Unraveling Universal Correlations

Gaussian Characterization of Systems Near the Unitary Limit

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Outline

(Enlarged) Unitary Window

- S-matrix
- Effective Range Expansion
- Zero-shape Universality
- Gaussian (Eckart) characterization

Three-body sector

- Efimov Effect
- Level Functions - Gaussian Characterization
- Moving along the universal curve
- Note on DSI

More particles

- LO Gaussian Potential - Two- and Three-Body Force

References

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S-Matrix

- Two-body Schrödinger equation $\Longleftrightarrow$ S-matrix

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  - Scaling (zero-range) limit

$$r_B/a_B \to 0 \quad \text{with } a_B \text{ fixed}$$

$$S(k) \sim -\frac{k + i/a_B}{k - i/a_B}$$

Effective Range Expansion

- S-matrix

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$$r_B = a - a_B$$

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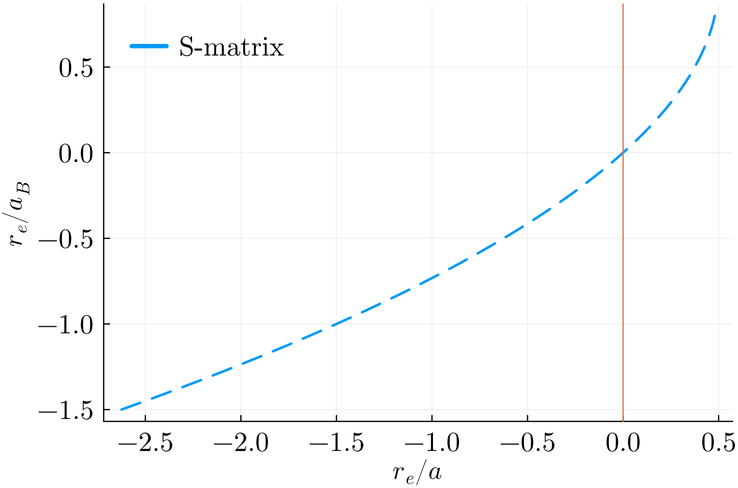
$$r_B = a - a_B$$

  - Effective range relation

$$r_e a = 2 r_B a_B$$

Zero-shape universality



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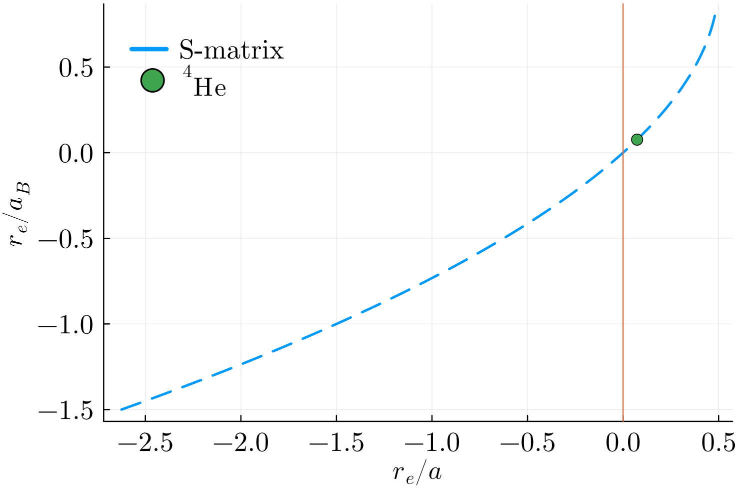

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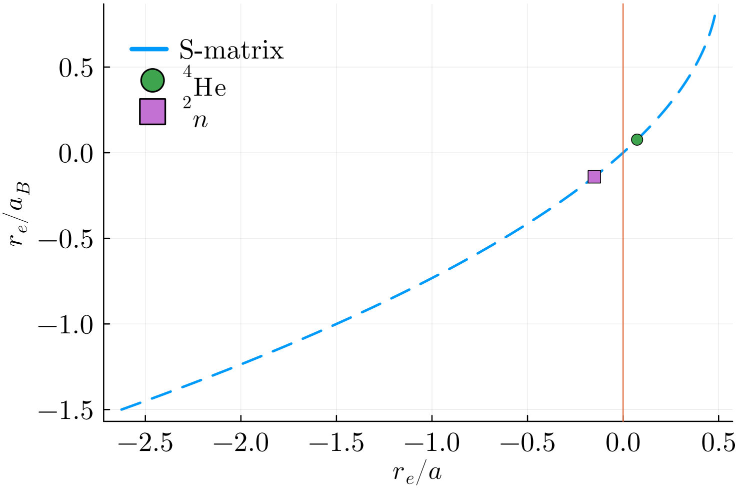

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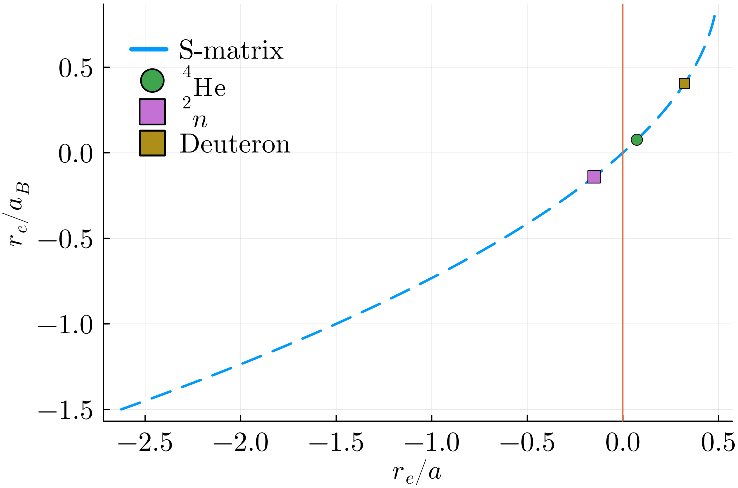

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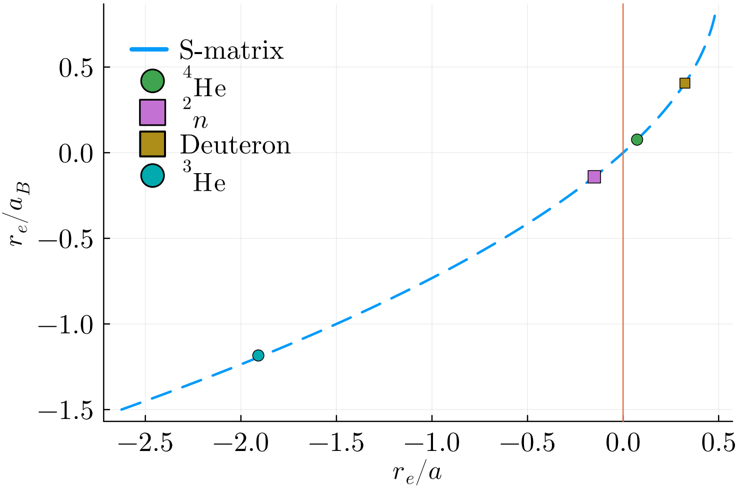

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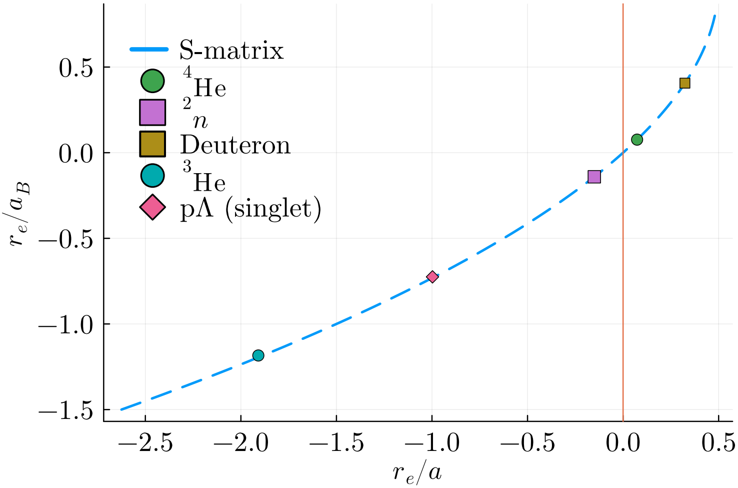

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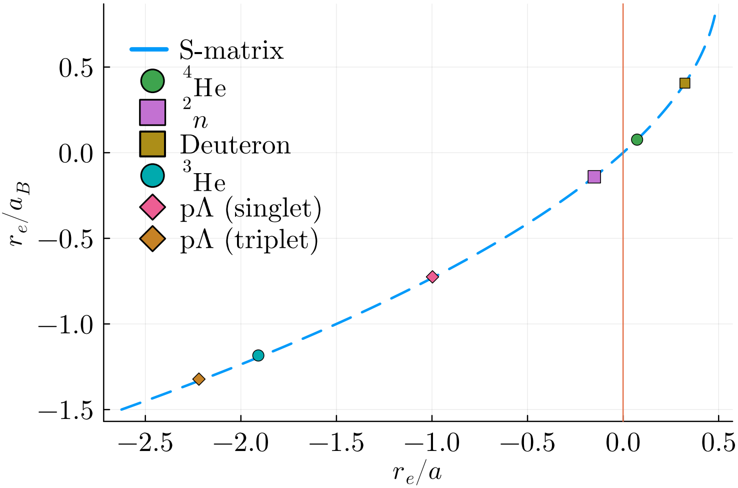

Eckart Potential

$$V(r) = -2\beta\lambda^2 \frac{\mathsf{e}^{-\lambda r}}{(1+\beta \mathsf{e}^{-\lambda r})^2}$$

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- Binding length

$$a_B = \frac{1}{\lambda}\frac{2(\beta+1)}{\beta-1}$$

- Scattering length

$$a = \frac{4\beta}{\lambda(\beta-1)}$$

- Effective range

$$r_e = \frac{2(\beta+1)}{\lambda\beta}$$

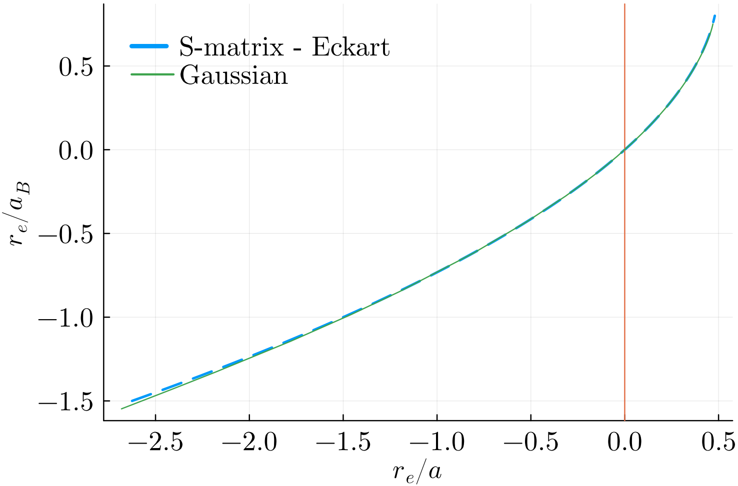
- "Interaction pole"

$$r_B = \frac{2}{\lambda}$$

Gaussian characterization

Effective Description using Gaussian Potential

$$V(r) = V_0\, e^{-(r/r_0)^2}$$



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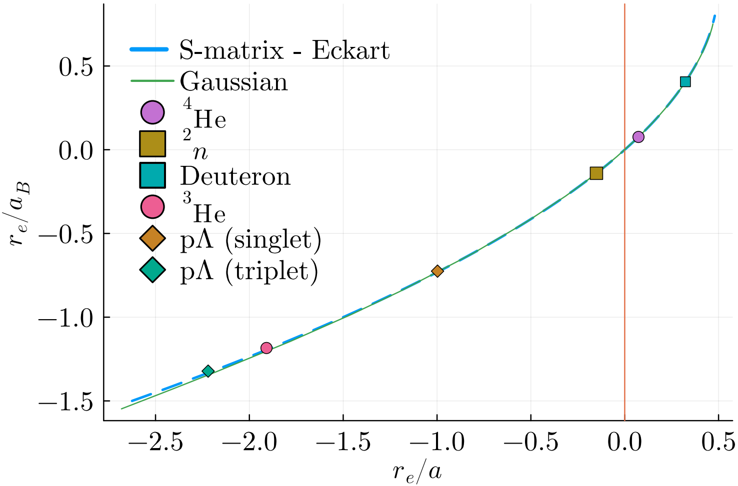
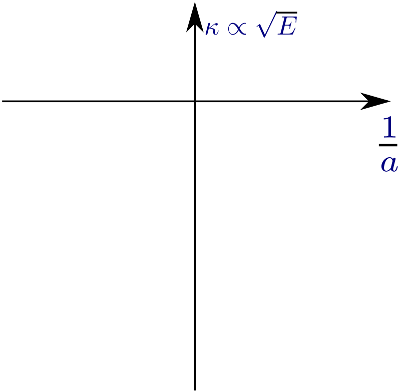


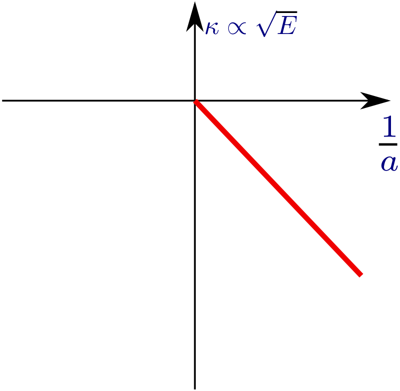
Outline

Efimov Effect



Efimov Effect

$$E_2 \propto \frac{1}{a^2}$$



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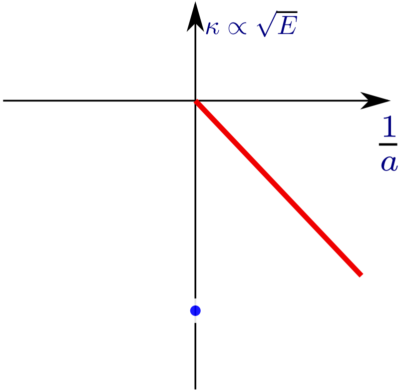

$$@1/a = 0 \quad \left\{ \quad E_3^0 \propto \frac{1}{\ell^2} \right.$$

Efimov Effect

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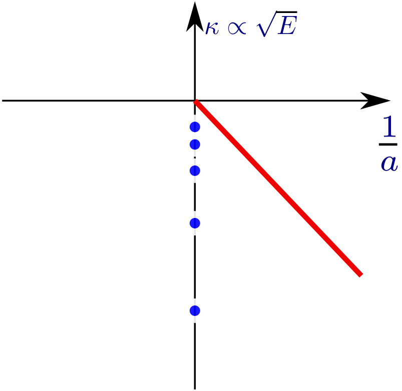

$$@1/a = 0 \quad \begin{cases} E_3^0 \propto \frac{1}{\ell^2} \\ E_3^n \to 0 \qquad n \to \infty \end{cases}$$

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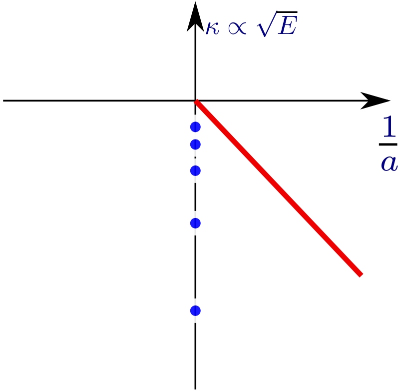

$$@1/a = 0 \quad \begin{cases} E_3^0 \propto \frac{1}{\ell^2} \\ E_3^n \to 0 \qquad n \to \infty \\ E_3^{n+1}/E_3^n \to 1/515 \\ E_3^n \sim (1/515)^n \kappa_*^2 \end{cases}$$

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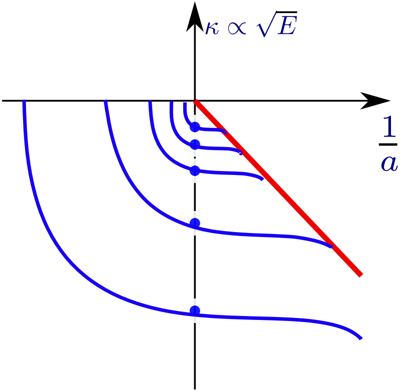

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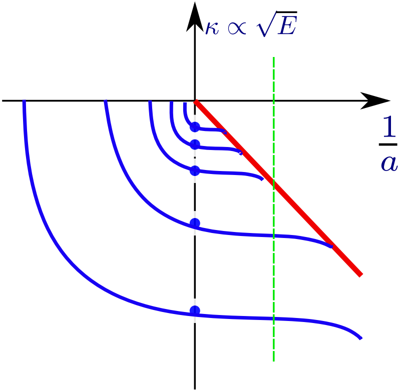

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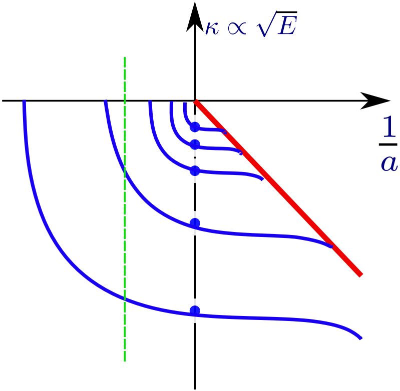
Finite # E_3's

Efimov Effect



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Borromean states

Efimov Effect

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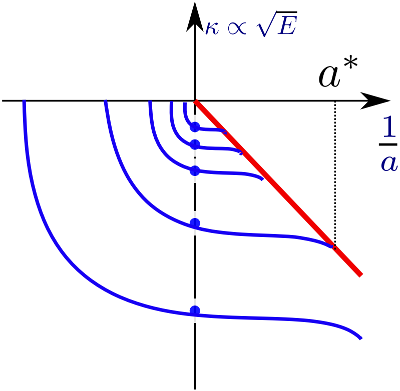

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$@\ a = a^*$

$$P_3 \to P_2 + P$$

Efimov Effect

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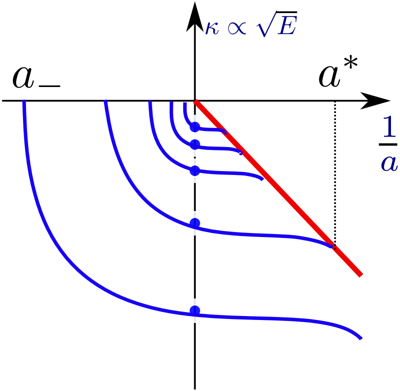

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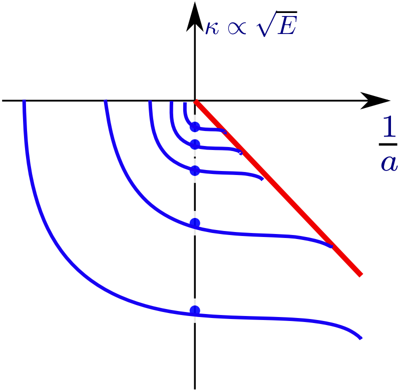
Efimov Levels - Discrete Scale Invariance



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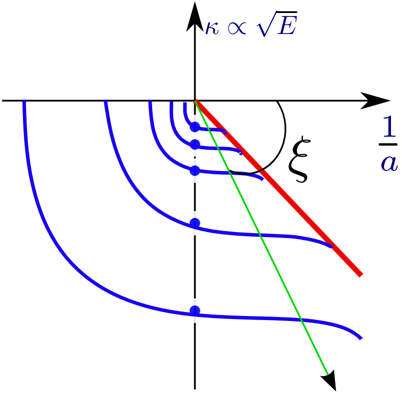

Polar coordinates

$$(H)^2 = (E_3 + E_2)/(\hbar^2/m)$$
$$\tan^2 \xi = E_3/E_2$$

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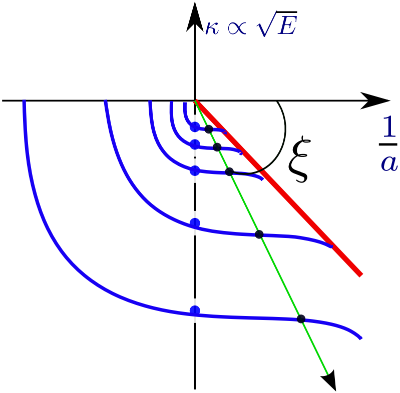

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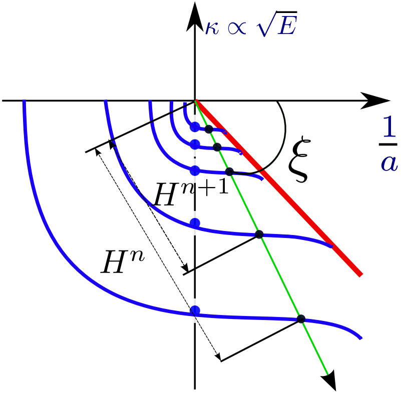

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For each ξ

$$H^{n+1}/H^n \to 1/22.7$$

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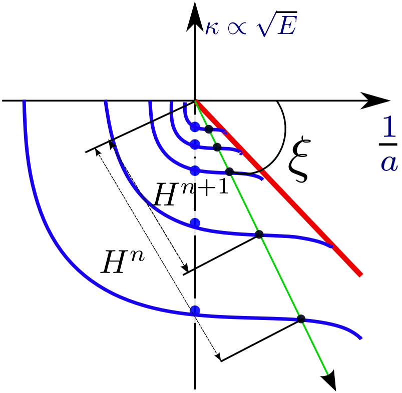

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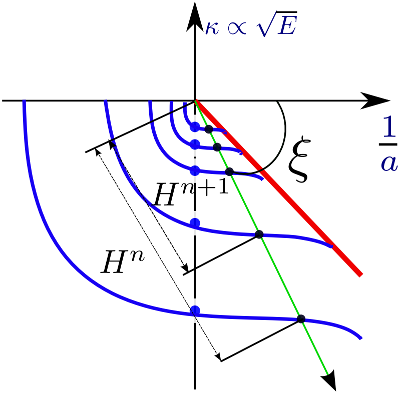
$$E_3^n + \frac{\hbar^2}{ma^2} = \frac{\hbar^2 \kappa_*^2}{m} \mathrm{e}^{-2(n-n^*)\pi/s_0} \mathrm{e}^{\Delta(\xi)/s_0}$$

Efimov Levels - Discrete Scale Invariance



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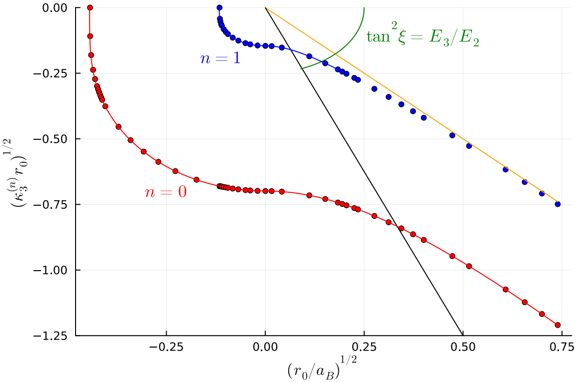
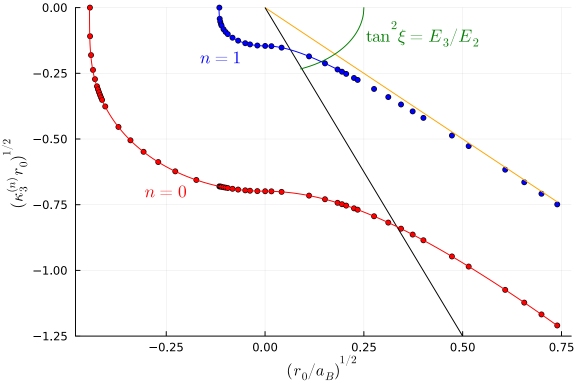
$$\begin{cases} E_3^n/(\hbar^2/ma^2) = \tan^2 \xi \\ \kappa_* a = \mathrm{e}^{(n-n^*)\pi/s_0} \dfrac{\mathrm{e}^{-\Delta(\xi)/2s_0}}{\cos \xi} \end{cases}$$

Gaussian Level Function - Universality



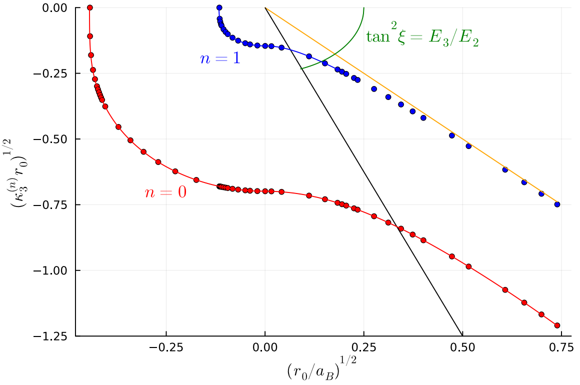
Gaussian Level Function - Universality

- Zero Range $\kappa_* a = \mathrm{e}^{-\Delta(\xi)/2s_0} / \cos\xi$



Gaussian Level Function - Universality

- Zero Range $\quad \kappa_* a = \mathrm{e}^{-\Delta(\xi)/2s_0} / \cos\xi$
- Potential $\quad \kappa_* a_B = \mathrm{e}^{-\widetilde{\Delta}(\xi)/2s_0} / \cos\xi$



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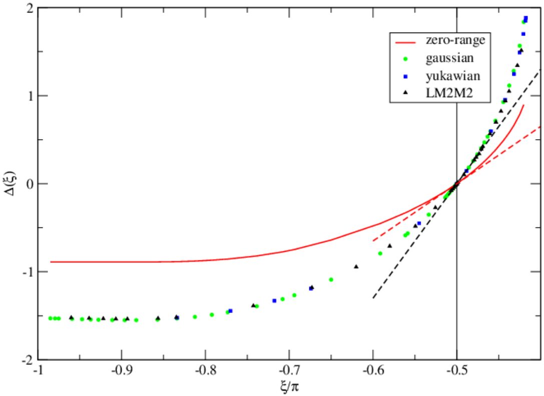

Moving along the curve

- Scale Invariance

$$\kappa_* a_B\Big|_{\text{Gaussian}} = \mathcal{F}(\xi)$$

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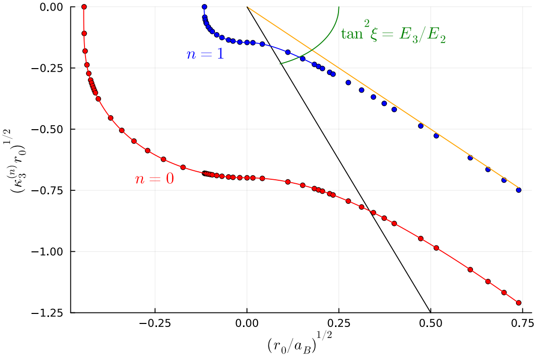

Moving along the curve

- Scale Invariance

$$\kappa_* a_B \Big|_{\text{Gaussian}} = \mathcal{F}(\xi) = \kappa_* a_B \Big|_{\text{Other finite range potentials}}$$

- Unique r_0

Potential	E_2(mK)	E_3(mK)	E_4(mK)	$r_0^{(3)}(a_0)$	$r_0^{(4)}(a_0)$
HFD-HE2	0.8301	117.2	535.6	11.146	11.840
LM2M2	1.3094	126.5	559.2	11.150	11.853
HFD-B3-FCH	1.4475	129.0	566.1	11.148	11.853
CCSAPT	1.5643	131.0	571.7	11.149	11.851
PCKLJS	1.6154	131.8	573.9	11.148	11.852
HFD-B	1.6921	133.1	577.3	11.149	11.854
SAPT96	1.7443	134.0	580.0	11.147	11.850

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- Energy at the unitary limit given by r_0

$$E_3^* \approx 83\,\text{mK}$$

$$E_4^* \approx 433\,\text{mK}$$

Moving along the curve

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- Unique r_0

Potential	E_2(mK)	E_3(mK)	E_4(mK)	$r_0^{(3)}(a_0)$	$r_0^{(4)}(a_0)$
HFD-HE2	0.8301	117.2	535.6	11.146	11.840
LM2M2	1.3094	126.5	559.2	11.150	11.853
HFD-B3-FCH	1.4475	129.0	566.1	11.148	11.853
CCSAPT	1.5643	131.0	571.7	11.149	11.851
PCKLJS	1.6154	131.8	573.9	11.148	11.852
HFD-B	1.6921	133.1	577.3	11.149	11.854
SAPT96	1.7443	134.0	580.0	11.147	11.850

- Energy at the unitary limit given by r_0

$$E_3^* \approx 83\,\text{mK}$$

$$E_4^* \approx 433\,\text{mK}$$

- Universal numbers

$$\kappa_3^* a_-^3 = -2.13 \qquad \kappa_4^* a_-^4 = -2.32$$

Note on DSI

- DSI $\Rightarrow$ Log-periodic functions

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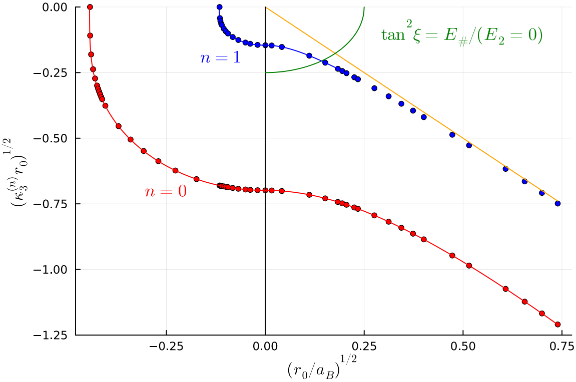

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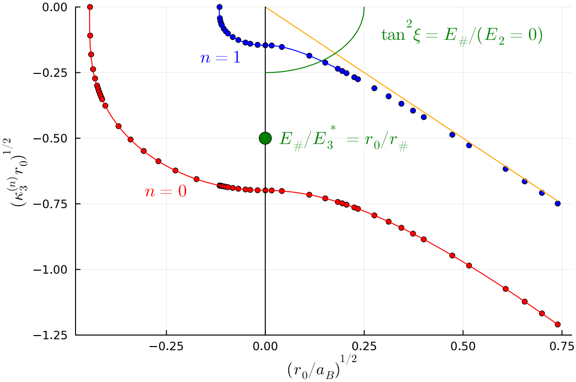

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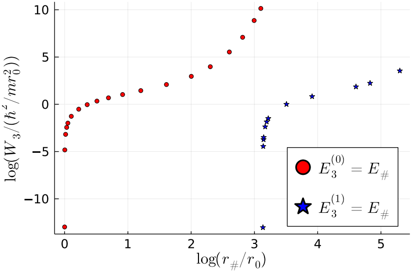

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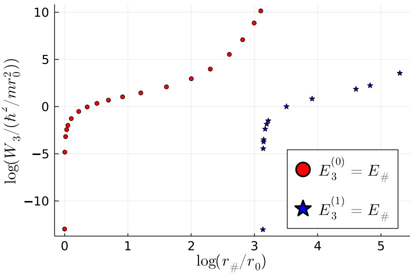

- Non analyticity

$$W_3 \sim \frac{\hbar^2}{mr_0^2}\, e^{(r_\#/r_0)^{1.13}}$$

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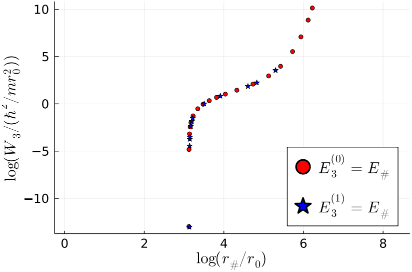

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- Log-periodicity

$$\log(r_\#/r_0) \to \log(r_\#/r_0) - \pi/s_0$$

Outline

Effective Gaussian Description of ^{4}He

- "*Reference*" ^{4}He given by LM2M2 potential

$$\bar{a} = 189.415\, a_0, \bar{r}_e = 13.845\, a_0, \text{and} \;\; r_B = 7.194\, a_0$$

N	$\bar{E}_N$(mK)	$\bar{E}^*_N$(mK)
2	-1.30348	
3	-126.40	-2.2706
4	-558.98 [Hiyama 2012]	-127.33 [Hiyama 2012]
5	-1300 [Bazak 2020]	
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- Effective Gaussian Potential

$$V_{\text{LO}}(r) = V_0\, e^{-(r/r_0)^2}$$

- Small parameter

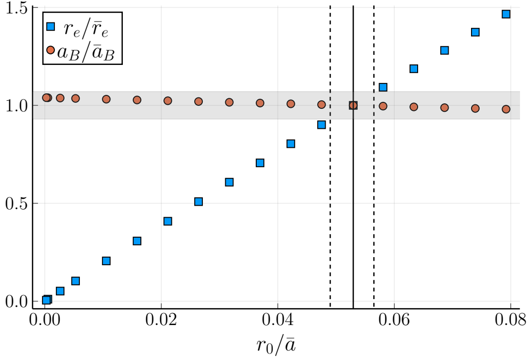
$$\varepsilon = \bar{r}_e/\bar{a} \approx 7\%$$

Two Body

- Effective Gaussian Potential

$$V_{\rm LO}(r) = V_0\, e^{-(r/r_0)^2}$$

- Fix only $\bar{a}$



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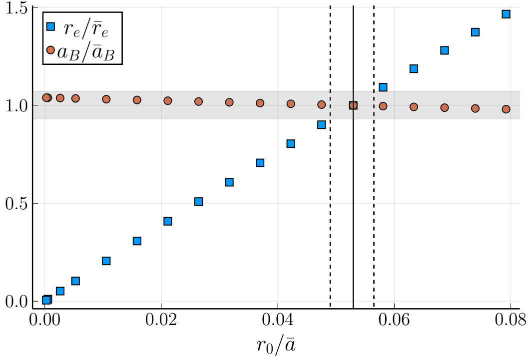
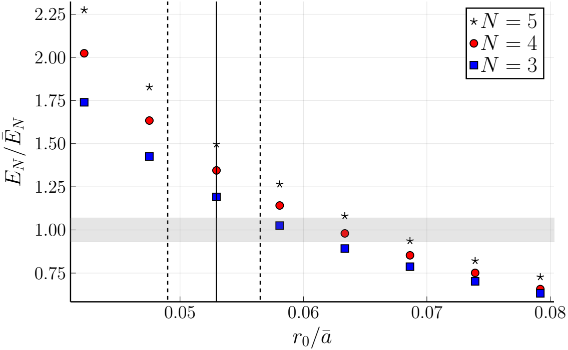


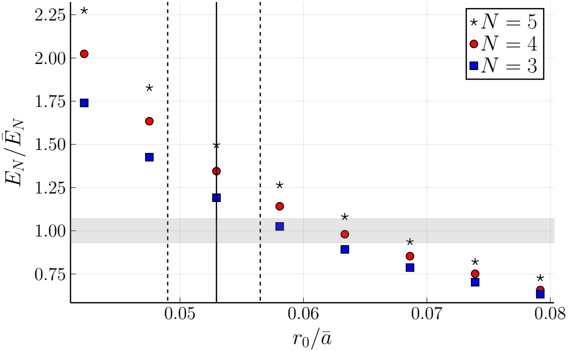
- Look for $\varepsilon = \bar{r}_e/\bar{a} \approx 7\%$ description also for r_e

Few Body



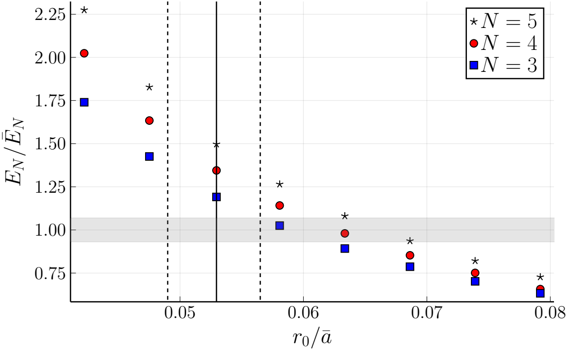
2.25
2.00
1.75
1.50
1.25
1.00
0.75
$E_N/\bar{E}_N$
0.05
0.06
0.07
0.08
$r_0/\bar{a}$
$N = 5$
$N = 4$
$N = 3$

Few Body



- Not inside the $\varepsilon = 7\%$ band

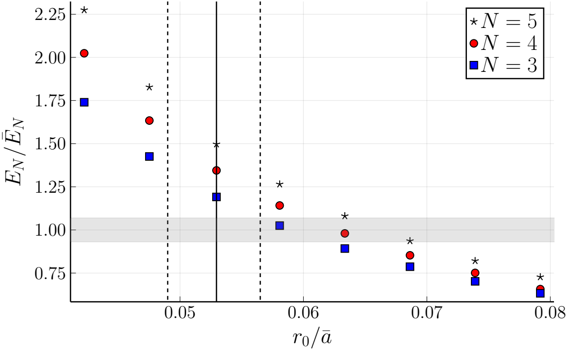
Few Body



- Not inside the $\varepsilon = 7\%$ band
- Collapse as $N \to \infty$

$$\frac{E_N}{N} = \frac{V_0}{2} N$$

Few Body



- Not inside the $\varepsilon = 7\%$ band
- Collapse as $N \to \infty$

$$\frac{E_N}{N} = \frac{V_0}{2} N$$

- Need for a three-body force

Few Body

- Three-body force

$$W_{\mathrm{LO}} = W_0\, e^{-(r_{12}^2 + r_{13}^2 + r_{23}^2)/\rho_0^2}$$

Few Body

- Three-body force

$$W_{\mathrm{LO}} = W_0\, e^{-(r_{12}^2+r_{13}^2+r_{23}^2)/\rho_0^2}$$

- A family of values (W_0, ρ_0) which fix $\bar{E}_3$

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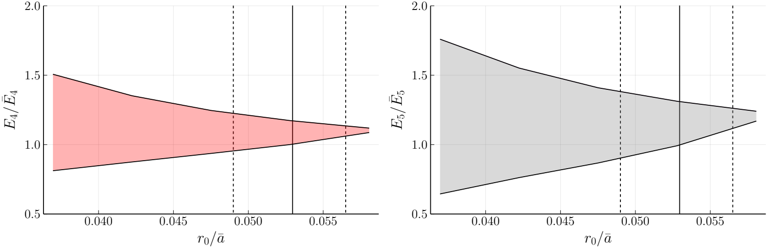
- Variation in $\bar{E}_N$



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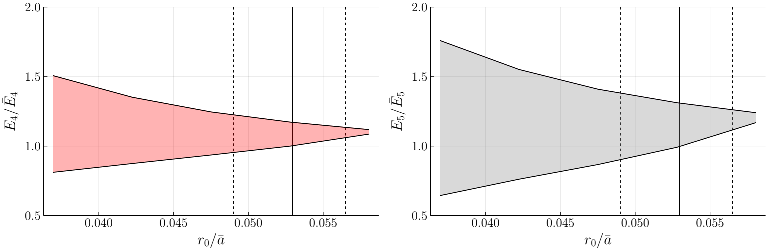
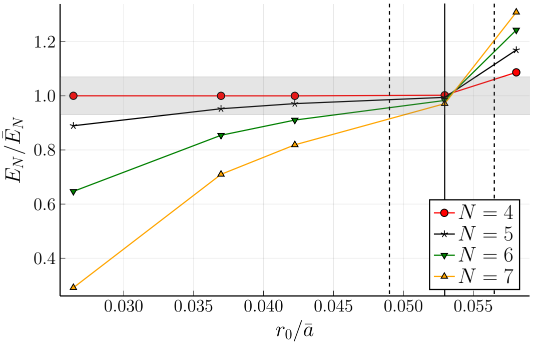


- We can use (W_0, ρ_0) to best fix $\bar{E}_4$

LO Gaussian Description

- LO Potential

$$V_0\, e^{-(r/r_0)^2} + W_0\, e^{-(r_{12}^2+r_{13}^2+r_{23}^2)/\rho_0^2}$$



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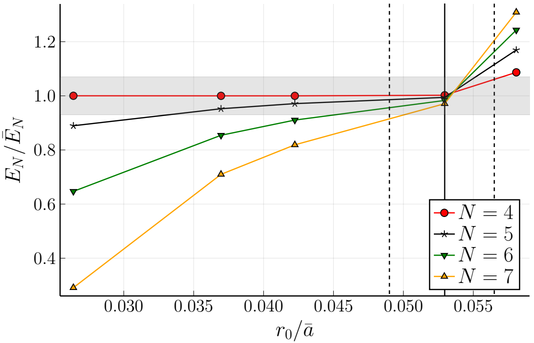

- Best point is where we reproduce $\bar{a}$, $\bar{a}_B$, and r_e!!

LO Gaussian Description

- Description within the ε-LO band up to liquid

	Physical point	
	SGP	HFD-HE2
$r_0[a_0]$	10.0485	
V_0[K]	1.208018	
$\rho_0[a_0]$	8.4853	
W_0[K]	3.011702	
E_4[K]	0.536	0.536
E_5[K]	1.251	1.266
E_6[K]	2.216	2.232
$E_{10}/10$[K]	0.792(2)	0.831(2)
$E_{20}/20$[K]	1.525(2)	1.627(2)
$E_{40}/40$[K]	2.374(2)	2.482(2)
$E_{70}/70$[K]	3.07(1)	3.14(1)
$E_{112}/112$[K]	3.58(2)	3.63(2)
$E_N/N(\infty)$[K]	7.2(3)*	7.14(2)
HFD-B [K]		7.33(2)

Outline

(Enlarged) Unitary Window

- S-matrix
- Effective Range Expansion
- Zero-shape Universality
- Gaussian (Eckart) characterization

Three-body sector

- Efimov Effect
- Level Functions - Gaussian Characterization
- Moving along the universal curve
- Note on DSI

More particles

- LO Gaussian Potential - Two- and Three-Body Force

References

References

- For the finite-range universality

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A. Kievsky, M. Gattobigio, L. Girlanda, M. Viviani
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- For the universal Level functions

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Phys. Rev. A 93, 062701 (2016) ▸ Link

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- For the LO- and NLO-Gaussian description of ^{4}He

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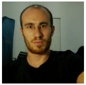
Phys. Rev. A **106**, 022812 (2022) ▸ Link

Collaborators



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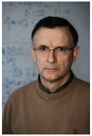

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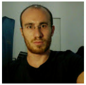

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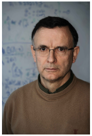

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Thanks!

NLO Gaussian Description - Two body

- NLO two-body force

$$V_{\text{NLO}}(r) = V_0\, e^{-(r/r_0)^2} + V_1\, \frac{r^2}{r_0^2} e^{-(r/r_0)^2}$$

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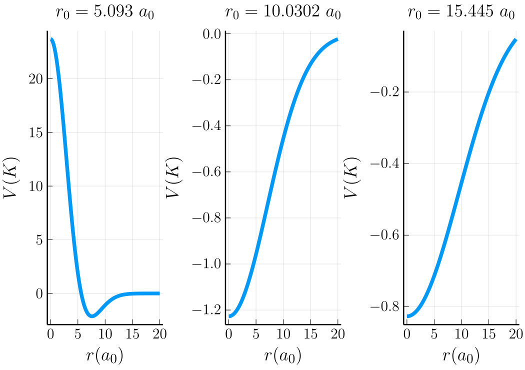
- We fix both $\bar{a}$ and $\bar{r}_e$



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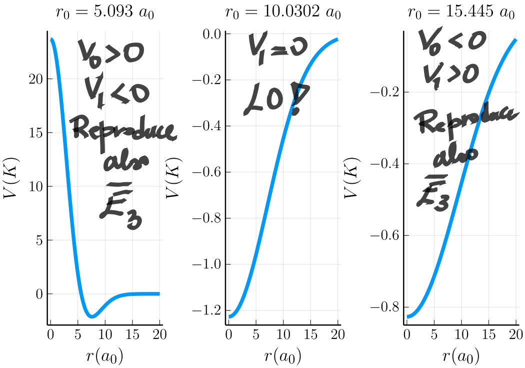
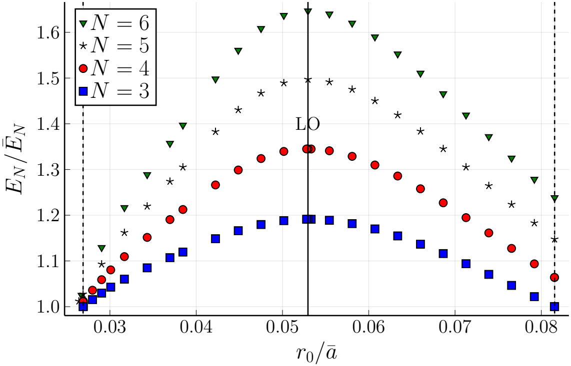


NLO Two body - Few-body energies



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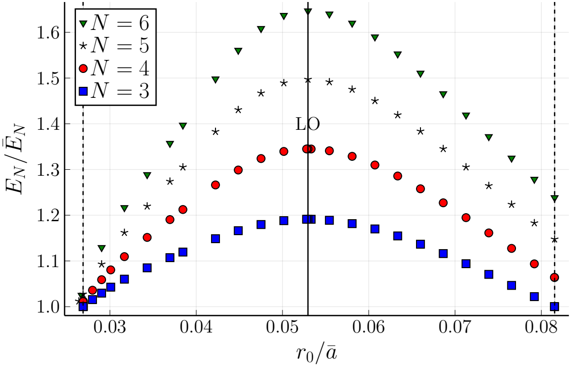


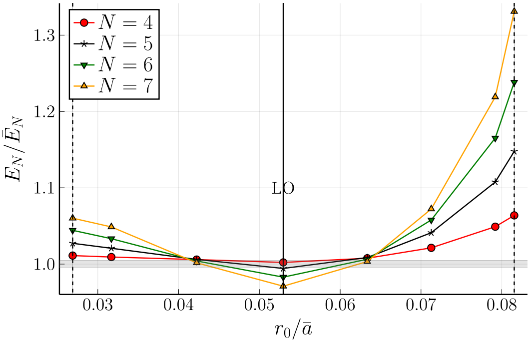
- Without 3-body force the system is unstable

$$\frac{E_N}{N} \propto N$$

NLO Two body + LO Three body

- With the LO 3-body force

$$W_{\text{LO}} = W_0\, e^{-(r_{12}^2+r_{13}^2+r_{23}^2)/\rho_0^2}$$



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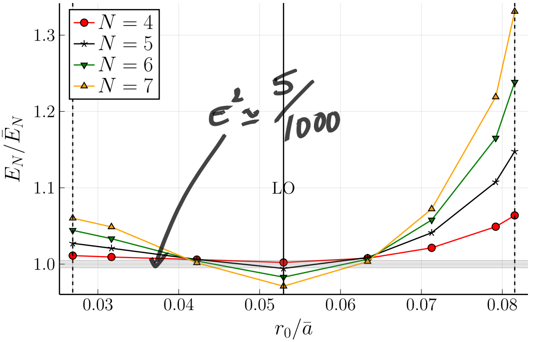

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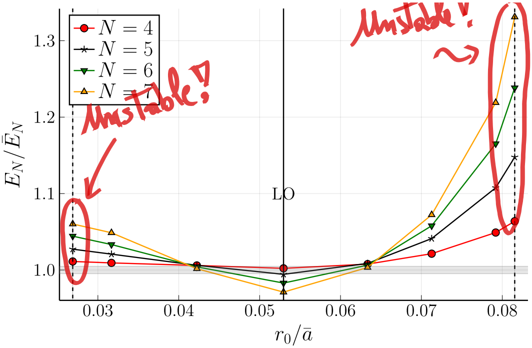
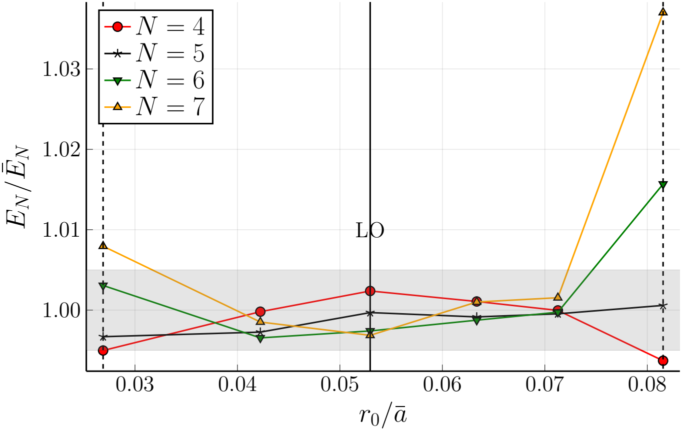


- Need another force at NLO!!!

Analysis with NLO 3-Body

- NLO Three-body force

$$W_{\text{NLO}} = W_0 e^{-r_{123}^2/\rho_0^2} + W_1 \left(\frac{r_{123}}{\rho_0}\right)^2 e^{-r_{123}^2/\rho_0^2}$$



Analysis with NLO 3-Body

- What happens to different three-body observables?

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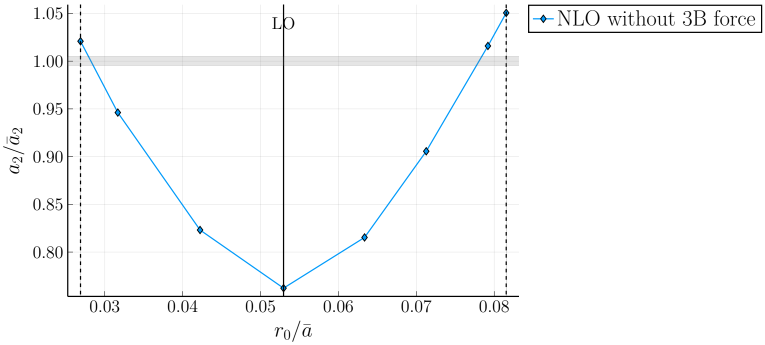

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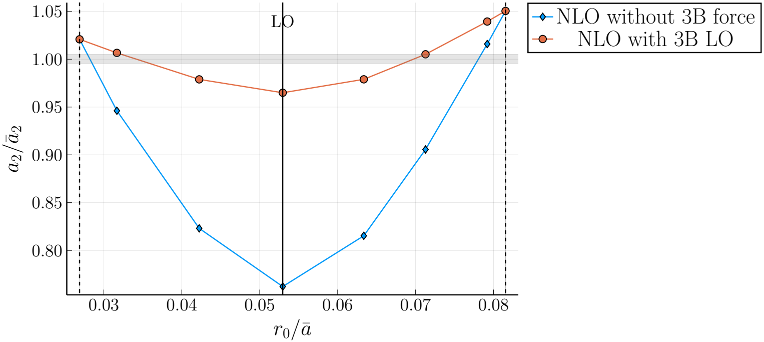

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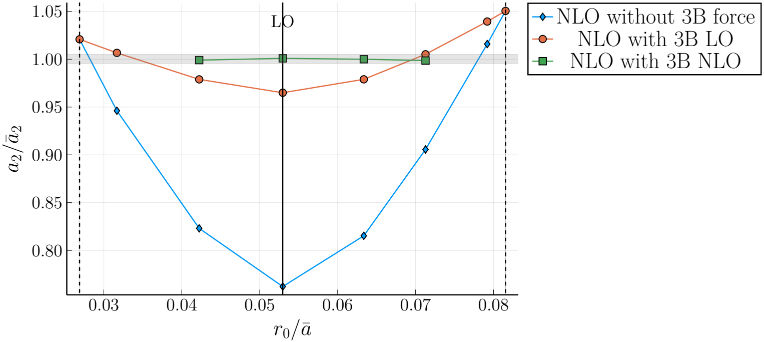

- Different 3-Body potential strengths

Analysis with NLO 3-Body

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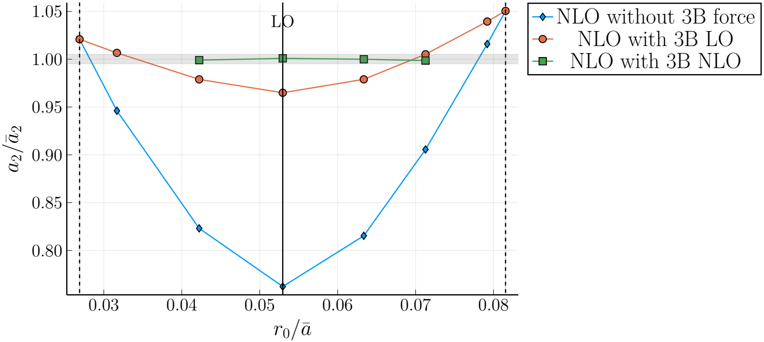

- Different 3-Body potential strengths
- Space for a NLO 4-Body potential