Discrete scale symmetry and non-integer dimensions in few-body systems at the unitarity limit

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25th European conference on few-body problems in physics(EFB25)
Mainz, Germany, July 30th to August 4th, 2023.

Context

- cold atoms;
- short-range potentials (range→0);
- scatt. lengths→ infinity (unitarity limit);
- weakly bound systems;
- noninteger dim./squeezed traps 3D→2D

3-Bosons
3D→2D
Efimov-Thomas→ No E.-T.

Discrete Scale Symmetry → No DScS

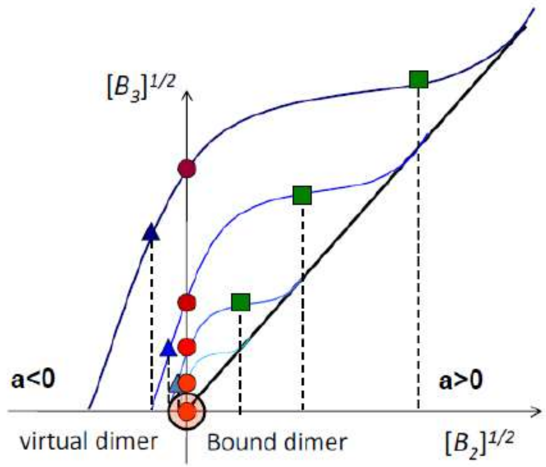
limit cycles → scaling with two-body energy

How that happens?
Start with what we know...

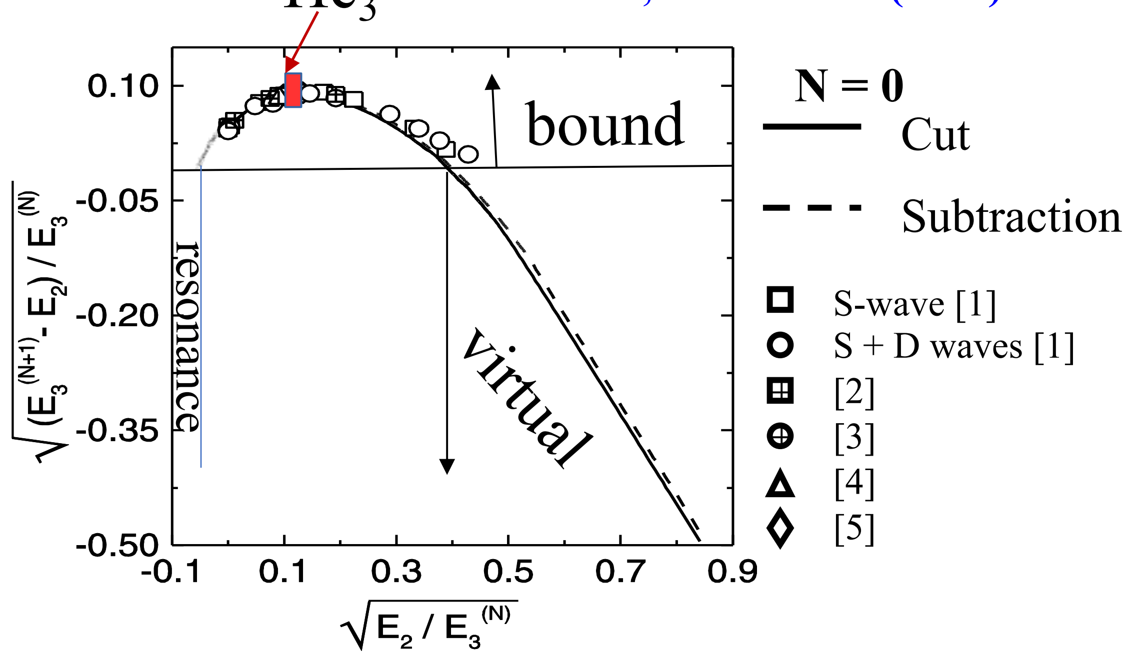
What do we know in 3D for three-bosons:
Scaling function as a limit cycle for the trimer bound-state

TF, Tomio, Delfino, Amorim. *Phys. Rev.* **A60**, R9 (1999).   Yamashita et al PRA66(2003)052702

Thomas-Efimov states
Discrete scale symmetry →→→ ***Scaling plot - universal correlation***



$^4He_3$ **Kunitski et al, Science 348 (2015) 551**



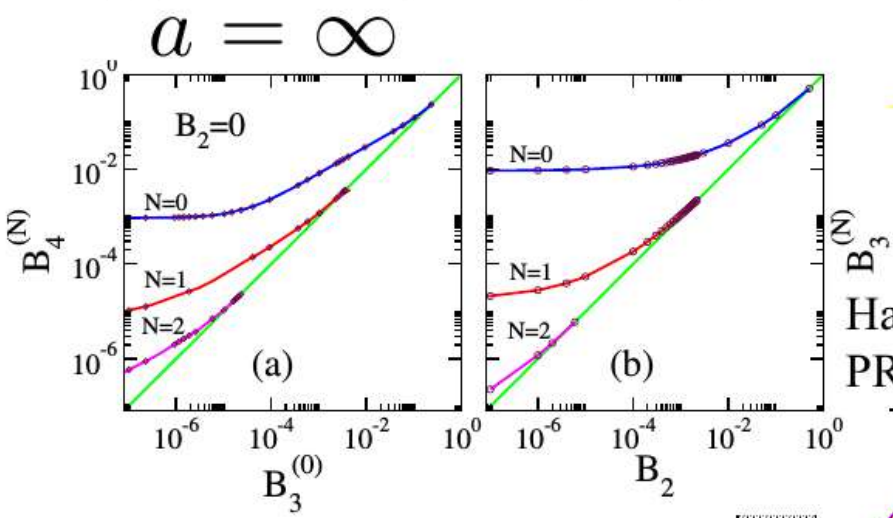
Efimov/zero-range limits

$|a|/r_o \rightarrow \infty$

[1] Cornelius & Glöckle. JCP 85, 1 (1996).
[2] Huber. PRA31, 3981 (1985).
[3] Barletta & Kievsky. PRAA64, 042514 (2001).
[4] Fedorov & Jensen. JPA34, 6003 (2001).
[5] Kolganova, Motovilov, Sofianos. PRA56, R1686 (1997).

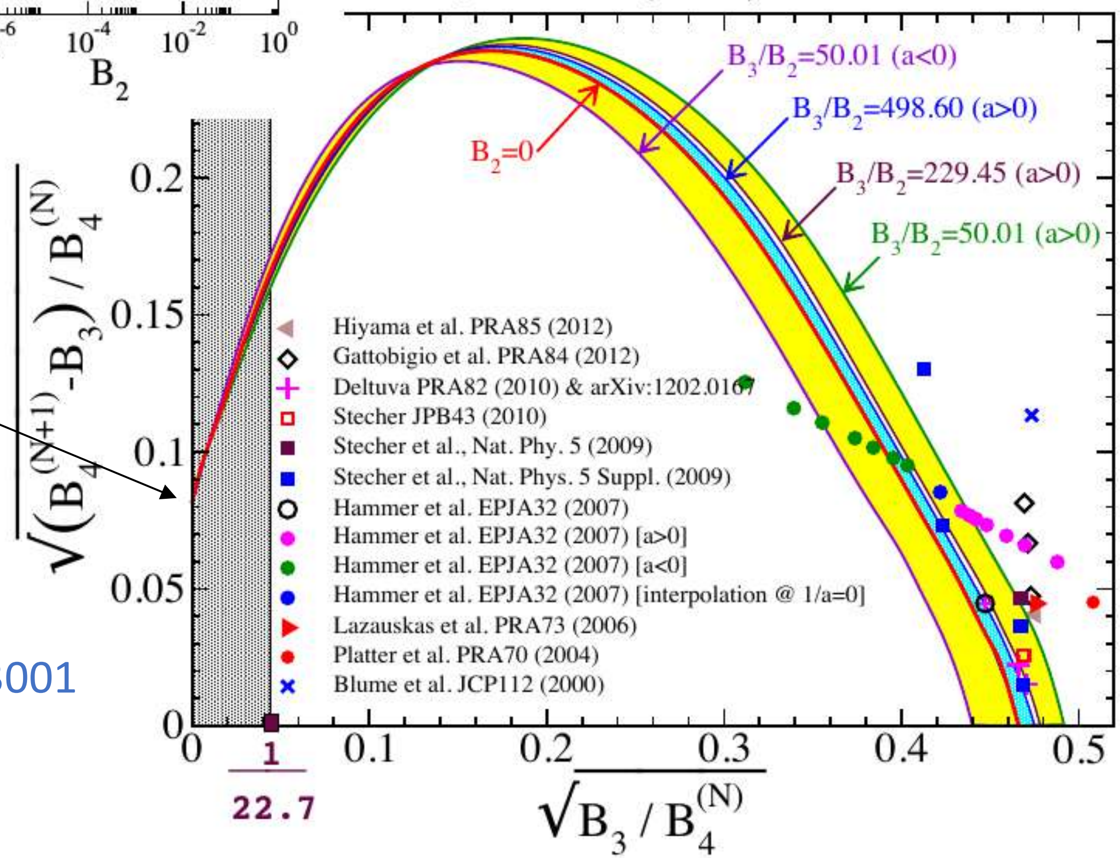
Range correction: Thogersen, Fedorov, Jensen PRA78(2008)020501(R)

What do we know in 3D for four-bosons:
Scaling function (limit cycle) for the bound-state & Interwoven cycles



Hadizadeh, Yamashita, Tomio, Delfino, TF, PRL107, 135304 (2011)

$$B_4^{(N)}/B_4^{(N+1)} \sim 151$$

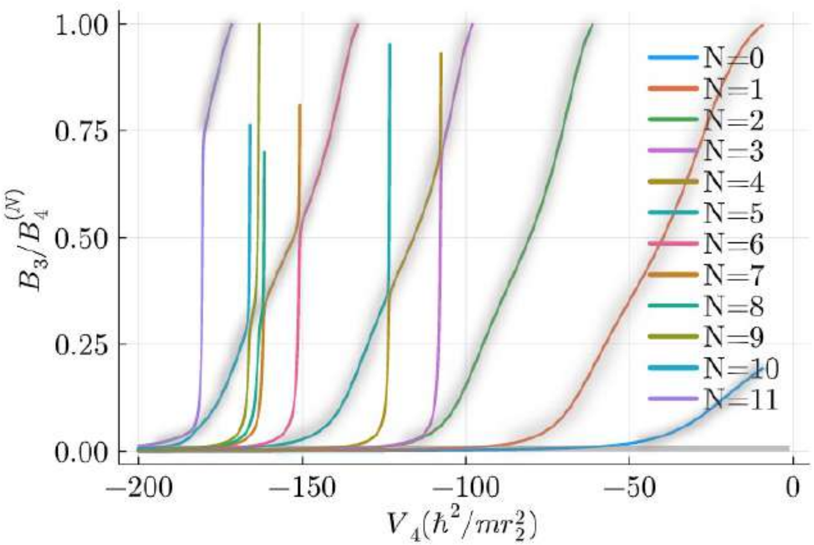


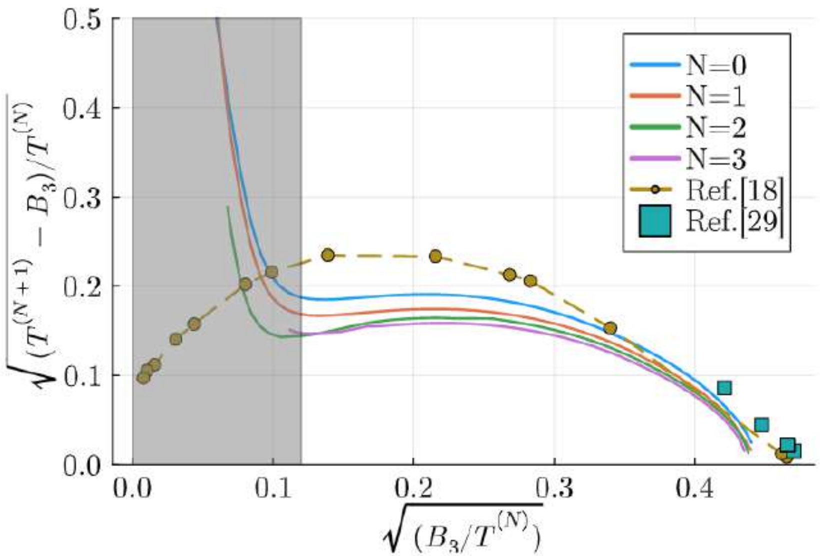
In EFT 4-body scale @ NLO
- Bazak et al. PRL 122 (2019) 143001

Universal tetramer limit-cycle at the unitary limit

TF & M. Gattobigio, arXiv:2303.14952 [physics.atm-clus]

Gaussian two, three and four-body potentials @ unitarity





- - - Hadizadeh et al PRL 107, 135304 (2011)

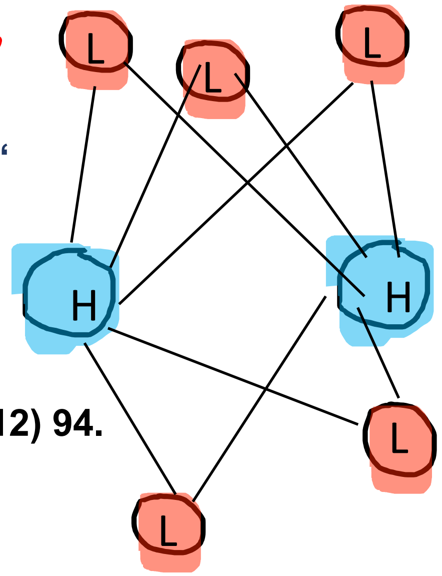
■ Deltuva PRA82 (2010) & arXiv 1202.0167

Limit Cycle in potential models

LLHH, LLLHH … systems: B.O. approximation

LLHH Naidon, Few-Body Syst. 59, 64 (2018)

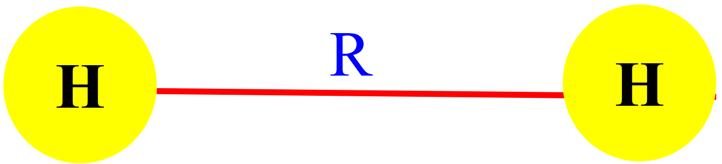
$m_H >> m_L$ *L-H interaction only*



"Interwoven limit cycles in the spectra of mass imbalanced many-boson system"
De Paula, Delfino, TF, Tomio, JPB53, 205301 (2020)

Born-Oppenheimer approx.

Fonseca, Redish, Shanley, Nucl. Phys. A320 (1979) 273
Bhaduri, Chatterjee, van Zyl, Am. J. Phys. 79 (2011) 274-281; Am. J. Phys. 80 (2012) 94.



$$\left[\frac{d^2}{dR^2}+\frac{s_N^2+\frac{1}{4}}{R^2}-\mathcal{B}_N\right]u=0 \quad (N\geq 3)$$

$$s_N \equiv s_N(A) \equiv \sqrt{\left(\tfrac{2+A}{4A}\right)(N-2)\gamma^2-\tfrac{1}{4}}$$

$A = m_L / m_H$

$\gamma = e^{-\gamma} = 0.5671433$

$(N-2)$ Light bosons

New limit cycles beyond 3-body

What do we know 2D for 2, 3 and 4 bosons in the limit of $E_2 \rightarrow 0$

$$\langle k \mid t(E) \mid k \rangle = \frac{2}{\pi}(-\cot\delta + i)^{-1} ,$$

with[14]

$$\cot\delta = a_2 + \frac{1}{\pi}\ln E + bE + cE^2 + \cdots .$$

[14] S. K. Adhikari, W. G. Gibson, and T. K. Lim, J. Chem. Phys. **85**, 5580 (1986); S. K. Adhikari, Am. J. Phys. **54**, 362 (1986).

zero-range limit:

$$\cot\delta = \not{a}_2 + \frac{1}{\pi}\ln\left[\frac{E}{E_2}\right] + b'\left[\frac{E}{E_2}\right] + \cdots$$

only scale that remains!

Bruch &Tjon, Phys. Rev. A 19, 425 (1979): **NO EFIMOV EFFECT IN 2D!**

3-bosons: $E_{3,0} = 16.1\ E_2$ and $E_{3,1} = 1.25\ E_2$

Platter, Hammer, Meißner FBS 35 (2004) 169

4-bosons $E_{4,0} = 197.3(1) E_2$ and $E_{4,1} = 25.5(1)\ E_2$

Dimensional reduction 3D→2D

- *Non-integer dimension:* E. Nielsen, et al. Phys. Rep. 347, 373 (2001)

- *Harmonic confinement: 3D→2D*
  Levinsen, Massignan, Parish, PRX 4, 031020 (2014)

- *Compactification (periodic bound. conditions) 3D→2D (3body)*
  Sandoval et al, JPB 51 (2018) 065004

- *Danilov's equations in non-integer dimensions (3body)*
  Rosa, TF, Krein, Yamashita, PRA97 (2018) 050701(R)

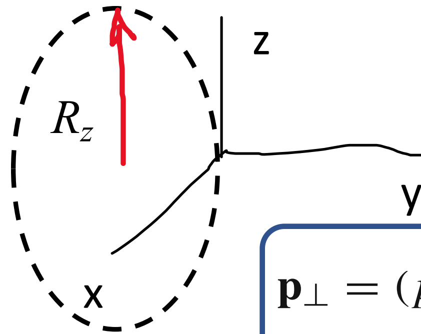
- *Bethe-Peierls boundary-conditions & hyperspherical method (3body)*
  Rosa, TF, Krein, Yamashita, PRA106 (2022) 023311 & arXiv:2305.18064

- EFT compactification & dim reg 4D→3D→2D, 4D→2D (2body)
  Beane, Jafry, JPB52(2019) 035001

Dimensional reduction: compactification

Sandoval et al. JPB 51 (2018) 065004

- $R_z \rightarrow$ infty $D \rightarrow 3$
- $R_z \rightarrow 0$ $D \rightarrow 2$



$$\mathbf{p}_\perp = (p_x, p_y) \qquad p_z = \frac{n}{R_z}, \quad \text{with } n = 0, \pm 1, \pm 2, \ldots .$$

2-body scatt amplitude zero-range interaction:

$$R_z\, \tau^{-1}_{A\beta;R_z}(E) = 2\, m_{A\beta} \sum_n \left\{ \int \frac{d^2 p_\perp}{\tilde{E} - p_\perp^2 - \frac{n^2}{R_z^2}} - \int \frac{d^2 p_\perp}{\tilde{E}_{A\beta} - p_\perp^2 - \frac{n^2}{R_z^2}} \right\}$$

$$\tau_{A\beta;R_z}(E) = R_z \left[4\pi\, m_{A\beta} \ln \left(\frac{\sinh \pi \sqrt{-2\, m_{A\beta} E}\; R_z}{\sinh \pi \sqrt{-2\, m_{A\beta} E_{A\beta}}\; R_z} \right) \right]^{-1}$$

D → 2 $\qquad \tau_{A\beta,R_z}(E)|_{R_z \to 0} = R_z \left[4\pi m_{A\beta} \ln \left(\sqrt{-E} / \sqrt{-E_{A\beta}} \right) \right]^{-1}$

D → 3 $\qquad \tau_{A\beta,R_z}(E)|_{R_z \to \infty} = \left[4\pi^2 m_{A\beta} \left(\sqrt{-2m_{A\beta}E} - \sqrt{-2m_{A\beta}E_{A\beta}} \right) \right]^{-1}$

Solution of the compactified SKM equations for AAB systems

Sandoval et al. JPB 51 (2018) 065004

$$\int d^3p \frac{1}{E_2^{3D} - \frac{p^2}{2M}} - \frac{1}{R_y}\sum_n \int d^2p_\perp \frac{1}{E_2^{ZR} - \frac{p_\perp^2}{2M} - \frac{n^2}{2MR_y^2}} = 0$$

$$E_2^{ZR} = -\frac{a_{3D}^2}{(\pi R_y)^2} \ln^2\left(\frac{e^{\pi R_y/a_{3D}}}{2} + \sqrt{\frac{e^{2\pi R_y/a_{3D}}}{4} + 1}\right)$$

b oscillator length

$$E_2(b_y) = -\frac{4a_{3D}^2}{\alpha + \beta b_y^2} \ln^2\left(\frac{e^{b_y/2a_{3D}}}{2} + \sqrt{\frac{e^{b_y/a_{3D}}}{4} + 1}\right)$$

$$E_2(b_\omega \to 0) \equiv E_2^{2D}$$

$$E_2(b_\omega \to \infty) \equiv E_2^{3D}$$

$m_B/m_A = 6/133$

J H Sandoval *et al*

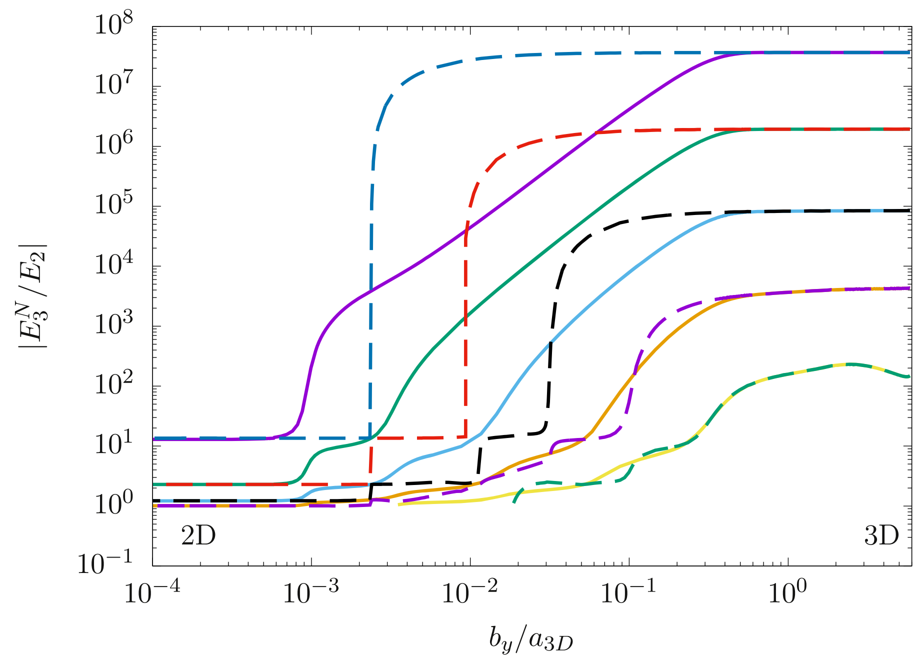


Figure 2. Trimer energies plotted in units of the two-body energy for $m_B/m_A = 6/133$ as functions of b_y/a_{3D}. For the solid lines the two-body energy varies with b_y while for the dashed lines it is kept constant (see text for discussion). Solid and dashed lines have different colors for visibility.

Non-integer dimension and harmonic trap

Garrido & Jensen, Phys. Rev. Res. 2 (2020) 033261

b_{ho} oscillator length

$$\frac{b_{\text{ho}}}{r_{2\text{D}}} = \sqrt{\frac{3(d-2)}{(d-1)(3-d)}}.$$

r_{2D} 3-body rms radius in 2D

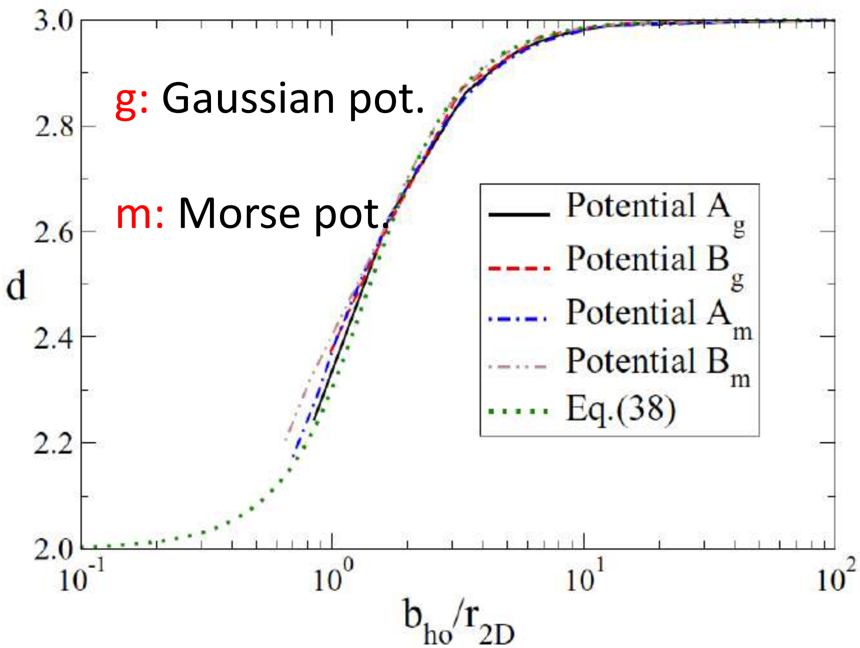


FIG. 5. Numerical relation between the dimension d and the harmonic oscillator parameter b_{ho} obtained after making E_{ext} in Fig. 3 and the ground-state energy E_d in Fig. 4 equal. The oscillator parameter b_{ho} is normalized to the root-mean-square radius of the 2D three-body calculation. The cases of potentials A_g, B_g, A_m, and B_m are shown by the solid, dashed, dot-dashed, and dot-dot-dashed curves, respectively. These results are compared with the estimate given in Eq. (38), which is shown by the dotted curve.

Danilov's equations in non-integer dimensions for AAB systems

zero-range 3B equations in non-integer D: Rosa, TF, Krein, Yamashita PRA97, 050701(R) (2018)

$$\mathcal{A} = m_B/m_A$$

Region where there is a real solution for the scaling factor *s*

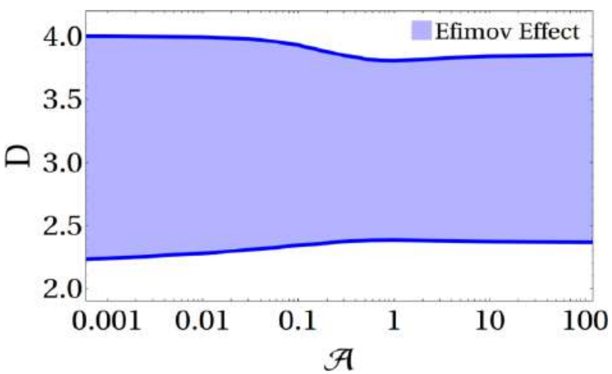


FIG. 1. Regions (in blue) where there is a real solution for the scaling factor s, solution to Eq. (8); outside this "dimensional band," the Efimov effect does not exist. For $\mathcal{A} = 1$ we reproduce exactly the result in Ref. [7], where the dimensional limits are given by $2.3 < D < 3.8$.

$\mathcal{A} = 1;\ 2.3 < D < 3.8$

E. Nielsen, et al. Phys. Rep. 347, 373 (2001).

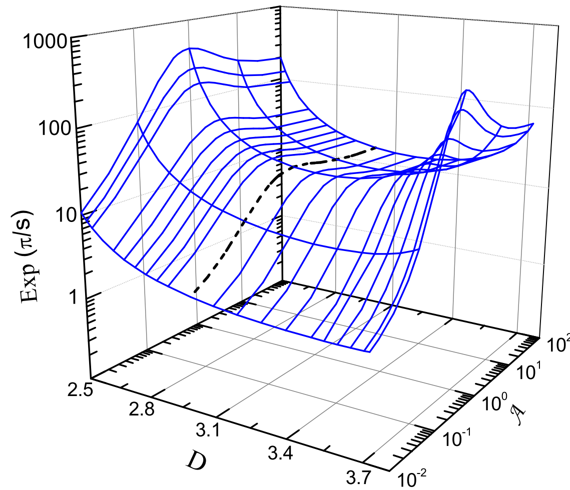
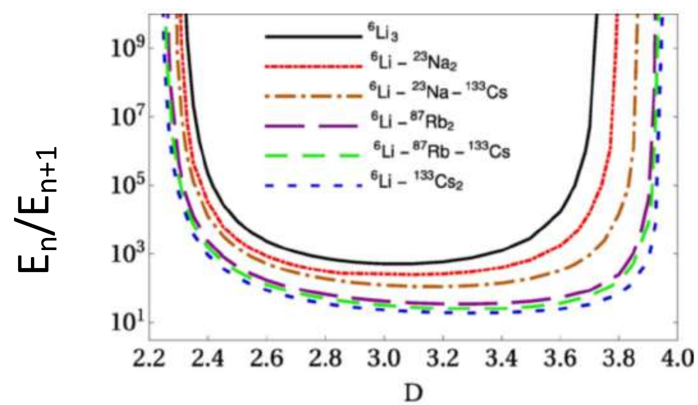


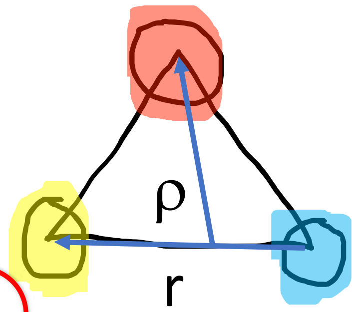
FIG. 2. Discrete scaling factor as a function of the mass ratio $\mathcal{A} = m_B/m_A$, and dimension D. The black dashed line shows the well-known situation of $D = 3$.



Analytic bound-state wave function for abc systems [scatt lengths→∞]

Rosa, TF, Krein, Yamashita PRA 106, 023311 (2022)

Faddeev equation: $\Psi = \sum_{i=1}^{3} G_0 V_i \Psi = \sum_{i=1}^{3} \psi_i$



Bethe-Peierls Boundary condition

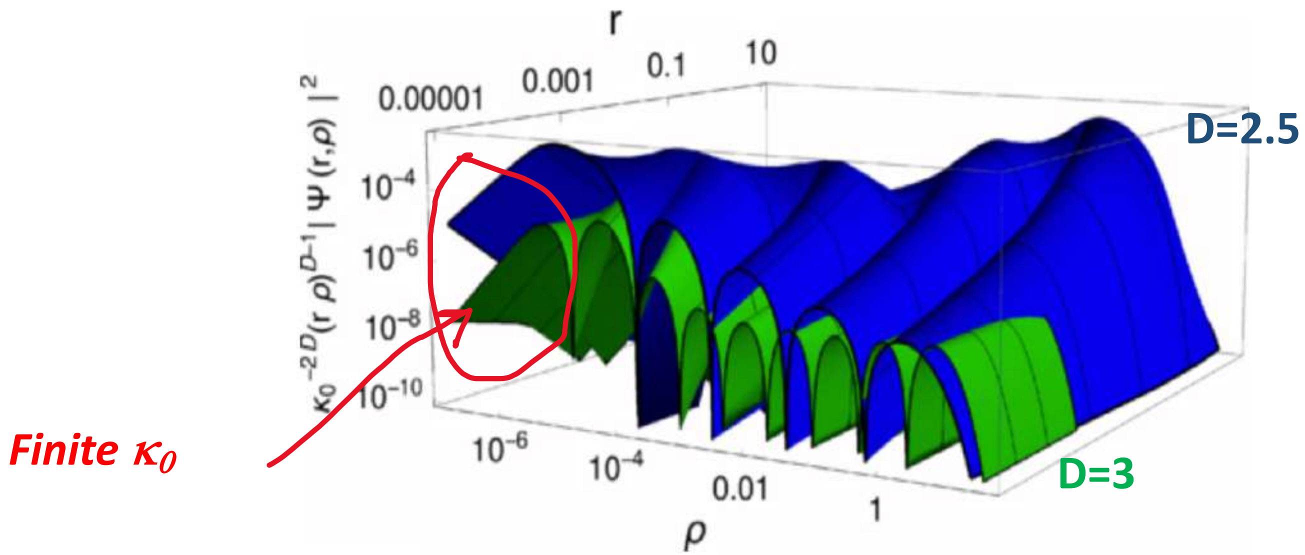
$$\left[\frac{\partial}{\partial r_i} r_i^{\frac{D-1}{2}} \Psi(\boldsymbol{r}_i, \boldsymbol{\rho}_i)\right]_{r_i \to 0} = \frac{3-D}{2}\left[\frac{\Psi(\boldsymbol{r}_i, \boldsymbol{\rho}_i)}{r_i^{\frac{3-D}{2}}}\right]_{r_i \to 0}$$

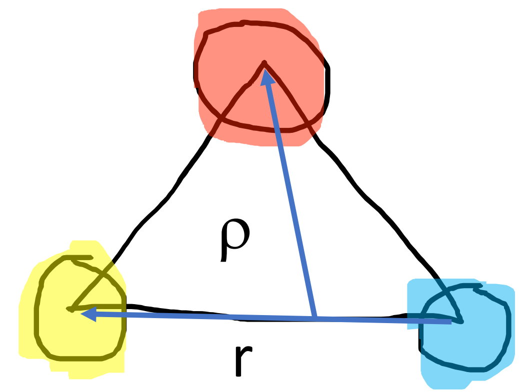
$$\psi^{(i)}(r_i', \rho_i') = C^{(i)} \frac{K_{s_n}\left(\kappa_0 \sqrt{r_i'^2 + \rho_i'^2}\right)}{\left(r_i'^2 + \rho_i'^2\right)^{D/2-1/2}} \frac{\sqrt{\sin\left(2\arctan\left(r_i'/\rho_i'\right)\right)}}{\left[\cos\left(\arctan\left(r_i'/\rho_i'\right)\right)\ \sin\left(\arctan\left(r_i'/\rho_i'\right)\right)\right]^{D/2-1/2}}$$
$$\times \left[P^{D/2-1}_{s_n/2-1/2}\left(\cos\left(2\arctan(r_i'/\rho_i')\right)\right) - \frac{2}{\pi}\tan\left(\pi(s_n-1)/2\right) Q^{D/2-1}_{s_n/2-1/2}\left(\cos\left(2\arctan(r_i'/\rho_i')\right)\right)\right]$$

where K_{s_n} is the modified Bessel function of the second kind. $P_n^m(x)$ and $Q_n^m(x)$ are the associated Legendre functions

$$-\kappa_0^2 = 2E$$

$^{6}Li - {}^{133}Cs - {}^{87}Rb$





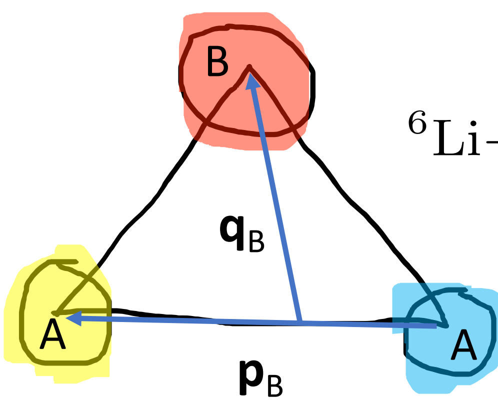
Dimensionless radial distribution as a function of dimensionless quantities $r = \kappa_0 r_3$ ($^{133}Cs - {}^{87}Rb$ relative distance) and $\rho = \kappa_0 \rho_3$ (^{6}Li relative distance to the $^{133}Cs - {}^{87}Rb$ system). We consider the three-body system $^{6}Li - {}^{133}Cs - {}^{87}Rb$ for $D = 2.5$ (blue) with $b_{ho}/r_{2D} = \sqrt{2}$, and $D = 3.0$ (green). The angle between $\vec{r}$ and $\vec{\rho}$ is fixed to $\pi/3$.

Single-particle momentum distribution of AAB Efimov states in non-integer dimensions

Rosa, TF, Krein, Yamashita,
arXiv preprint arXiv:2305.18064

$$\langle \mathbf{q}_B \mathbf{p}_B | \Psi \rangle = \frac{1}{E_3 + p_B^2/2\eta_B + q_B^2/2\mu_B} \Big[\chi^{(B)}(\mathbf{q}_B) + \chi^{(A)}\left(\left| \mathbf{p}_B - \frac{\mathbf{q}_B}{2} \right| \right) + \chi^{(A)}\left(\left| \mathbf{p}_B + \frac{\mathbf{q}_B}{2} \right| \right) \Big],$$

spectator functions



$^6\mathrm{Li}{-}^{133}\mathrm{Cs}_2$

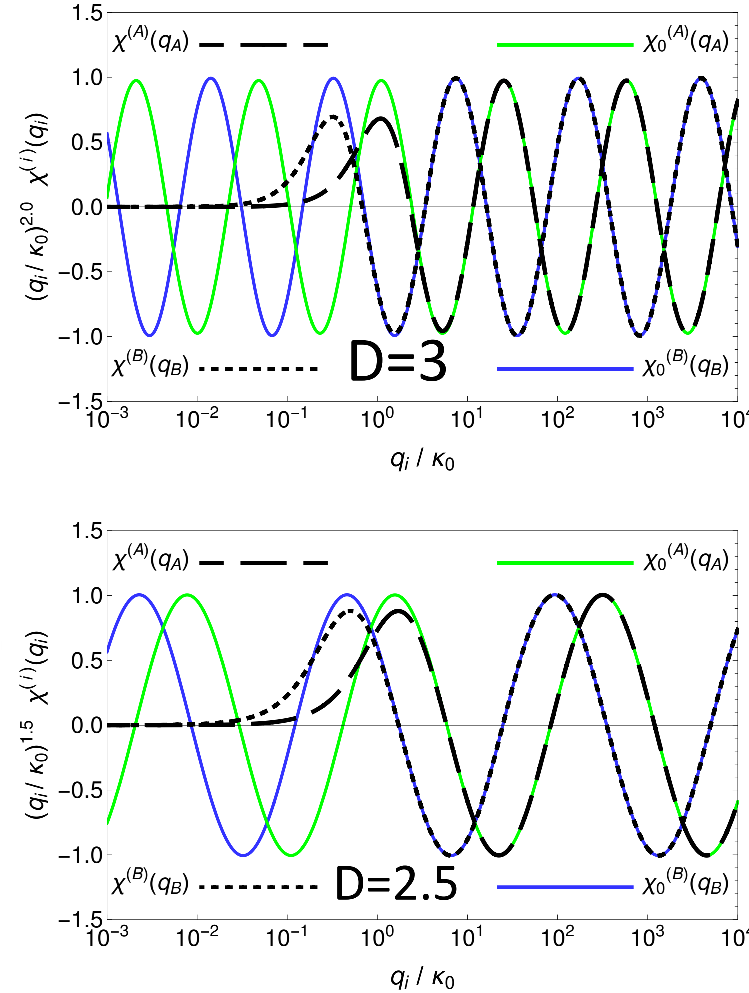


FIG. 1. Spectator functions in momentum space for the $^6\mathrm{Li}{-}^{133}\mathrm{Cs}_2$ system with finite three-body energy, $\chi^{(i)}(q_i)$ ($i = A \equiv {}^{133}\mathrm{Cs}$ or $B \equiv {}^6\mathrm{Li}$), computed with Eq. (2.14) for $\chi^{(A)}(q_A)$ (long-dashed line) and $\chi^{(B)}(q_B)$ (short-dashed line), compared to the zero-energy case from Eq. (2.16) for $\chi_0^{(A)}(q_A)$ (green solid line) and $\chi_0^{(B)}(q_B)$ (blue solid line). Top: three dimensions. Bottom: $D = 2.5$, which corresponds to a harmonic-trap length of $b_{ho}/r_{2D} = \sqrt{2}$.

Single-particle momentum distribution

$$n_B(q_B) = \int d^D p_B \; |\langle \mathbf{q}_B \mathbf{p}_B | \Psi \rangle|^2 \qquad \int d^D q_B \, n_B(q_B) = 1$$

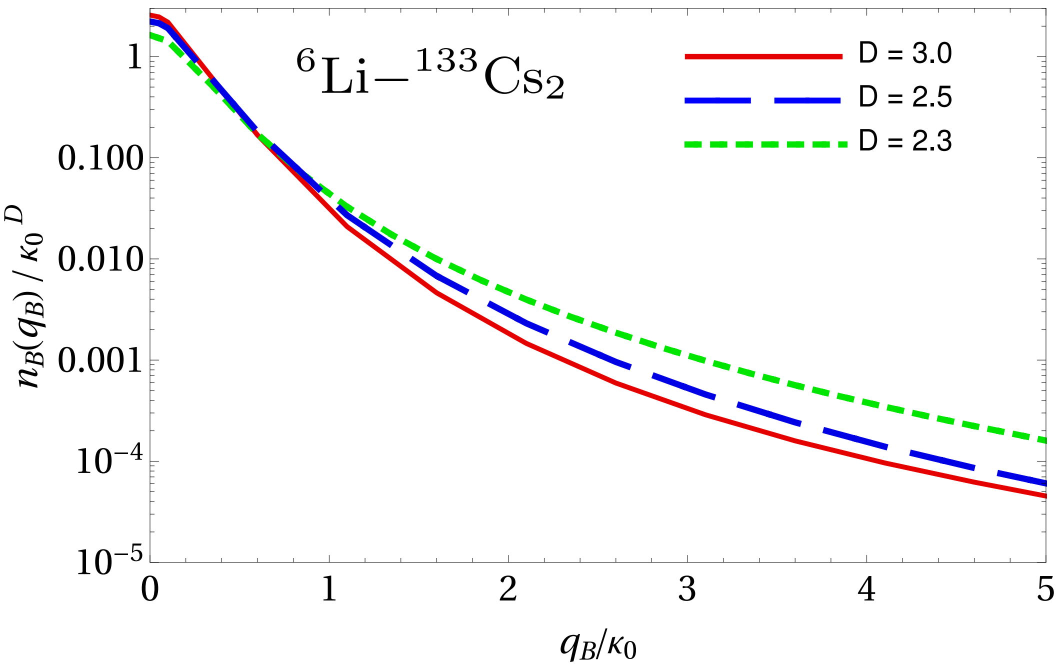


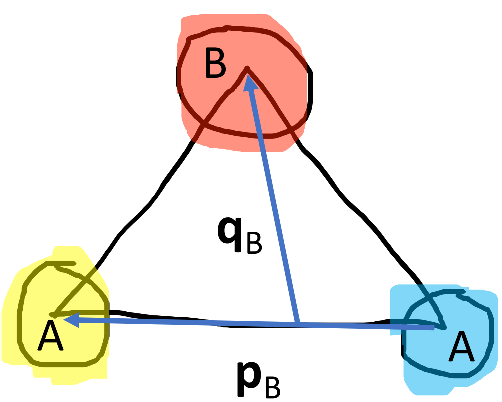
FIG. 2. Single particle momentum distribution, $n_B(q_B)$ of an ^{6}Li$-^{133}$Cs$_2$ Efimov state in $D = 3$ (solid line), $D = 2.5$ (long-dashed line) and $D = 2.3$ (short-dashed line).

Contacts: $q_B \rightarrow$ infinity

AAA & D=3: Castin & Werner, PRA 83, 063614 (2011)

AAB & D=3: Yamashita et al, PRA 87, 062702 (2013)

$$n_B(q_B) = \frac{C_2}{q_B^4} + \frac{C_3'}{q_B^{D+2}} + \frac{C_3}{q_B^{D+2}} \cos\left[2s_0 \log\left(\frac{q_B/\kappa_0^*}{(4\mu_A\mu_B)^{1/4}}\right) + \Phi\right] + \cdots$$



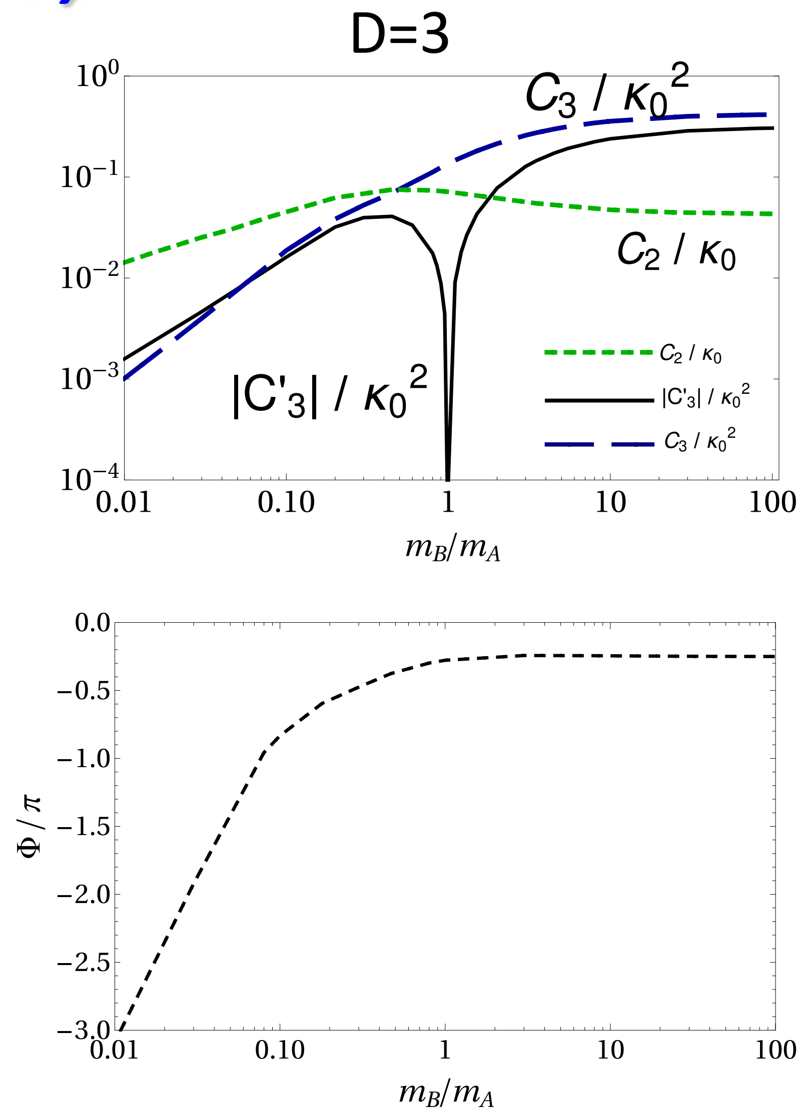


FIG. 4. Three- and two-body contact parameters (top panel) and phase (bottom panel), considering a *AAB* system with different mass ratios embedded in three dimensions.

Contacts: $q_B \rightarrow$ infinity

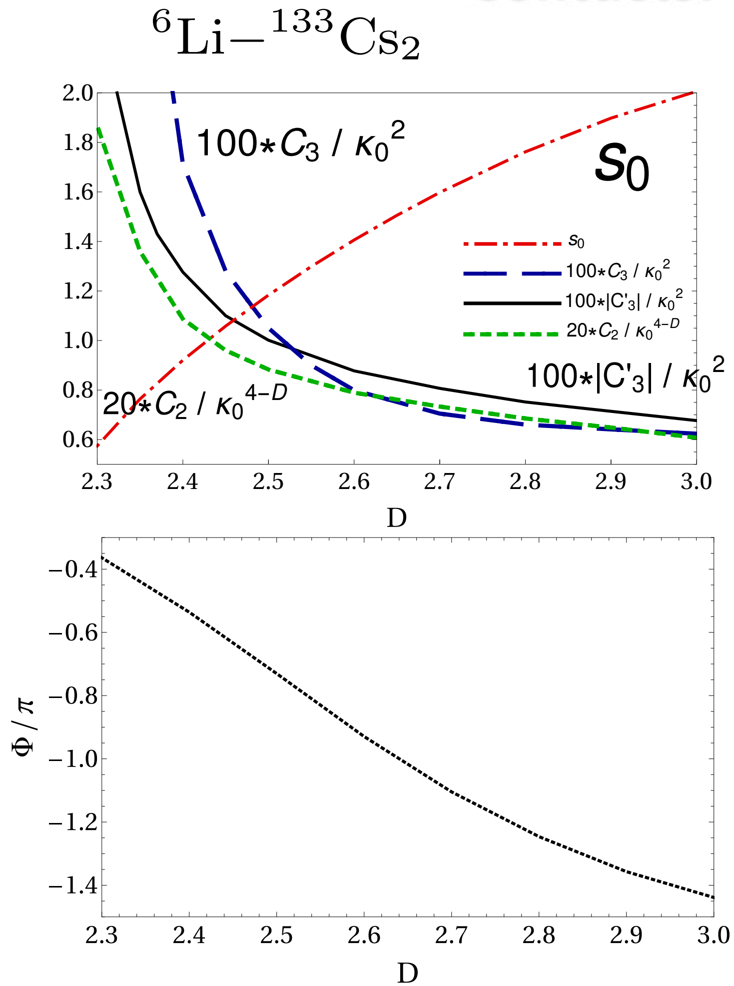


FIG. 5. Three- and two-body contact parameters and phase for the ^{6}Li$-^{133}$Cs$_2$ system in noninteger dimensions from 2.3 to three. Top panel: $100\, C_3'/\kappa_0^2$ (solid line), $100\, |C_3|/\kappa_0^2$ (long-dashed line), $20\, |C_2|/\kappa_0^{4-D}$ (short-dashed line) and s_0 (dot-dashed line). Lower panel: phase Φ/π (dotted line).

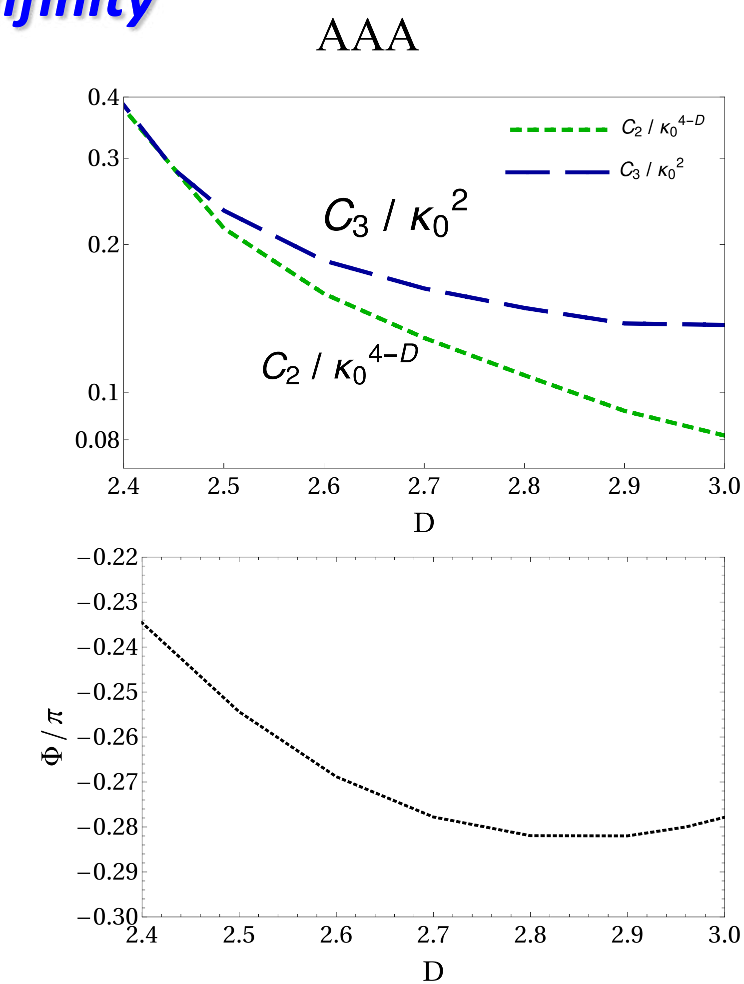
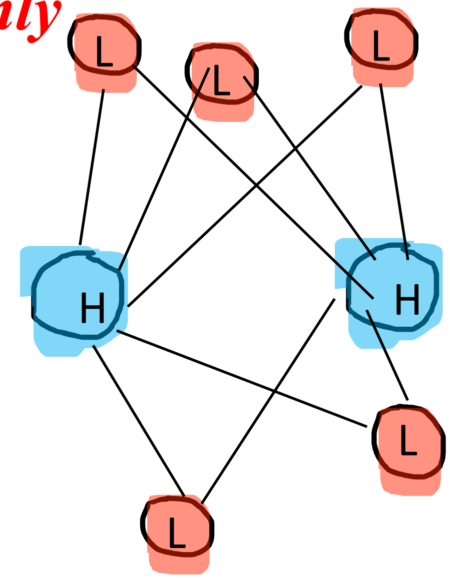


FIG. 6. Three- and two-body contact parameters and phase for three-identical bosons in noninteger dimensions. Top panel: C_3/κ_0^2 (long-dashed line) and C_2/κ_0^{4-D} (short-dashed line). Bottom panel: phase Φ/π (dotted line).

LLHH, LLLHH... systems: B.O. approximation in non-integer dimensions

L-H interaction only

Rosa, TF, Krein, Yamashita, JPB 52, 025101 (2018)
Francisco, Rosa , TF, PRA 106, 063305 (2022)



$$\left[-\frac{d^2}{dR^2} - \frac{m_A}{2\mu_{B,AA}}(N-2)\frac{g(D)}{R^2} + \frac{(D-3+2l)(D-1+2l)}{4R^2}\right]\chi(R) = 0,$$

Solve transcendental equation: $g(D) = \left[-\frac{\pi \csc(D\pi/2)}{2^{\frac{D}{2}}\Gamma(D/2)K_{\frac{D-2}{2}}(\sqrt{g(D)})}\right]^{\frac{4}{2-D}}$

$$s = \sqrt{\frac{m_A}{2\mu_{B,AA}}(N-2)g(D) - \frac{(D-2+2l)^2}{4}}$$

- 3 , 4, 5... independent cycles
- 4-body cycle and no 3-body cycle
- turn-off cycles with D !

Summary

- ✓ *Discrete scaling in non-integer dimension & Efimov and Thomas effects;*
- ✓ *Bethe-Peierls B.C.: analytic form of the wave function ABC system in D dimensions @unitarity (applications to halo nuclei)*
- ✓ *Contacts of the AAB system increase with D=3→2 (up to a factor ~ 2);*
- ✓ *Discrete scaling in N-boson systems - BO approx.: new scales and discrete cycles ;*
- ✓ *Dimension/squeezing for when D=3→2 manipulates the discrete N-body cycles independently.*

THANK YOU!!!!

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