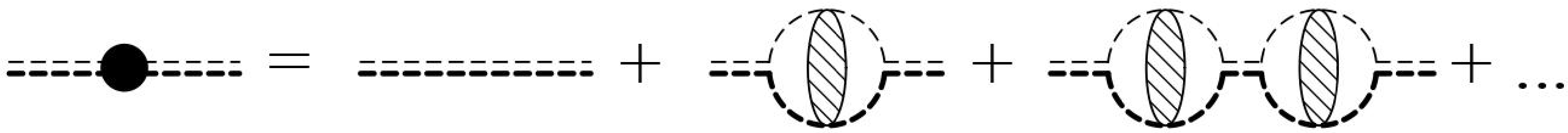
$$S_l = e^{2i\delta_l} \,, \qquad S_l = 1 + 2ip\tilde{A}_l \,.$$

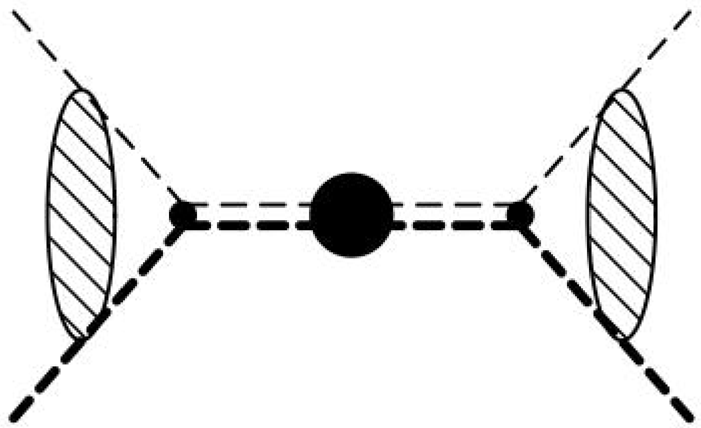
$$\delta_l = \delta_l^{(bs)} + \delta_l^{(rs1)} + \delta_l^{(rs2)} \,,$$

$$e^{2i\delta_l^{(ch)}} \;=\; 1 + 2ip\tilde{A}_l^{(ch)} \,,$$

$$\tilde{A}_l \;=\; \tilde{A}_l^{(bs)} + e^{2i\delta_l^{(bs)}}\tilde{A}_l^{(rs1)} + e^{2i(\delta_l^{(bs)}+\delta_l^{(rs1)})}\tilde{A}_l^{(rs2)} \,.$$

- Amplitudes





- Amplitude for bound states

$$\tilde{A}_l^{(bs)} = \frac{C_\eta^2 W_l(p)}{K_l(p) - 2\kappa H_l(p)},$$

$$C_\eta^2 = \frac{2\pi\eta}{\exp(2\pi\eta) - 1}, \quad W_l(p) = \left(\frac{\kappa^2}{l^2} + p^2\right) W_{l-1}(p), \quad W_0(p) = 1,$$

$$H_l(p) = W_l(p) H(\eta), \quad H(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \log(i\eta),$$

$$K_l(p) = -\frac{1}{a_l} + \frac{1}{2} r_l p^2 - \frac{1}{4} P_l p^4 + Q_l p^6 - R_l p^8,$$

ANCs, Asymptotic Normalization Coefficients

- Wave function normalization factor

$$\frac{1}{D_l(p)} = \frac{1}{K_l(p) - 2\kappa H_l(p)} = \frac{\mathcal{Z}_l}{E + B_l} + \cdots ,$$

$$\sqrt{\mathcal{Z}_l} = \left(2\mu \left| \frac{\partial D_l(p)}{\partial p^2} \right|_{p^2 = -\gamma_l^2} \right)^{-1/2} ,$$

- ANCs

$$|C_b|_l = \frac{\gamma_l^l}{l!} \Gamma(l + 1 + \kappa/\gamma_l) \sqrt{2\mu \mathcal{Z}_l} \, .$$

- Amplitudes for resonant states

$$\tilde{A}_l^{(rsN)} = -\frac{1}{p}\frac{\frac{1}{2}\Gamma_{(li)}(E)}{E - E_{R(li)} + R_{(li)}(E) + i\frac{1}{2}\Gamma_{(li)}(E)},$$

$$\Gamma_{(li)}(E) = \Gamma_{R(li)}\frac{pC_\eta^2 W_l(p)}{p_r C_{\eta_r}^2 W_l(p_r)},$$

$$R_{(li)}(E) = a_{(li)}(E - E_{R(li)})^2 + b_{(li)}(E - E_{R(li)})^3,$$

S matrices

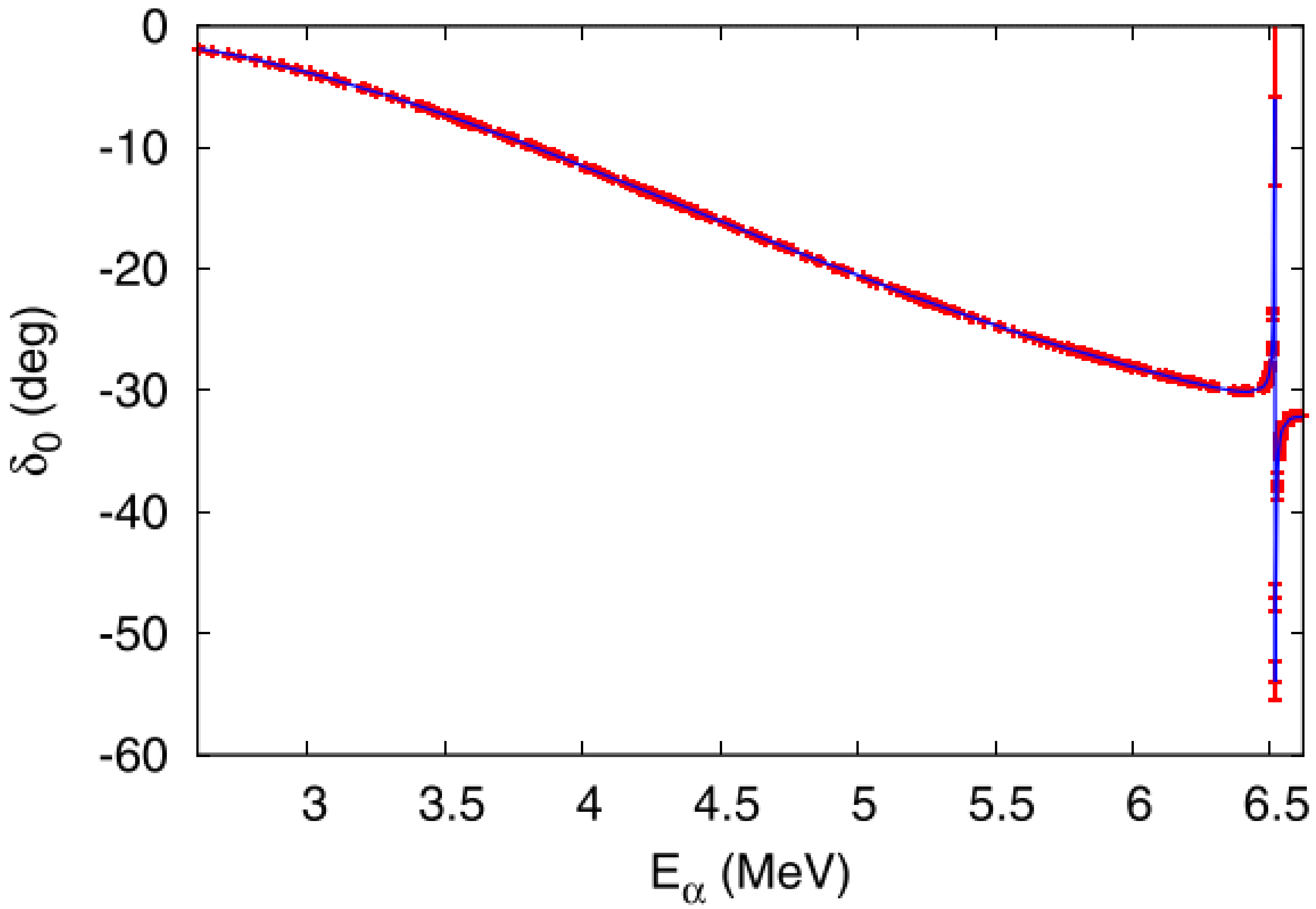
$$e^{2i\delta_l} = \frac{K_l(p) - 2\kappa \mathrm{Re} H_l(p) + ipC_\eta^2 W_l(p)}{K_l(p) - 2\kappa \mathrm{Re} H_l(p) - ipC_\eta^2 W_l(p)} \times \prod_i \frac{E - E_{R(li)} + R_{(li)}(E) - i\frac{1}{2}\Gamma_{(li)}(E)}{E - E_{R(li)} + R_{(li)}(E) + i\frac{1}{2}\Gamma_{(li)}(E)},$$

l	(Bound states)	$2.6\ \mathrm{MeV} \leq E_\alpha \leq 6.62\ \mathrm{MeV}$,	$6.62\ \mathrm{MeV} < E_\alpha$
0	0_2^+	0_3^+	0_4^+
1	1_1^-	1_2^-	1_3^-
2	2_1^+	2_2^+, 2_3^+	2_4^+
3	3_1^-	3_2^-	3_3^-
4	–	4_1^+, 4_2^+	4_3^+
5	–	–	5_1^-
6	(bg)	–	6_1^+

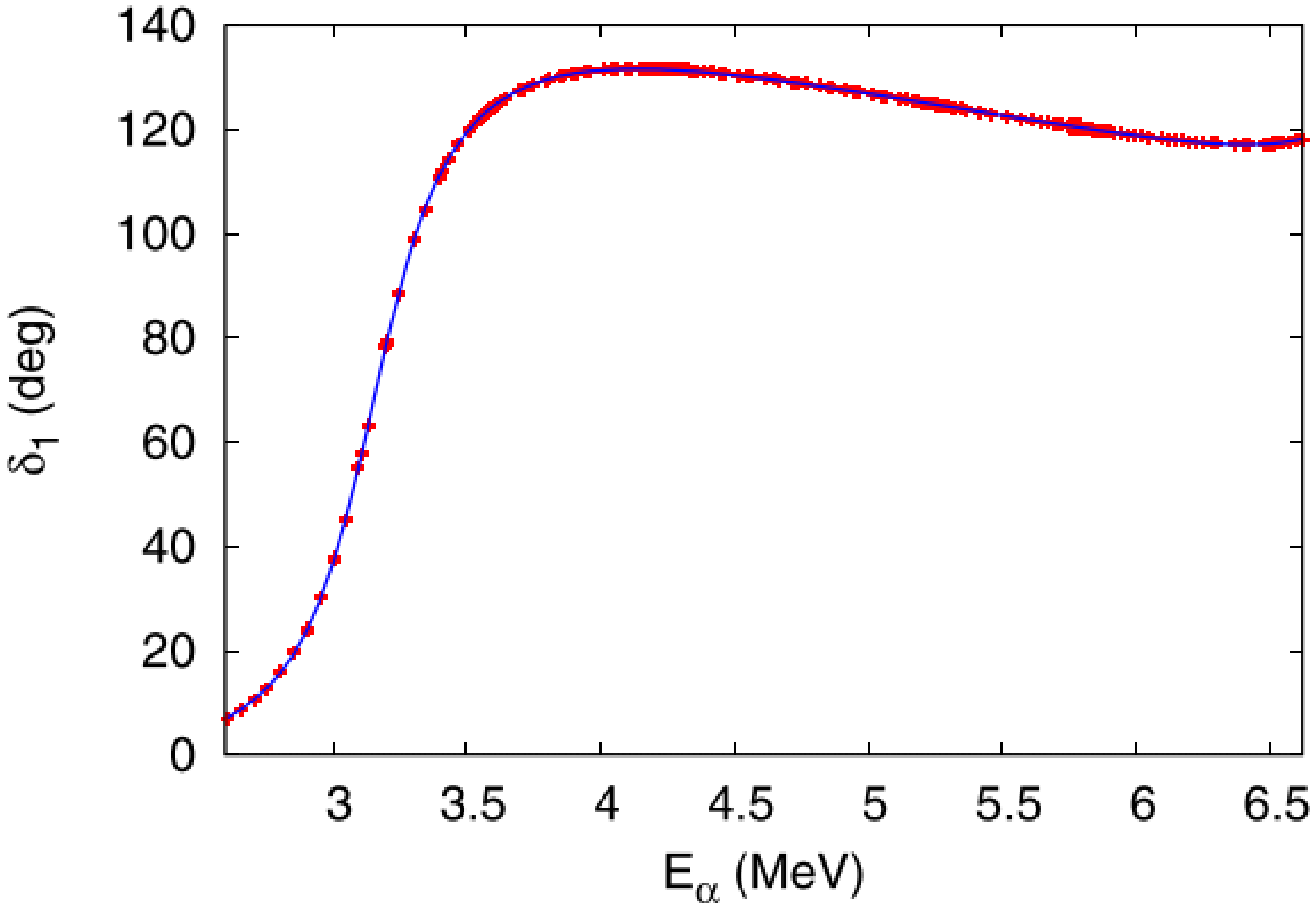
Numerical results

- Chi-square per # of data

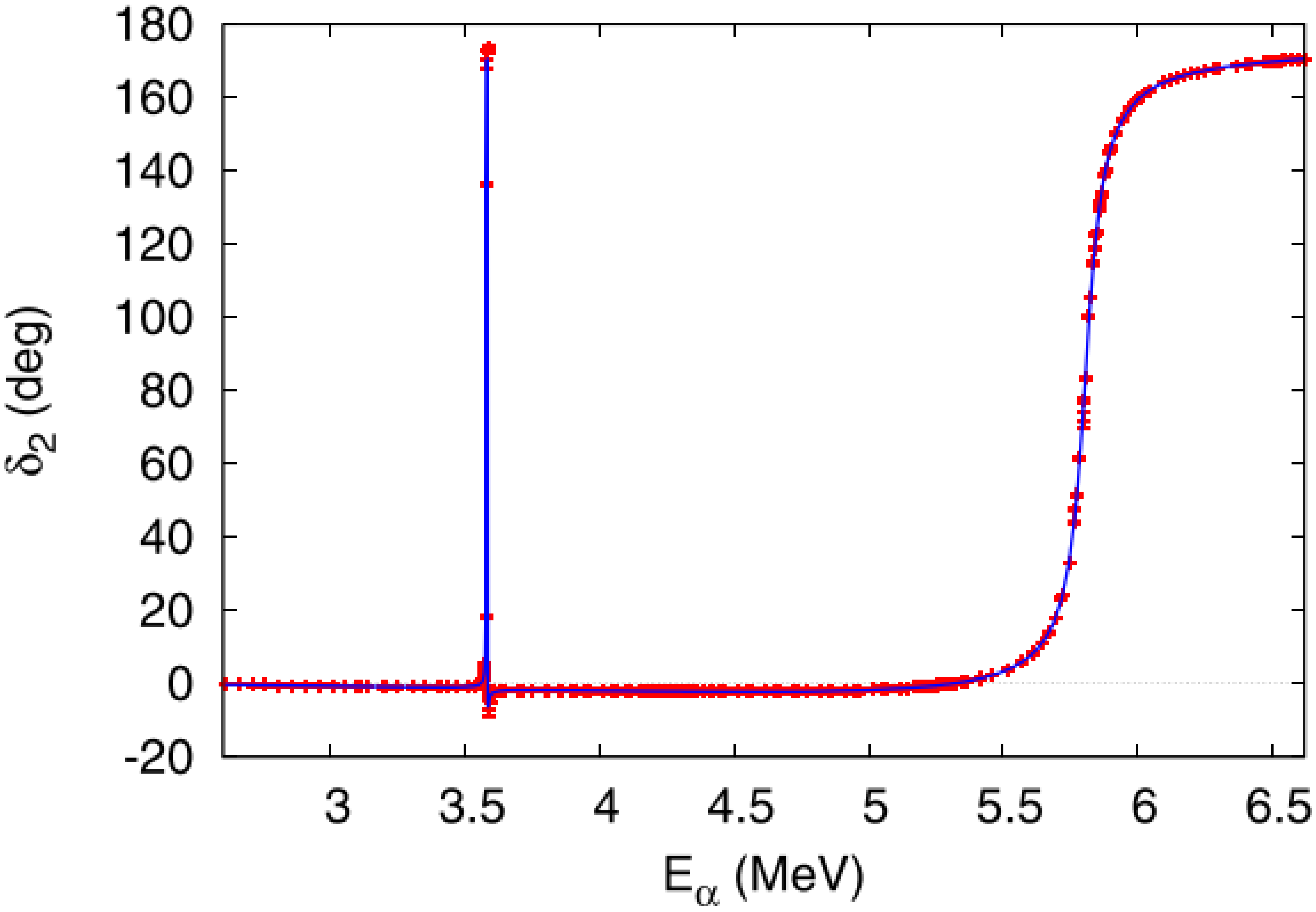
l	0	1	2	3	4	5	6
χ^2/N	0.013	0.089	0.66	0.87	0.47	0.094	0.026



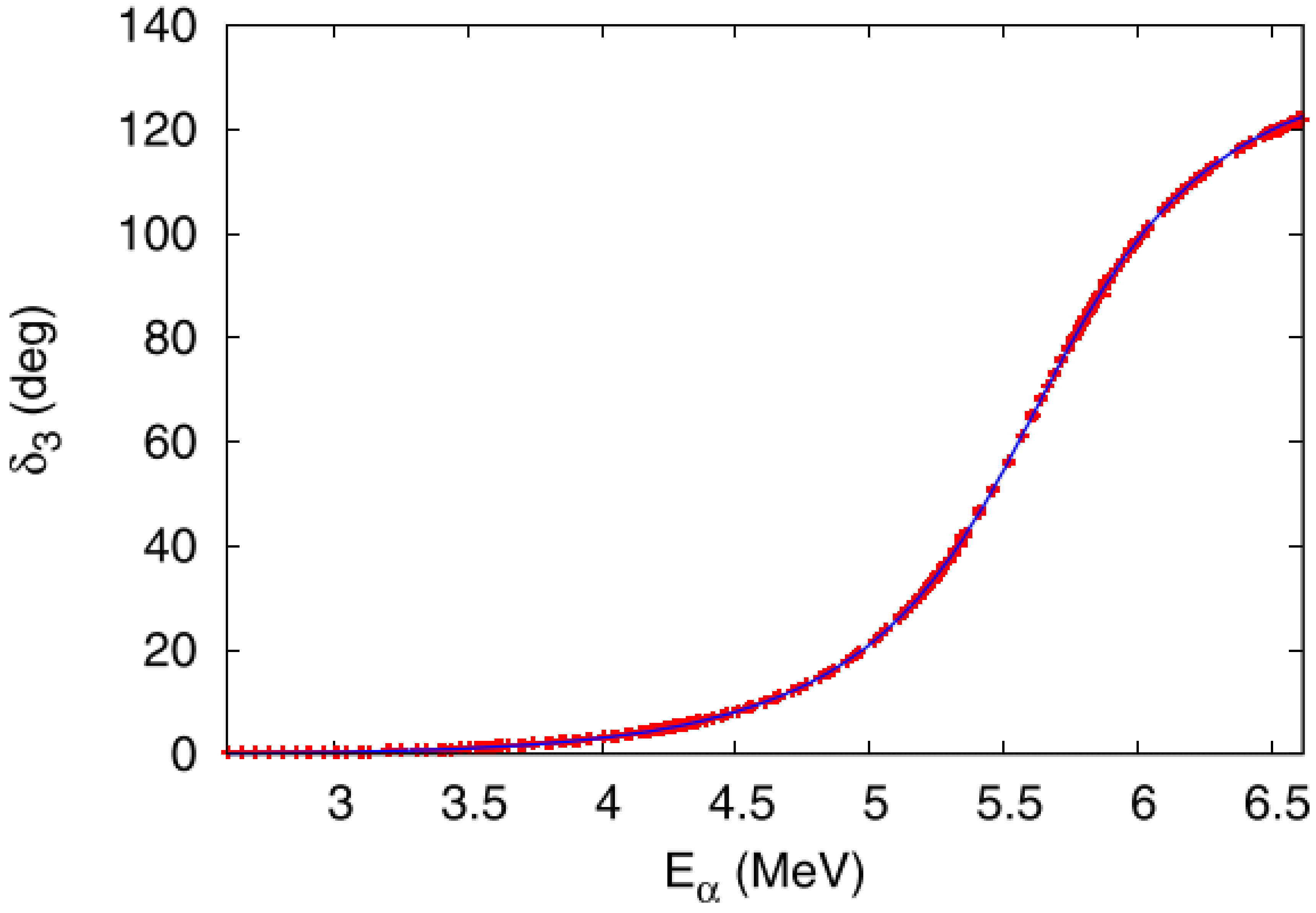
0
-10
-20
-30
-40
-50
-60
δ0 (deg)
3
3.5
4
4.5
5
5.5
6
6.5
Eα (MeV)



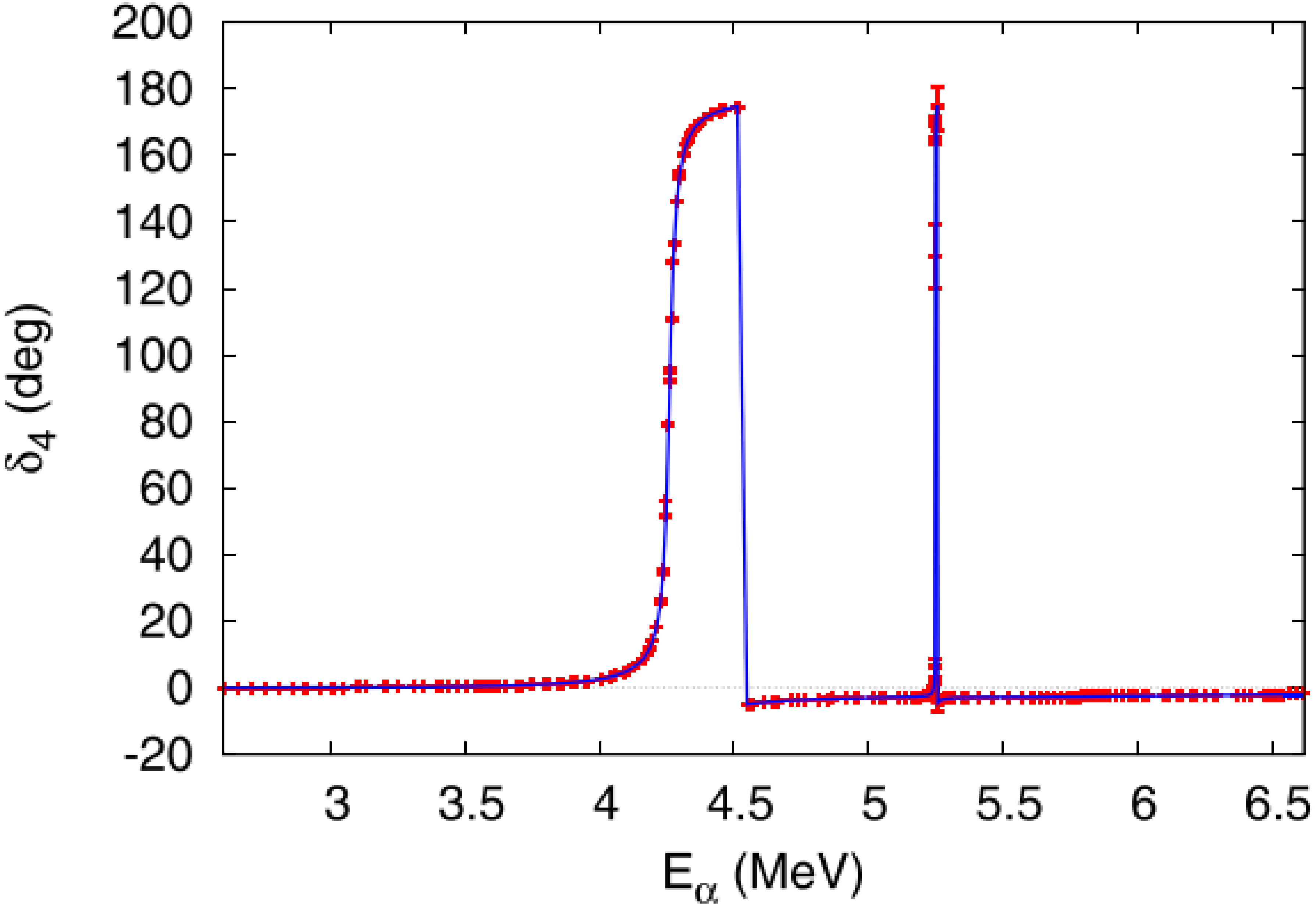
140
120
100
80
60
40
20
0
δ1 (deg)
3
3.5
4
4.5
5
5.5
6
6.5
Eα (MeV)



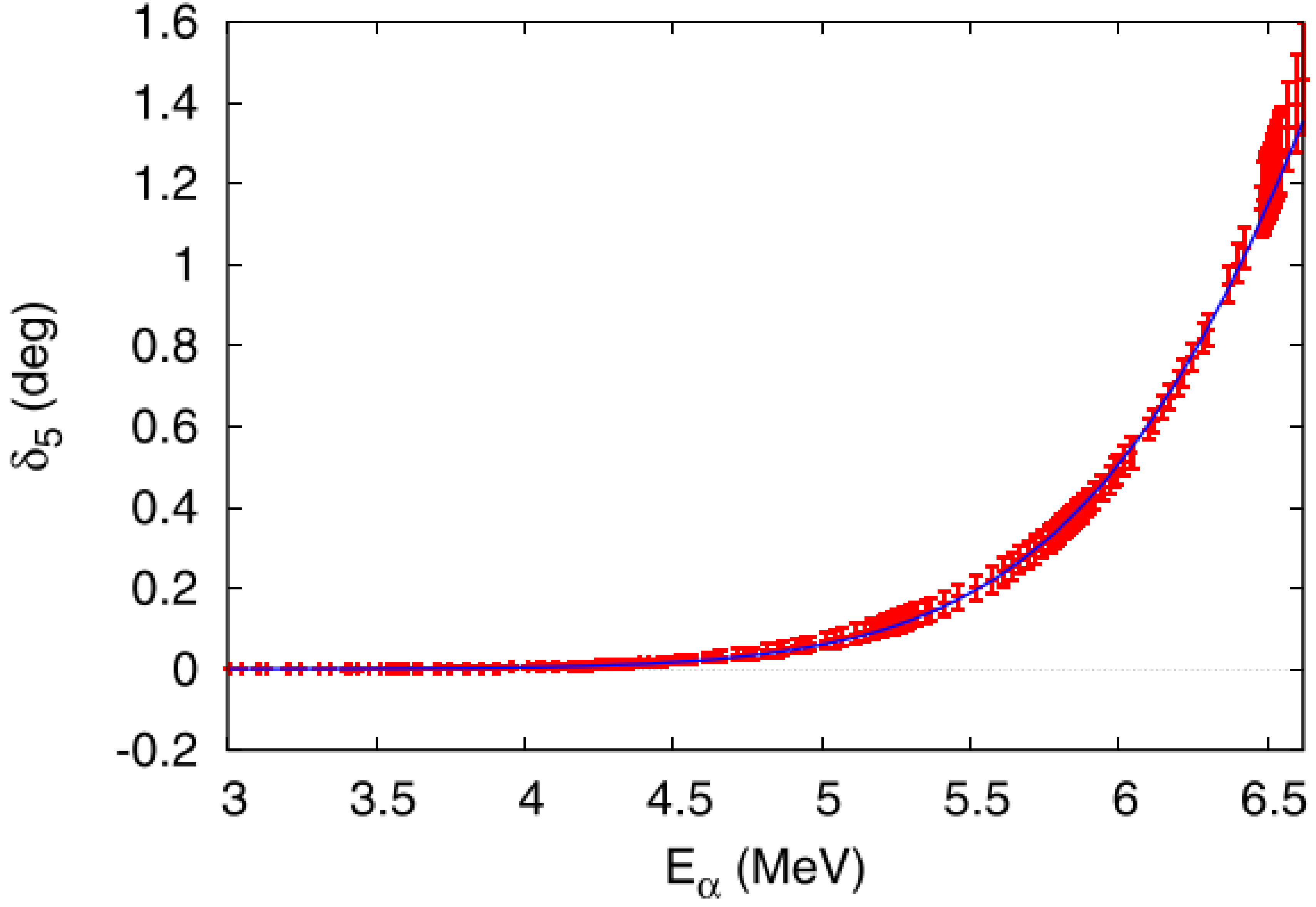
180
160
140
120
100
80
60
40
20
0
-20
δ2 (deg)
3
3.5
4
4.5
5
5.5
6
6.5
Eα (MeV)



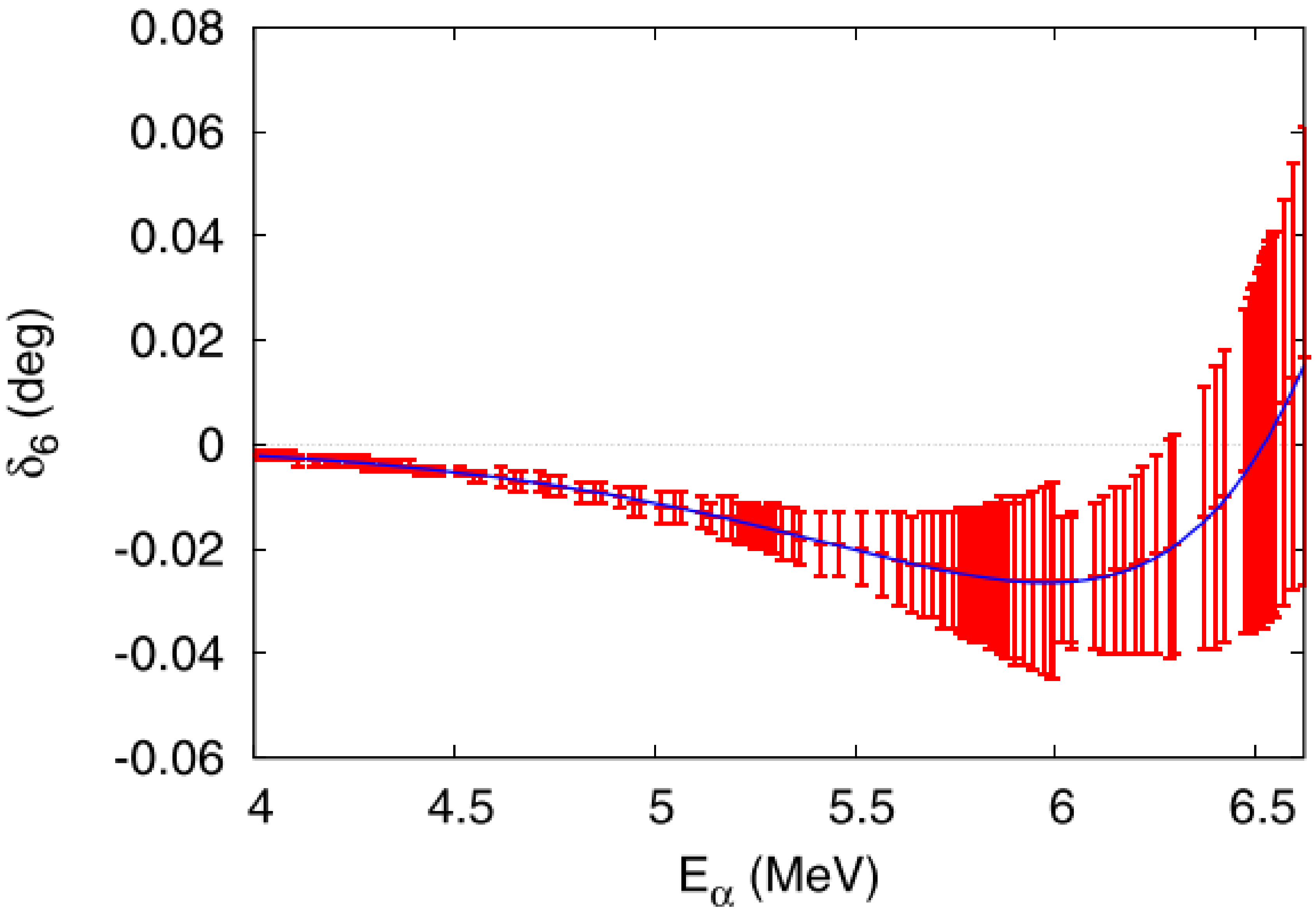
140
120
100
80
60
40
20
0
3
3.5
4
4.5
5
5.5
6
6.5
δ3 (deg)
Eα (MeV)



200
180
160
140
120
100
80
60
40
20
0
-20
3
3.5
4
4.5
5
5.5
6
6.5
δ4 (deg)
Eα (MeV)



1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
δ_5 (deg)
3
3.5
4
4.5
5
5.5
6
6.5
E_α (MeV)



0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
δ_6 (deg)
4
4.5
5
5.5
6
6.5
E_α (MeV)

ANCs of b.s. of ^{16}O for α-^{12}C clusters

TABLE IV. Values of ANC, $|C_b|$, for the subthreshold 0_2^+, 1_1^-, 2_1^+, 3_1^- states of ^{16}O.

$l_{i\text{th}}^{\pi}$	0_2^+	1_1^-	2_1^+	3_1^-
$\lvert C_b \rvert$ ($\text{fm}^{-1/2}$)	370(25)	$1.727(3) \times 10^{14}$	$3.1(6) \times 10^{4}$	113(8)

ANC (7.12 MeV, 1^-)

- This work $1.723(3) \times 10^{14}\ fm^{1/2}$
- The α transfer reaction, $^{12}C(^6Li,d)^{16}O$

Brune et al. (1999)	$2.08(20) \times 10^{14}$
Belhout et al.(2007)	$1.87(54) \times 10^{14}$
Oulebsir et al. (2012)	$2.00(35) \times 10^{14}$
Avila et al. (2015)	$2.09(14) \times 10^{14}$

Reanalysis: Hebborn et al. arXiv:2307.05636

- Analysis of the α transfer reaction, $^{12}C(^{6}Li,d)^{16}O$

  -- Three-body system: d+α+^{12}C clusters

  -- DWBA + ANC_d(1^{+},^{6}Li) → ANC_α(J^{π},^{16}O)

  -- The value of ANC_d(1^{+},^{6}Li) is updated and increased:

  ab initio calculation of $\alpha(d,\gamma)^{6}Li$, PRL129, 042503 (2022)

  -- Values of ANCs of the bound states of ^{16}O are decreased.

Results and discussion

- The simple and transparent expression of the S matrix to describe some sub-threshold and resonant states is discussed.
- The formalism of the S matrix is applied to the study of elastic α-^{12}C scattering at low energies. We find that the precise phase shift data are descried very well by using the method.
- I am now working on fitting the S factors of $^{12}C(\alpha,\gamma)^{16}O$.
- It is interesting to apply the present method to the study of other nuclear/hadronic reactions, e.g., ^{12}C-^{12}C fusion reaction.

Effective range exp. vs Wood-Saxson pot.

- Appendix in 2208.02404v2
- What to do

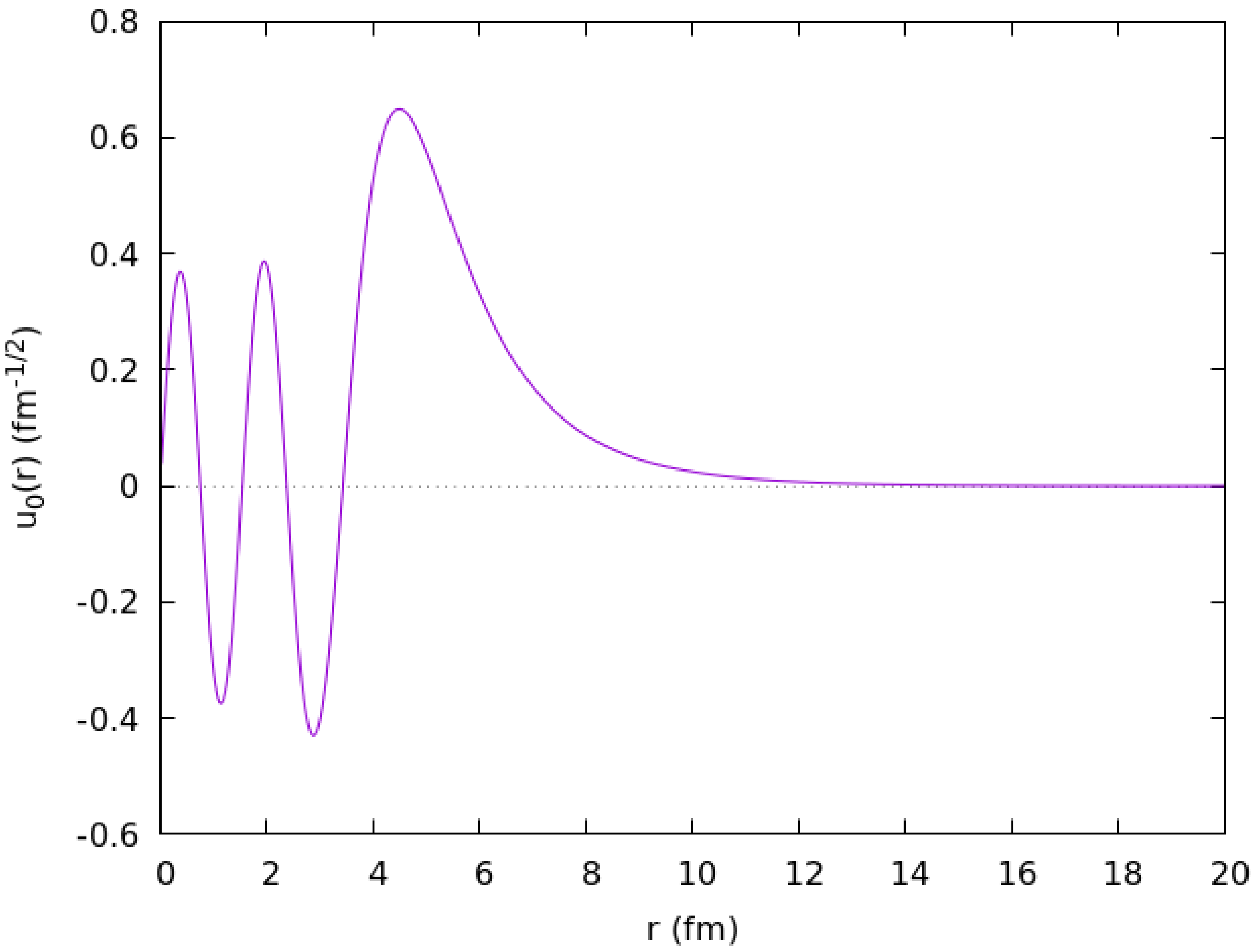
1) Fix V_0 from the binding energy of the second 0+ state
2) Calculate the ANC
3) Calculate the phase shift from the WS potential
4) Fit the phase shift by using the ER parameters
5) Calculate the ANC from the fitted ER parameters
6) Compare the ANCs from WS and ER

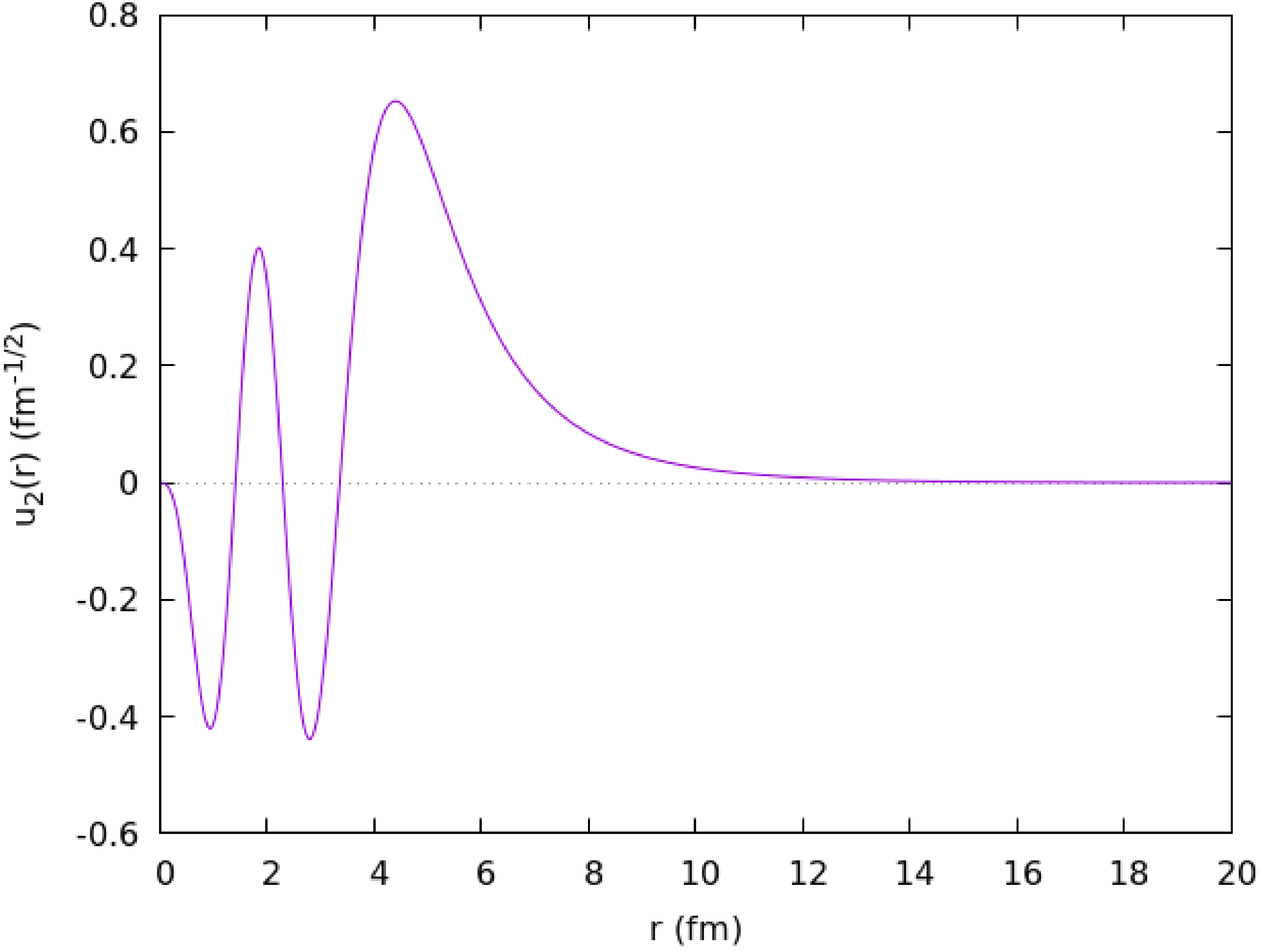
Wood-Saxon Potential

- $V(r) = \frac{V_0}{1+\exp(\frac{r-R}{a})}$, $R = R_0 A^{1/3} = 2.976$, $R_0 = 1.3$, a = 0.7 fm
- $V_0 = 131.1\,V$ for 0+ state, B = 1112.5 keV
- $V_0 = 130.8\,V$ for 2+ state, B = 244.8 keV
- Wave functions (next two pages): norm. $\int_0^\infty dr u_l(r)^2 = 1$
- ANCs definition

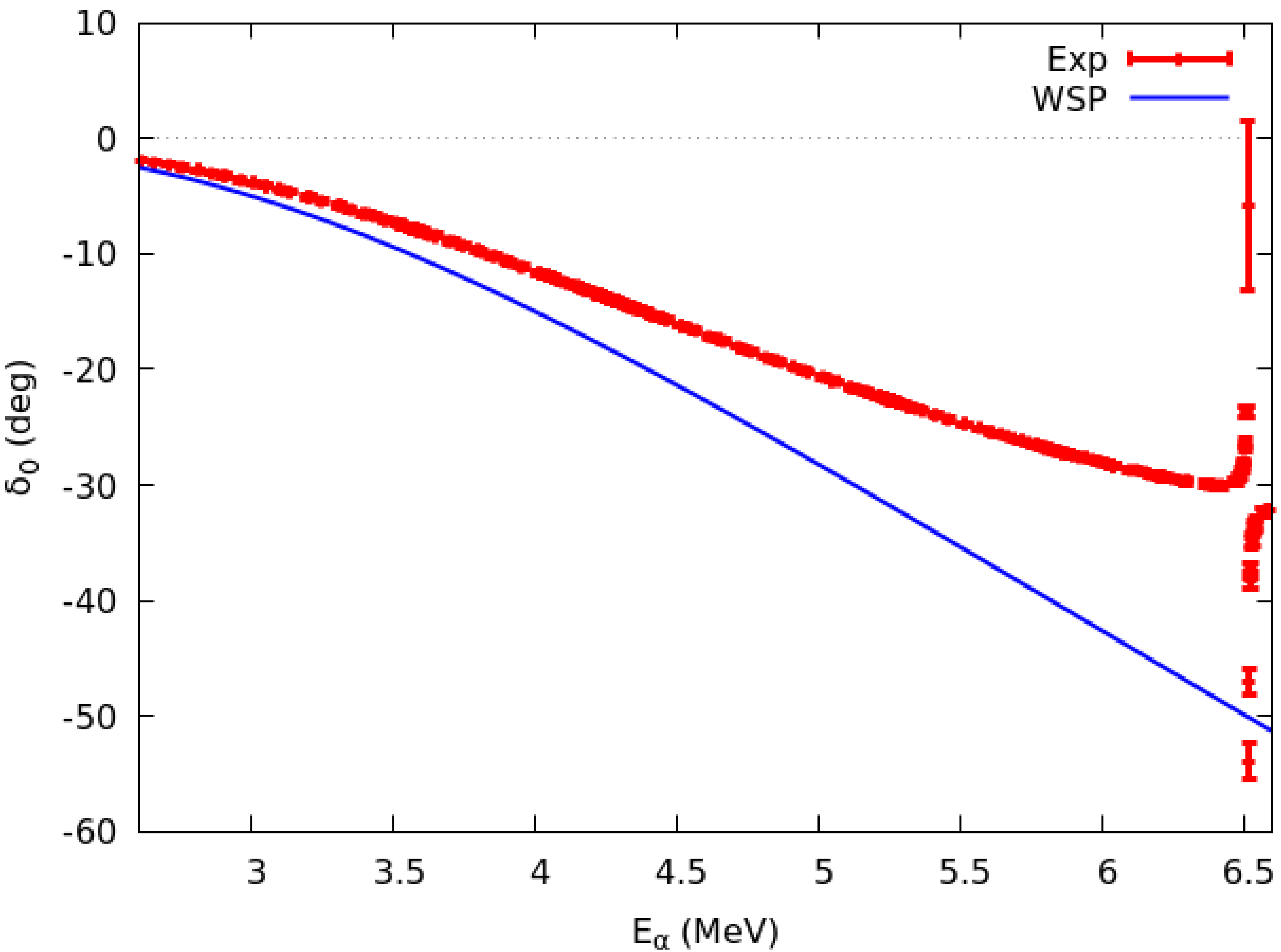
  $u_0(r) = |C_b|_0 W_{-\frac{\kappa}{\gamma_0},\frac{1}{2}}(2\gamma_0 r)$,

  $u_2(r) = |C_b|_2 W_{-\frac{\kappa}{\gamma_2},\frac{5}{2}}(2\gamma_2 r)$ at r > R

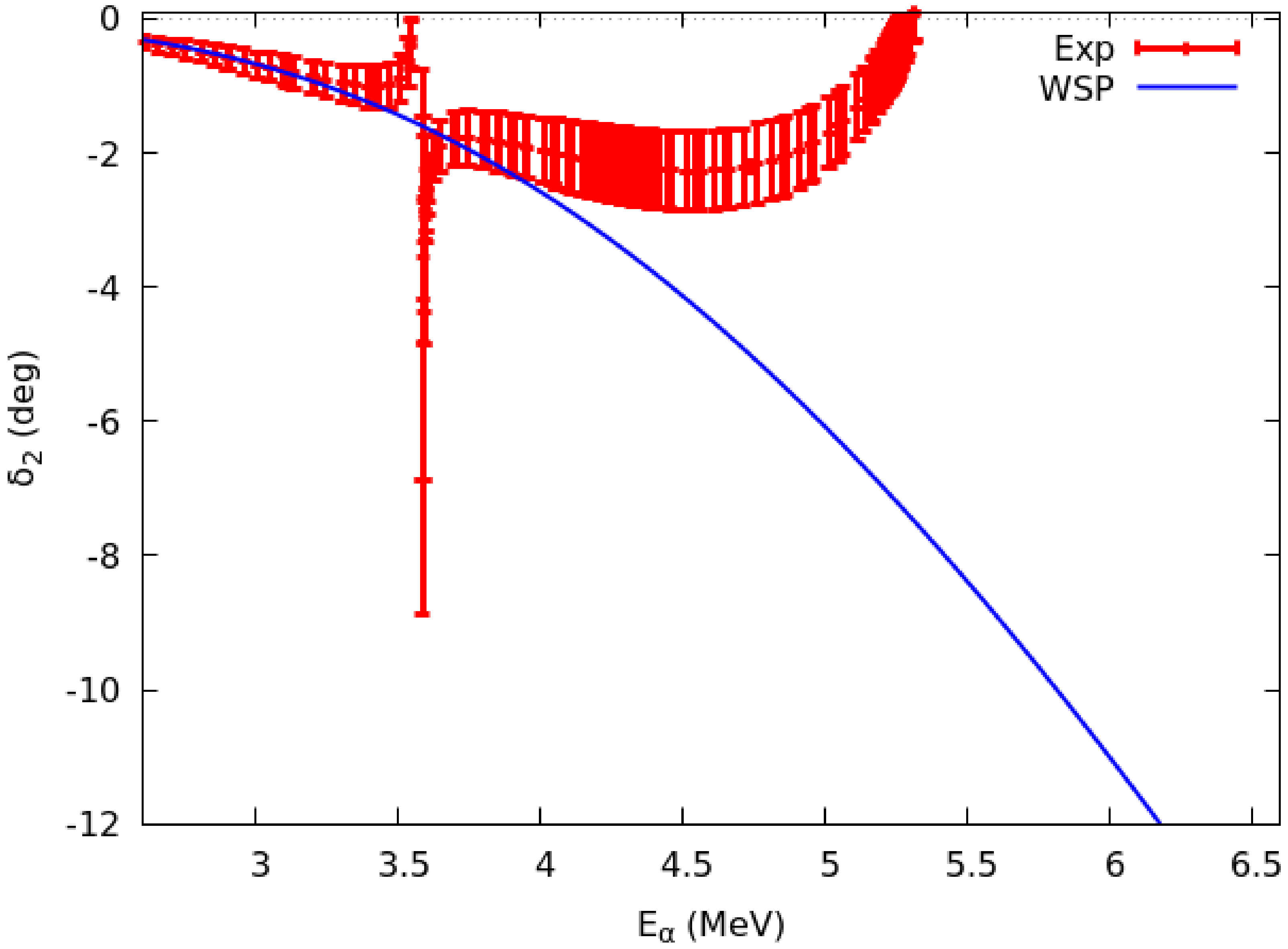

0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
$u_0(r)$ (fm$^{-1/2}$)
0
2
4
6
8
10
12
14
16
18
20
r (fm)



0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
2
4
6
8
10
12
14
16
18
20
u2(r) (fm-1/2)
r (fm)



Exp
WSP
δ_0 (deg)
E_α (MeV)
10
0
-10
-20
-30
-40
-50
-60
3
3.5
4
4.5
5
5.5
6
6.5



Exp
WSP
0
-2
-4
-6
-8
-10
-12
3
3.5
4
4.5
5
5.5
6
6.5
δ2 (deg)
Eα (MeV)

3) Fit the effective range parameters

- S-wave:
  1) Include the 0.20 degrees errors of the phase shifts
  2) the shapes of curves are different
  3) Use the different maximum energies of the data sets

$E_{\alpha,max}(MeV)$	4.0	4.5	5.0	5.5	WSP
$\|C_b\|_0\ (fm^{-1/2})$	1.1×10^3	2.2×10^3	3.7×10^3	5.8×10^3	2.6×10^3
χ^2/N	0.05	0.16	0.43	1.18	---

3) Fit the ER parameters (2)

- D-wave

  $|C_b|_2 = 3.4(54.4) \times 10^5\ fm^{-1/2}$ (ERP fit)

  $|C_b|_2 = 1.9 \times 10^5\ fm^{-1/2}$ (WSP)

Results and discussion

- About 2 times difference between the ERPF and WSP
- In the ERPF, when ANCs become large, sensitive to the fit; the error bar becomes large.
- The ERPF is not ideal to deduce large ANCs from the phase shift data for s-wave and d-wave.
- It may indicate a size of the model dependence of the ANCs from different methods.

- l=0 $(0_2^+;\ 0_3^+;\ 0_4^+)$

$$e^{2i\delta_0} = \frac{K_0(p) - 2\kappa \mathrm{Re} H_0(p) + ipC_\eta^2}{K_0(p) - 2\kappa \mathrm{Re} H_0(p) - ipC_\eta^2} \times \prod_{i=3}^{4} \frac{E - E_{R(0i)} + R_{(0i)}(E) - i\frac{1}{2}\Gamma_{(0i)}(E)}{E - E_{R(0i)} + R_{(0i)}(E) + i\frac{1}{2}\Gamma_{(0i)}(E)}, \tag{28}$$

with

$$K_0(p) = \frac{1}{2} r_0 \left(\gamma_0^2 + p^2\right) + \tfrac{1}{4} P_0 \left(\gamma_0^4 - p^4\right) + Q_0 \left(\gamma_0^6 + p^6\right) + 2\kappa H_0(i\gamma_0), \tag{29}$$

$$\Gamma_{(0i)}(E) = \Gamma_{R(0i)} \frac{pC_\eta^2 W_0(p)}{p_r C_{\eta_r}^2 W_0(p_r)}, \tag{30}$$

$$R_{(0i)}(E) = a_{(0i)}(E - E_{R(0i)})^2 + b_{(0i)}(E - E_{R(0i)})^3, \quad i = 3, 4, \tag{31}$$

l^{π}_{i-th}	p^0 order	p^2	p^4	p^6
0^+_2	a_0 (fm)	r_0 (fm)	P_0 (fm^3)	Q_0 (rm^5)
		0.26847(1)	−0.0363(4)	0.0011(1)
0^+_3	$E_{R(03)}$(MeV)	$\Gamma_{R(03)}$(keV)		
	4.8884(1)	1.34(3)		
0^+_4	$E_{R(04)}$ (MeV)	$\Gamma_{R(04)}$ (keV)	$a_{(04)}$ (MeV^{-1})	$b_{(04)}$ (MeV^{-2})
			0.75(1)	0.18(1)

- l=1 (1_1^-; 1_2^-; 1_3^-)

$$e^{2i\delta_1} = \frac{K_1(p) - 2\kappa \mathrm{Re} H_1(p) + ipC_\eta^2 W_1(p)}{K_1(p) - 2\kappa \mathrm{Re} H_1(p) - ipC_\eta^2 W_1(p)} \frac{E - E_{R(13)} + R_{(13)}(E) - i\frac{1}{2}\Gamma_{(13)}(E)}{E - E_{R(13)} + R_{(13)}(E) + i\frac{1}{2}\Gamma_{(13)}(E)},$$

$1_1^-, 1_2^-$	a_1 (fm^3)	r_1 (fm^{-1})	P_1 (fm)	Q_1 (fm^3)
		0.415314(7)	−0.57428(7)	0.02032(2)
1_3^-	$E_{R(13)}$ (MeV)	$\Gamma_{R(13)}$ (keV)	$a_{(13)}$ (MeV^{-1})	$b_{(13)}$ (MeV^{-2})
			0.43(25)	3.8(7)

- l=2 (2_1^+; 2_2^+, 2_3^+; 2_4^+)

$$e^{2i\delta_2} = \frac{K_2(p) + 2\kappa \mathrm{Re} H_2(p) + ipC_\eta^2 W_2(p)}{K_2(p) + 2\kappa \mathrm{Re} H_2(p) - ipC_\eta^2 W_2(p)} \times \prod_{i=2}^{4} \frac{E - E_{R(2i)} + R_{(2i)}(E) - i\frac{1}{2}\Gamma_{(2i)}(E)}{E - E_{R(2i)} + R_{(2i)}(E) + i\frac{1}{2}\Gamma_{(2i)}(E)},$$

2_1^+	a_2 (fm^5)	r_2 (fm^{-3})	P_2 (fm^{-1})	Q_2 (fm)
		0.149(4)	−1.19(5)	0.081(16)
2_2^+	$E_{R(22)}$ (MeV)	$\Gamma_{R(22)}$ (keV)		
	2.68308(5)	0.75(2)		
2_3^+	$E_{R(23)}$(MeV)	$\Gamma_{R(23)}$(keV)	$a_{(23)}$ (MeV^{-1})	$b_{(23)}$ (MeV^{-2})
	4.3545(2)	74.61(3)	0.46(12)	0.49(9)
2_4^+	$E_{R(24)}$ (MeV)	$\Gamma_{R(24)}$ (keV)		

- l=3 (3_1^-; 3_2^-; 3_3^-)

$$e^{2i\delta_3} = \frac{K_3(p) - 2\kappa \mathrm{Re} H_3(p) + ipC_\eta^2 W_3(p)}{K_3(p) - 2\kappa \mathrm{Re} H_3(p) - ipC_\eta^2 W_3(p)}$$

$$\times \frac{E - E_{R(33)} + R_{(33)}(E) - i\frac{1}{2}\Gamma_{(33)}(E)}{E - E_{R(33)} + R_{(33)}(E) + i\frac{1}{2}\Gamma_{(33)}(E)},$$

$3_1^-, 3_2^-$	a_3 (fm^7)	r_3 (fm^{-5})	P_3 (fm^{-3})	Q_3 (fm^{-1})	R_3 (fm)
		0.0335(2)	−0.446(9)	0.311(5)	−0.152(3)
3_3^-	$E_{R(33)}$ (MeV)	$\Gamma_{R(33)}$ (keV)	$a_{(33)}$ (MeV^{-1})	$b_{(33)}$ (MeV^{-2})	
			32(32)	$3.2(32) \times 10^2$	

- l=4 (; $4_1^+, 4_2^+$; 4_3^+)

$$e^{2i\delta_4} = \prod_{i=1}^{3} \frac{E - E_{R(4i)} + R_{(4i)}(E) - i\frac{1}{2}\Gamma_{(4i)}(E)}{E - E_{R(4i)} + R_{(4i)}(E) + i\frac{1}{2}\Gamma_{(4i)}(E)},$$

4_1^+	$E_{R(41)}$(MeV)	$\Gamma_{R(41)}$(keV)	$a_{(41)}$ (MeV^{-1})	$b_{(41)}$ (MeV^{-2})
	3.19606(1)	25.91(1)	0.740(3)	0.304(5)
4_2^+	$E_{R(42)}$(MeV)	$\Gamma_{R(42)}$(keV)		
	3.93655(2)	0.425(4)		
4_3^+	$E_{R(43)}$ (MeV)	$\Gamma_{R(43)}$ (keV)	$a_{(43)}$ (MeV^{-1})	$b_{(43)}$ (MeV^{-2})
			0.889(6)	0.216(3)

- l=5 (; ; 5_1^-)

$$e^{2i\delta_5} = \frac{E - E_{R(51)} + R_{(51)}(E) - i\frac{1}{2}\Gamma_{(51)}(E)}{E - E_{R(51)} + R_{(51)}(E) + i\frac{1}{2}\Gamma_{(51)}(E)},$$

5_1^-	$E_{R(51)}$ (MeV)	$\Gamma_{R(51)}$ (keV)	$a_{(51)}$ (MeV^{-1})	$b_{(51)}$ (MeV^{-2})
			0.572(6)	0.104(2)

- L=6((bg); ; 6_1^+)

$$e^{2i\delta_6} = \frac{K_6(p) - 2\kappa \mathrm{Re} H_6(p) + ipC_\eta^2 W_6(p)}{K_6(p) - 2\kappa \mathrm{Re} H_6(p) - ipC_\eta^2 W_6(p)} \times \frac{E - E_{R(61)} + R_{(61)}(E) - i\frac{1}{2}\Gamma_{(61)}(E)}{E - E_{R(61)} + R_{(61)}(E) + i\frac{1}{2}\Gamma_{(61)}(E)},$$

(bg)		r_6 (fm^{-11})	P_6 (fm^{-9})	
		−0.3(2)	2(1)	
6_1^+	$E_{R(61)}$ (MeV)	$\Gamma_{R(61)}$ (keV)	$a_{(61)}$ (MeV^{-1})	$b_{(61)}$ (MeV^{-2})
			0.8(1)	0.18(4)

Effective Lagrangian

$$\mathcal{L} = \phi_\alpha^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m_\alpha} \right) \phi_\alpha + \phi_C^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m_C} \right) \phi_C$$

$$+ \sum_{l=0}^{6} \sum_{i} \sum_{k=0}^{3} C_{(li)k} d_{(li)}^\dagger \left[iD_0 + \frac{\vec{D}^2}{2(m_\alpha + m_C)} \right]^k d_{(li)}$$

$$- \sum_{l=0}^{6} \sum_{i} y_{(li)} [d_{(nr)}^\dagger (\phi_\alpha O_{(l)} \phi_C) + (\phi_\alpha O_{(l)} \phi_C)^\dagger d_{(li)}],$$

Formalism: overview

- Cluster effective field theory (EFT)

  -- Choose a large scale Λ_H to separate relevant degrees of freedom at low energy from irrelevant degrees of freedom at high energy

  -- Construct an effective Lagrangian in powers of the number of derivatives

  -- Perturbative expansion in powers of Q/Λ_H where Q is a typical momentum scale of a reaction

  --- Coefficients of the Lagrangian should be fixed by experimental data