Disconnected beyond 2+2

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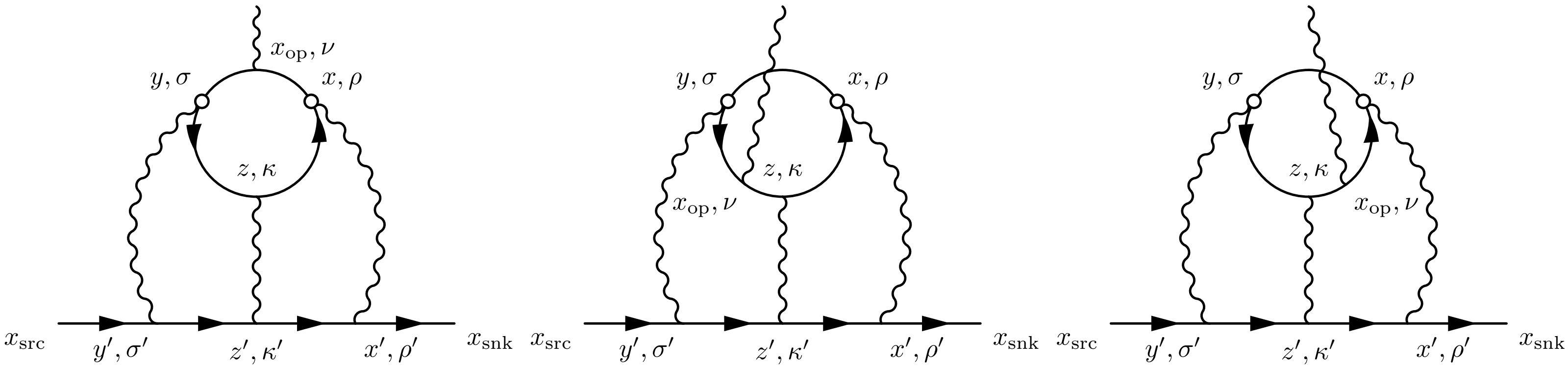
Helmholtz-Institut Mainz

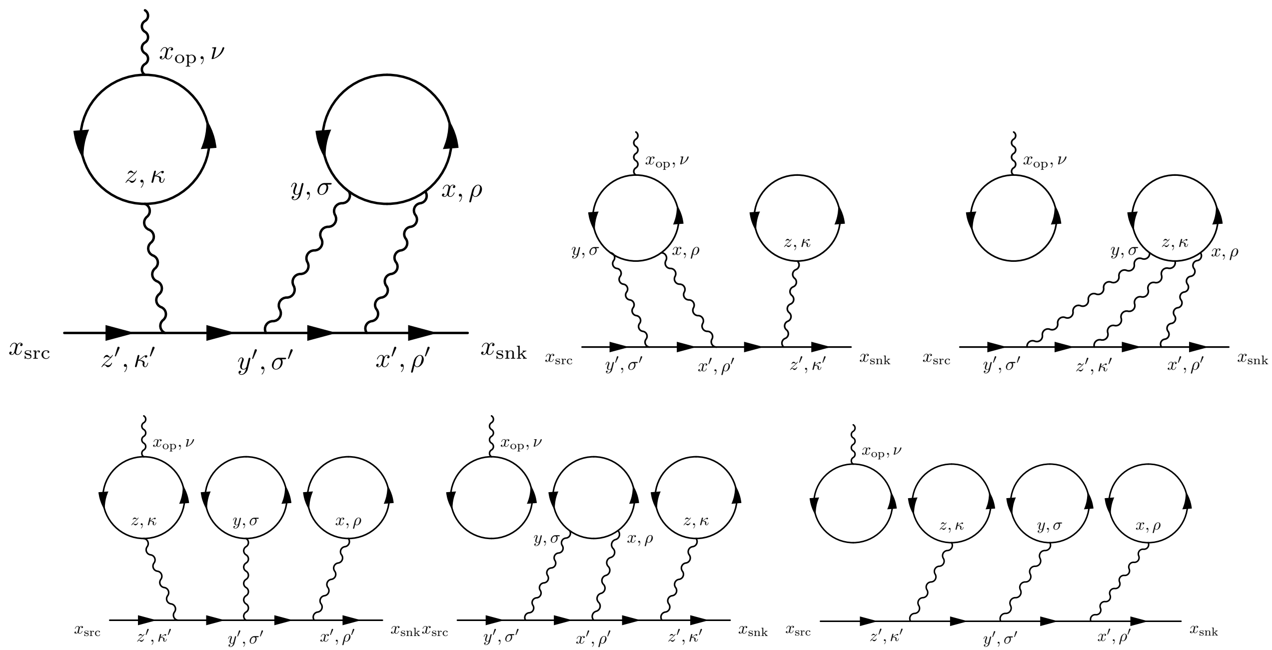
Second Plenary Workshop of the Muon g-2 Theory Initiative



- Permutations of the three internal photons are not shown.
- There should be gluons exchange between and within the quark loops, but are not drawn.

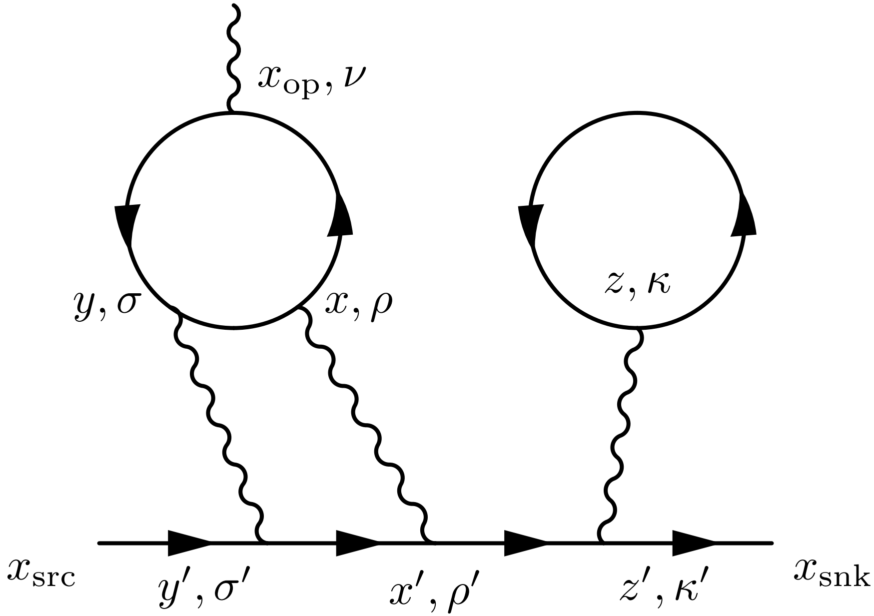
Disconnected diagrams

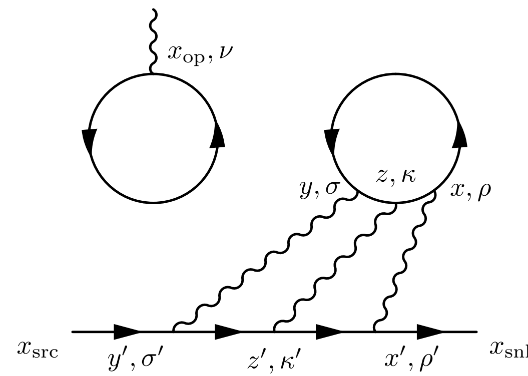
- One diagram (the biggest diagram below, referred to as $2+2$) do not vanish even in the $\mathrm{SU}(3)$ limit.
- We extend the method and computed this leading disconnected diagram as well.



- Permutations of the three internal photons are not shown.
- There should be gluons exchange between and within the quark loops, but are not drawn.
- We need to make sure that the loops are connected by gluons by "vacuum" subtraction. So the diagrams are 1-particle irreducible.

Disconnected diagram beyond 2+2





- Still working in progress.
- The right loop has been calculated by Christoph Lehner (can also be used to calculated disconnected HVP) and saved to disk.
- The left loop can be evaluated by two point source propagators at x and y. We can then randomly sample x and y, similar to the way we evaluted the connected diagrams.

Pion Transition Form Factor (TFF) on Lattice: RBC results

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We will be working in Euclidean space by default.

- **Pion TFF formulation**
- Model and Lattice results
- Contribution to HLbL with pion TFF

$$\langle 0|T i\, J_\mu(u)\, i\, J_\nu(v)|\pi^0(\vec{p})\rangle \tag{1}$$

Momentum space TFF $F(q_1^2, q_2^2)$ (X.D. Ji, C. Jung, [hep-lat/0101014]):

$$\int d^4u\, e^{-iq_1\cdot u - iq_2\cdot v}\, \langle 0|T i\, J_\mu(u)\, i\, J_\nu(v)|\pi^0(\vec{p})\rangle \;=\; \frac{i}{4\,\pi^2\,F_\pi}\,\epsilon_{\mu,\nu,\rho,\sigma}\, q_{1,\rho}\, q_{2,\sigma}\, F(q_1^2, q_2^2). \tag{2}$$

Coordinate space TFF $F_c(x, z^2)$ (previously presented at UConn by Cheng Tu):

$$\begin{aligned} &\langle 0|T i\, J_\mu(u)\, i\, J_\nu(v)|\pi^0(\vec{p})\rangle \\ =\;& \frac{i}{4\,\pi^2\,F_\pi}\,\epsilon_{\mu,\nu,\rho,\sigma}\,(-i\,\partial_\rho^u)\,(-i\,\partial_\sigma^v)\, F'(p\cdot(u-v), (u-v)^2)\, e^{ip\cdot v}, \end{aligned} \tag{3}$$

Let $r = u - v$, $F_c(x, r^2)$ is the Fourier transformation of $F'(p\cdot r, r^2)$:

$$F'(p\cdot r, r^2) \;=\; \int_{-\infty}^{\infty} dx\, F_c(x, r^2)\, e^{ixp\cdot r}. \tag{4}$$

Interestingly, we can prove that:

$$F_c(x, r^2) \;=\; 0 \quad \text{if } x < 0 \text{ or } x > 1. \tag{5}$$

Pion TFF formulation

$$
\begin{aligned}
& \langle 0 | T\, i\, J_\mu(u)\, i\, J_\nu(v) | \pi^0(\vec{p}) \rangle \\
= \; & \frac{i}{4\,\pi^2\,F_\pi}\,\epsilon_{\mu,\nu,\rho,\sigma}(-i\,\partial^u_\rho)\,(-i\,\partial^v_\sigma) \\
& \quad \times \left\langle 0 \middle| \int_0^1 dx\, F_c(x,(u-v)^2)\, \pi^0(x\,u+(1-x)\,v) \middle| \pi^0(\vec{p}) \right\rangle \qquad (6) \\
= \; & \frac{i}{4\,\pi^2\,F_\pi}\,\epsilon_{\mu,\nu,\rho,\sigma} \\
& \quad \times \left\langle 0 \middle| \int_0^1 dx\, [-\partial^u_\rho\, F_c(x,(u-v)^2)]\, \partial_\sigma \pi^0(x\,u+(1-x)\,v) \middle| \pi^0(\vec{p}) \right\rangle \qquad (7)
\end{aligned}
$$

The coordinate space form factor $F_c(x,r^2)$ can be interpreted this way:

- The dependence on x describe the distribution of the pion source along the segment between the two EM currents.
- In the $r^2 \to 0$ limit, the function can be factorized into **PION DISTRIBUTION AMPLITUDES (PDA)**. At tree level, $F_c(x,r^2)$ is the same as PDA after normalization.

  $F_c(x,r^2) \sim x(1-x)$.
- The parameter $r=(u-v)$ is the separation between the two EM currents.

Define

$$\hat{P}_{\pi^0} \; = \; \int \frac{d^3p}{(2\pi)^3} \, |\pi^0(\vec{p})\rangle \, \frac{1}{2\,E_{\pi^0,\vec{p}}} \, \langle \pi^0(\vec{p})| . \tag{8}$$

$$\left\langle 0 \middle| \pi^0(x)\, \hat{P}_{\pi^0}\, \pi^0(y) \middle| 0 \right\rangle \; = \; G(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{p^2+m_\pi^2} \tag{9}$$

$$\begin{aligned} & \left\langle 0 \middle| T\left[i\,J_\mu(u) i\,J_\nu(v)\right] \hat{P}_{\pi^0}\, \pi^0(w) \middle| 0 \right\rangle \\ = \; & \left\langle 0 \middle| \frac{i}{4\,\pi^2\,F_\pi}\, \epsilon_{\mu,\nu,\rho,\sigma} \int_{-\infty}^{\infty} dx \left[-\partial_\rho^u\, F_c(x,(u-v)^2)\right] \partial_\sigma \pi^0(x\,u + (1-x)\,v) \hat{P}_{\pi^0} \pi^0(w) \middle| 0 \right\rangle \\ = \; & \frac{i}{4\,\pi^2\,F_\pi}\, \epsilon_{\mu,\nu,\rho,\sigma} \int_{-\infty}^{\infty} dx \left[-\partial_\rho^u\, F_c(x,(u-v)^2)\right] \partial_\sigma G(x\,u + (1-x) v - w) . \end{aligned} \tag{10}$$

Let $w = x\,u + (1-x)v$, the above expression should not be singular when $x > 1$ or $x < 0$. Therefore $F_c(x,(u-v)^2)$ should be zero for x outside of $[0,1]$.

Pion TFF formulation

Let $f(|r|)$ be the function which describes the strength of the $\pi^0\gamma\gamma$ coupling:

$$\int_0^1 dx\,[-\partial_\rho^r F_c(x,r^2)] \;=\; 2\,z_\rho\left[\frac{2F_\pi^2}{3}\frac{1}{(r^2)^2}\right] f(|r|). \tag{11}$$

- Based on Chiral anomaly, $(\pi^2/2)\int_0^\infty (2F_\pi^2/3)\, f(r)\, 2\,r\,dr = 1$, ($F(0,0)=1$).
- Based on OPE, in the $r \to 0$ limit, $f(|r|) \to 1$, ($F(Q^2,Q^2) \to 8\pi^2 F_\pi^2/(3\,Q^2)$).

For HLbL, the long distance contribution should be dominated by the π^0 exchange process, where the π^0 propagator for a relatively long distance, while the two photons created/annihilated the pion are fairly close.

Therefore, the x dependence of $F_c(x,r^2)$ is less important. Instead, we should focus on the total strength $f(|r|)$.

Outline

We will be working in Euclidean space by default.

- Pion TFF formulation
- **Model and Lattice results**
- Contribution to HLbL with pion TFF

Vector Meson Dominance model

$$F^{\mathrm{VMD}}(q_1^2, q_2^2) = \frac{m_V^2}{q_1^2 + m_V^2} \frac{m_V^2}{q_2^2 + m_V^2} \tag{12}$$

Two Ends model

$$F^{\mathrm{TE}}(q_1^2, q_2^2) = \frac{m_V^2/2}{q_1^2 + m_V^2} + \frac{m_V^2/2}{q_2^2 + m_V^2} \tag{13}$$

Lowest Meson Dominance model

$$F^{\mathrm{LMD}}(q_1^2, q_2^2) = \frac{8\pi^2 F_\pi^2}{3 m_V^2} F^{\mathrm{TE}}(q_1^2, q_2^2) + \left(1 - \frac{8\pi^2 F_\pi^2}{3 m_V^2}\right) F^{\mathrm{VMD}}(q_1^2, q_2^2) \tag{14}$$

Relation between Momentum space form and Coordinate space form:

$$\begin{aligned} F(q_1^2, q_2^2) &= \int d^4 z \, e^{-i q_1 \cdot r} \int_0^1 dx \, F_c(x, r^2) \, e^{i x p \cdot r} \\ &= \int_0^1 dx \int d^4 r \, e^{-i((1-x) q_1 - x q_2) \cdot r} \, F_c(x, r^2) \end{aligned} \tag{15}$$

Pion TFF formulation: VMD model

$$\begin{aligned} F^{\mathrm{VMD}}(q_1^2, q_2^2) &= \frac{m_V^2}{q_1^2 + m_V^2} \frac{m_V^2}{q_2^2 + m_V^2} \\ &= \int_0^1 d\,x \, \frac{m_V^4}{[(1-x)(q_1^2 + m_V^2) + x(q_2^2 + m_V^2)]^2} \\ &= \int_0^1 d\,x \, \frac{m_V^4}{[[(1-x)\,q_1 - x\,q_2]^2 + m_V^2 - x(1-x)m_\pi^2]^2} \end{aligned} \tag{16}$$

Recall

$$F(q_1^2, q_2^2) = \int_0^1 d\,x \int d^4 r \, e^{-i((1-x)\,q_1 - x\,q_2)\cdot r} \, F_c(x, r^2) \tag{17}$$

$$F_c^{\mathrm{VMD}}(x, r^2) = \int \frac{d^4 p}{(2\pi)^4} \frac{m_V^4 e^{ip\cdot r}}{[p^2 + m_V^2 - x(1-x)m_\pi^2]^2} \tag{18}$$

The dependence on x is very weak.

Pion is uniformly created/annihilated between the two EM currents.

$$
\begin{aligned}
F^{\mathrm{TE}}(q_1^2, q_2^2) &= \frac{m_V^2/2}{q_1^2 + m_V^2} + \frac{m_V^2/2}{q_2^2 + m_V^2} \\
&= \int_0^1 d\,x\, \frac{\delta(x) + \delta(x-1)}{2} \frac{m_V^2}{((1-x)\,q_1 - x\,q_2)^2 + m_V^2}
\end{aligned}
\tag{19}
$$

Recall

$$
F(q_1^2, q_2^2) = \int_0^1 d\,x \int d^4 r\, e^{-i((1-x)\,q_1 - x\,q_2)\cdot r}\, F_c(x, r^2) \tag{20}
$$

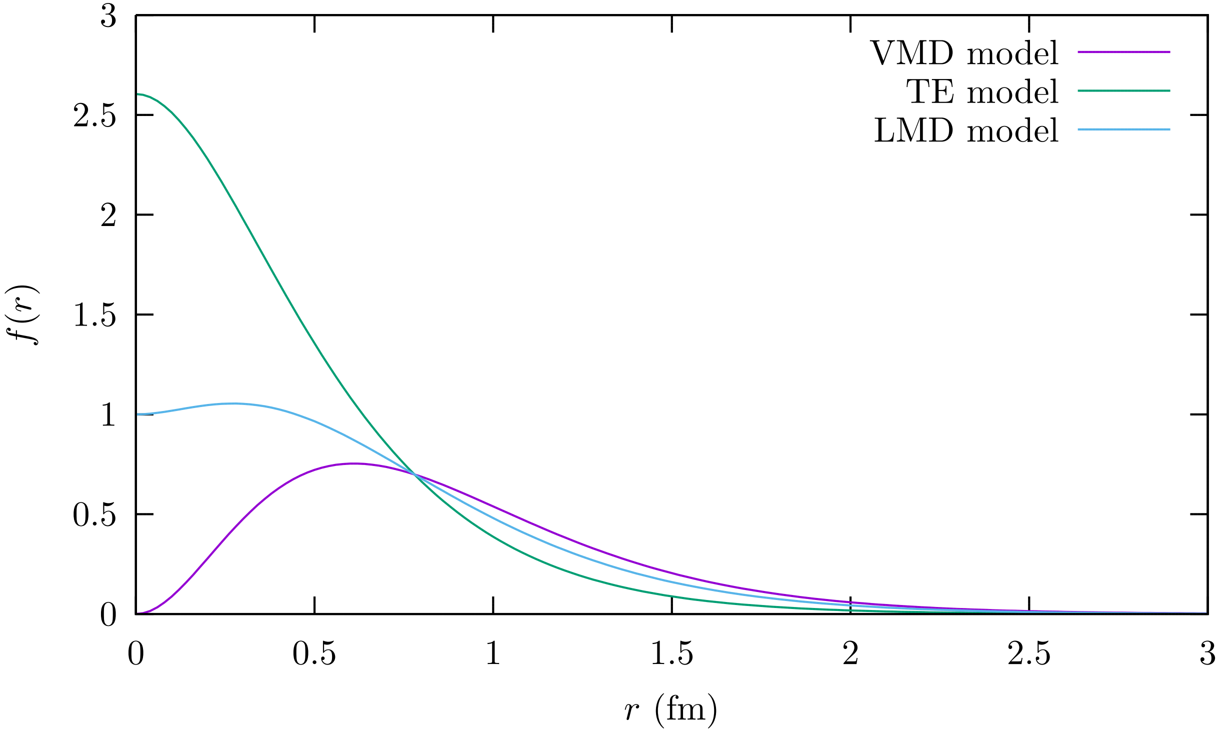
$$
F_c^{\mathrm{TE}}(x, r^2) = \frac{\delta(x) + \delta(x-1)}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{m_V^2 e^{ip\cdot r}}{p^2 + m_V^2} \tag{21}
$$

The value for x is either 0 or 1.

Pion is created/annihilated at the two ends of the segment between the two EM currents location.

Pion TFF formulation: LMD model

$$F^{\mathrm{LMD}}(q_1^2, q_2^2) \;=\; \frac{8\pi^2 F_\pi^2}{3m_V^2} F^{\mathrm{TE}}(q_1^2, q_2^2) + \left(1 - \frac{8\pi^2 F_\pi^2}{3m_V^2}\right) F^{\mathrm{VMD}}(q_1^2, q_2^2) \tag{22}$$

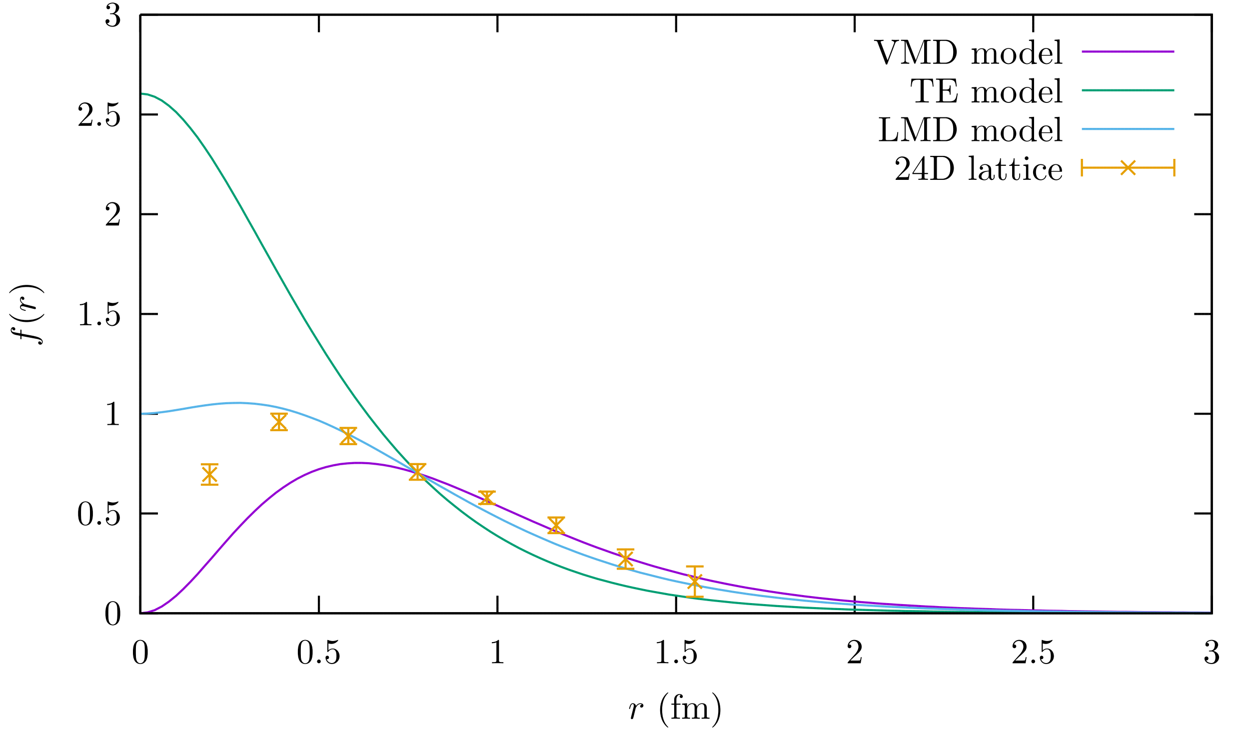


RBC/UKQCD $24^3 \times 64$ Iwasaki+DSDR ensemble: $m_\pi = 139\,\mathrm{MeV}$, $a^{-1} = 1.015\,\mathrm{GeV}$.

With $z_t = 0$, $f(|z|)$ can be evaluated with ($t_{\mathrm{sep}} = 10a$)

$$\begin{aligned} &\langle 0|T i\, J_\mu(z)\, i\, J_\nu(0)|\pi^0(\vec{p}=0)\rangle \\ &\quad = \frac{i}{4\,\pi^2\, F_\pi}\, \epsilon_{\mu,\nu,\rho,\sigma}\, 2\, z_\rho\, i\, p_\sigma \left[\frac{2F_\pi^2}{3}\, \frac{1}{(z^2)^2}\right] f(|z|), \end{aligned} \tag{23}$$

Using 16 configurations and the point source propagators generated by computing the leading disconnected contribution to HLbL, we obtained:



VMD model
TE model
LMD model
24D lattice
$f(r)$
r (fm)
0
0.5
1
1.5
2
2.5
3

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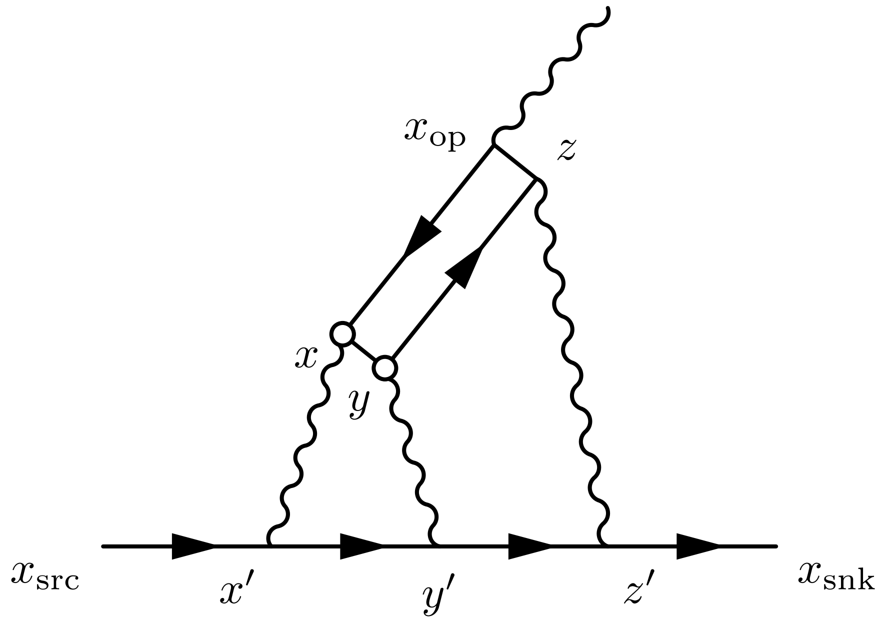
- **Contribution to HLbL with pion TFF**

Contribution to HLbL with pion TFF



$$R_{\rm max} = \max\{|x-y|, |y-z|, |z-x|\} \tag{24}$$

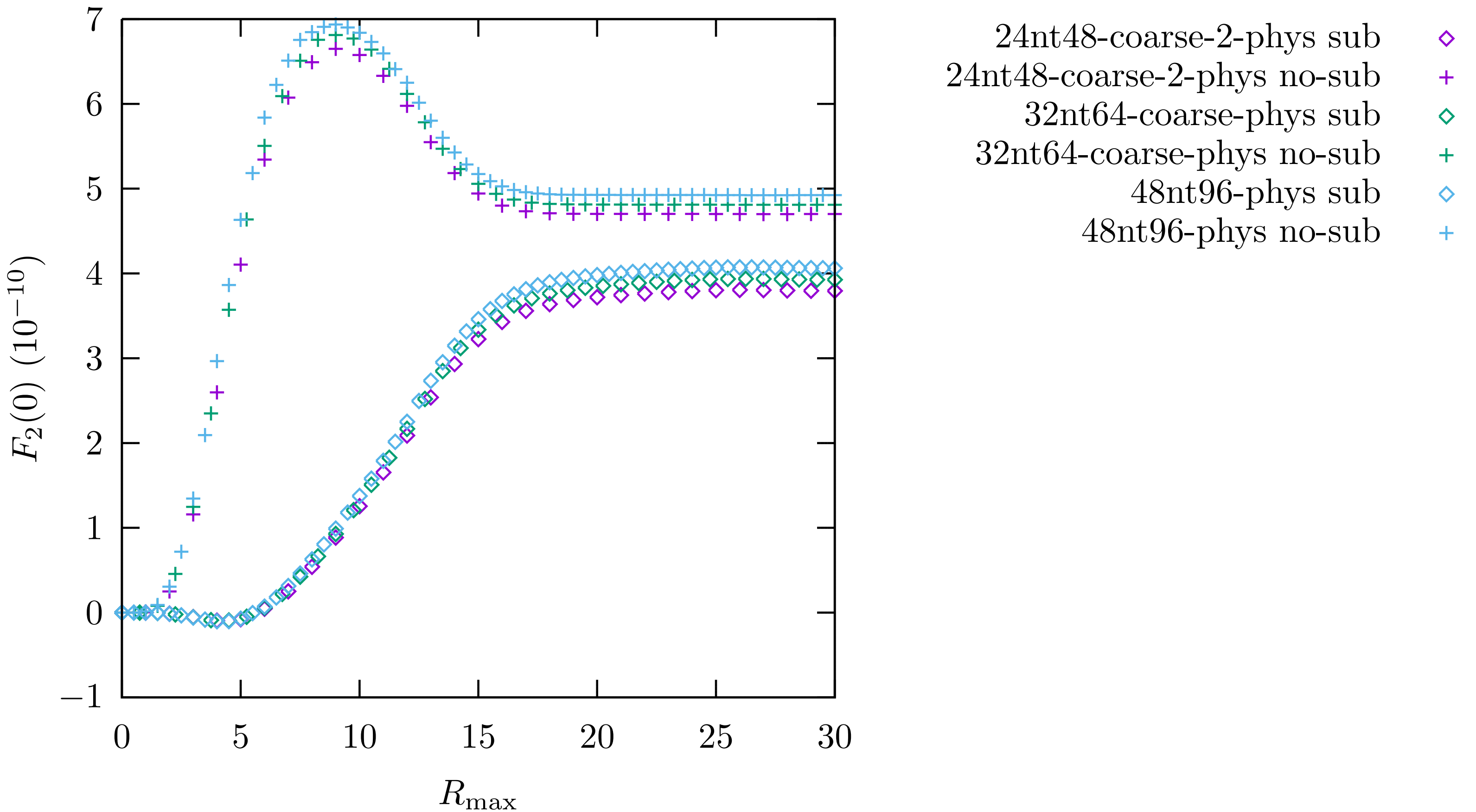
$$R_{\rm min} = \mathrm{minx}\{|x-y|, |y-z|, |z-x|\} \tag{25}$$

We will evaluate with the following parameter.

$$m_V = 770\,\mathrm{MeV} \qquad F_\pi = 93\,\mathrm{MeV} \tag{26}$$

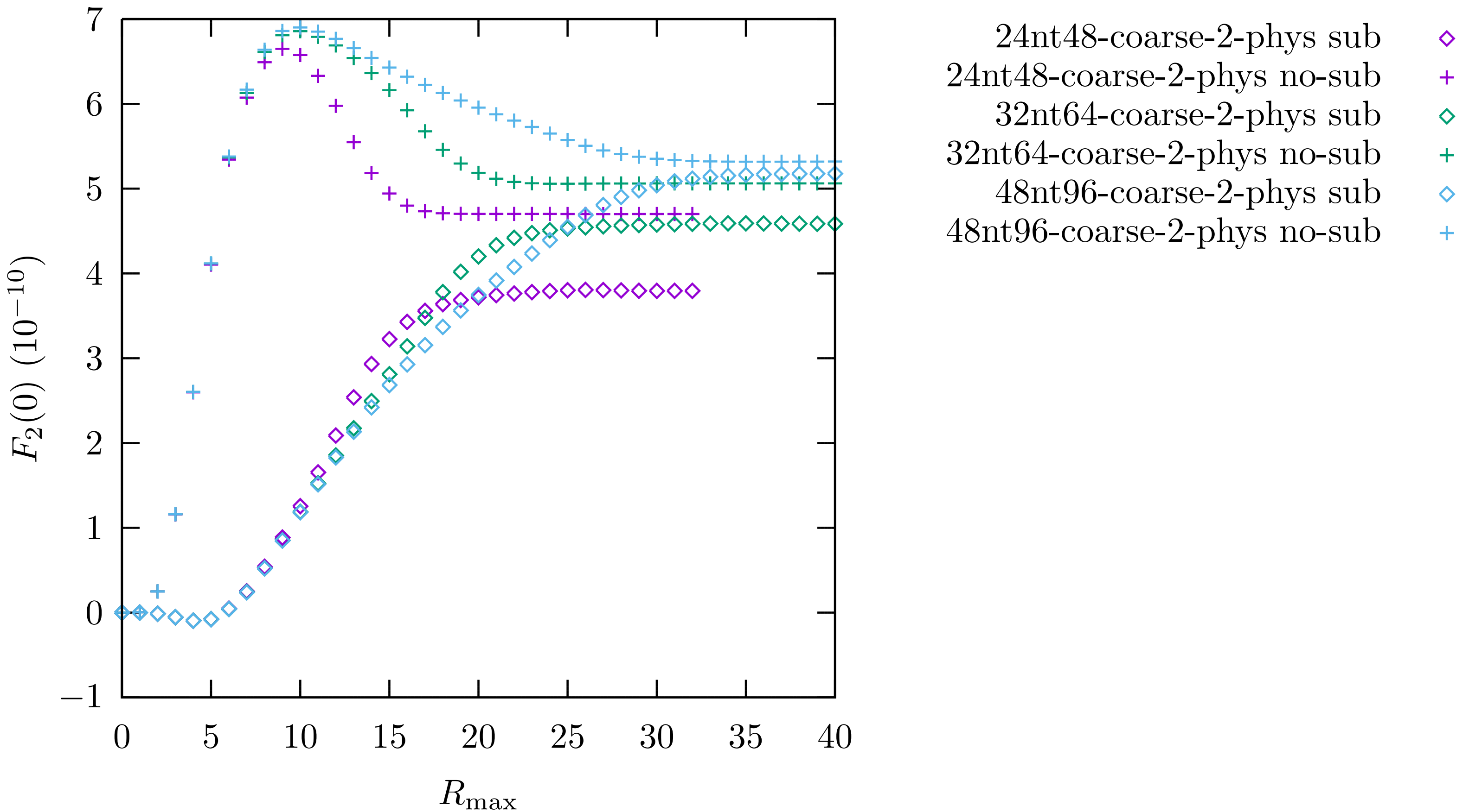
The QED part, we use the weighting function developed in [arXiv:1705.01067] by us.

Discretization effect: $a = 0.223\,\mathrm{fm}, 0.171\,\mathrm{fm}, 0.114\,\mathrm{fm}$.



The unit of x-axis is 0.223 fm, the lattice spacing of the coarsest lattice.

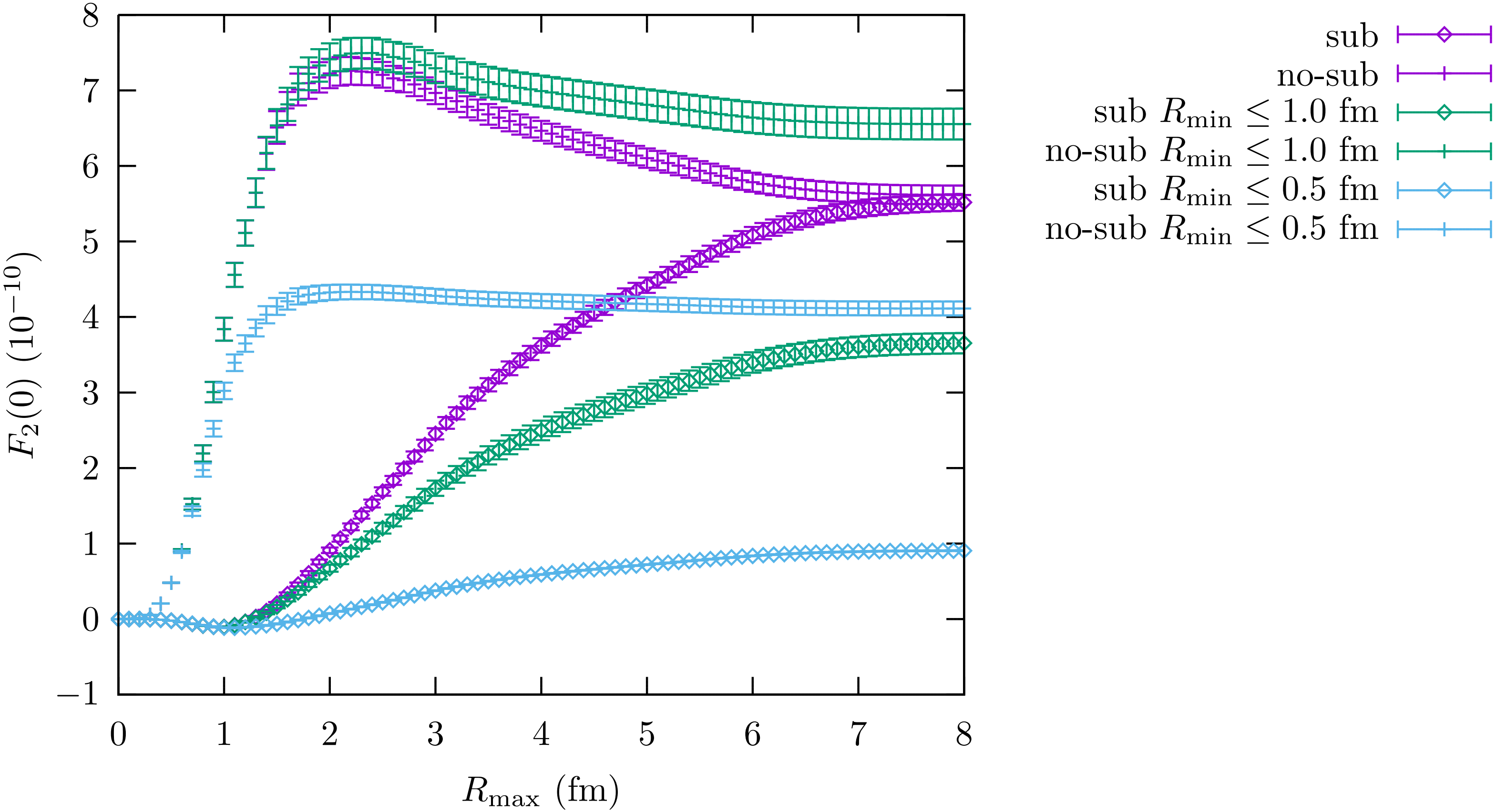
Finite volume effect: $a = 0.223\,\mathrm{fm}$, $L = 5.5\,\mathrm{fm}, 7.3\,\mathrm{fm}, 10.9\,\mathrm{fm}$.



The unit of x-axis is 0.223 fm, the lattice spacing of the lattices.

Extrapolations:

- Infinite volume: use the largest volume $L = 10.9\,\mathrm{fm}$ as approximation.
- Continuum: extrapolate from the two lattice spacing $a = 0.223\,\mathrm{fm}, 0.114\,\mathrm{fm}$, $\mathcal{O}(a^2)$ scaling.



Errorbars are statistical only.

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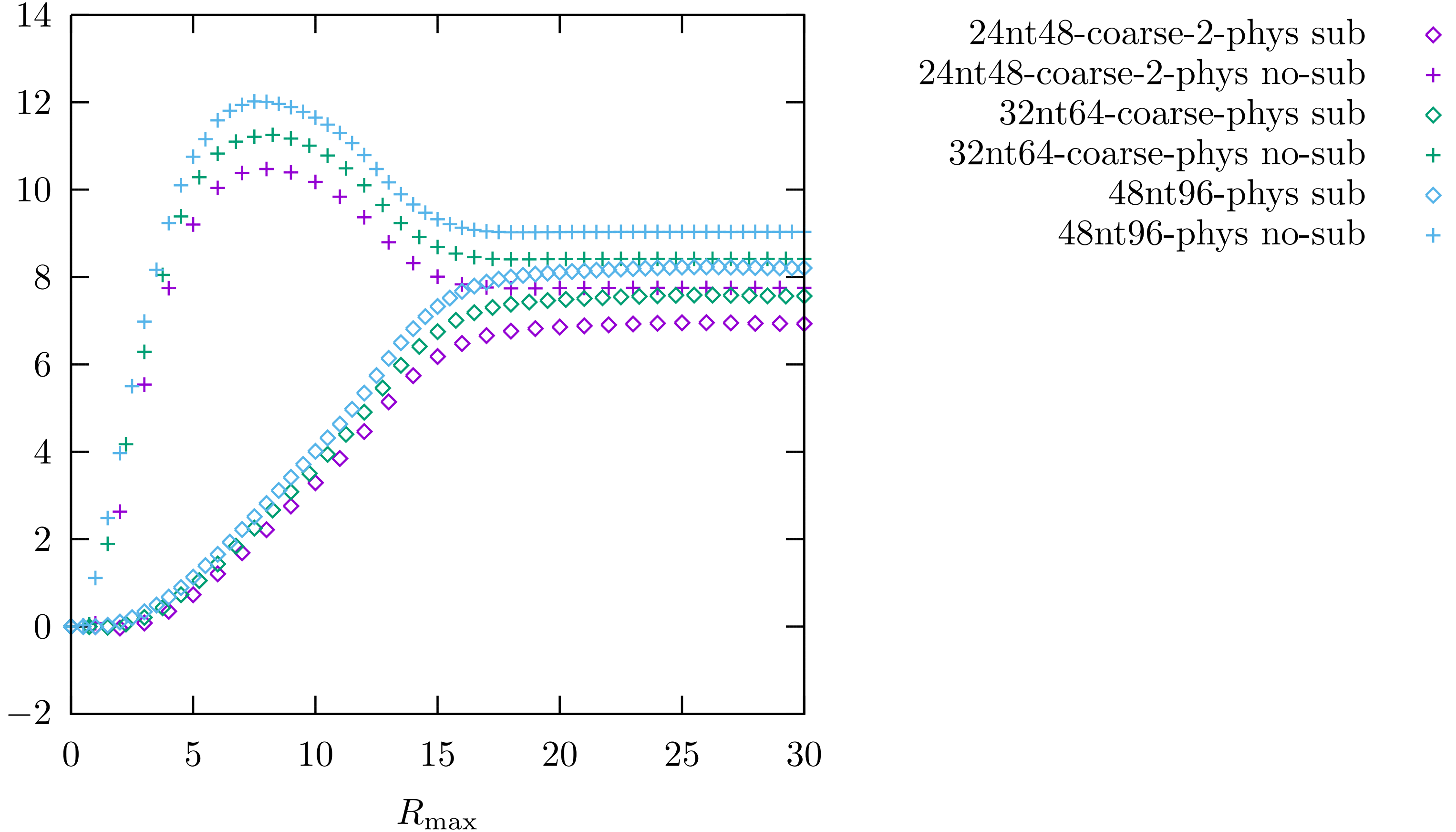

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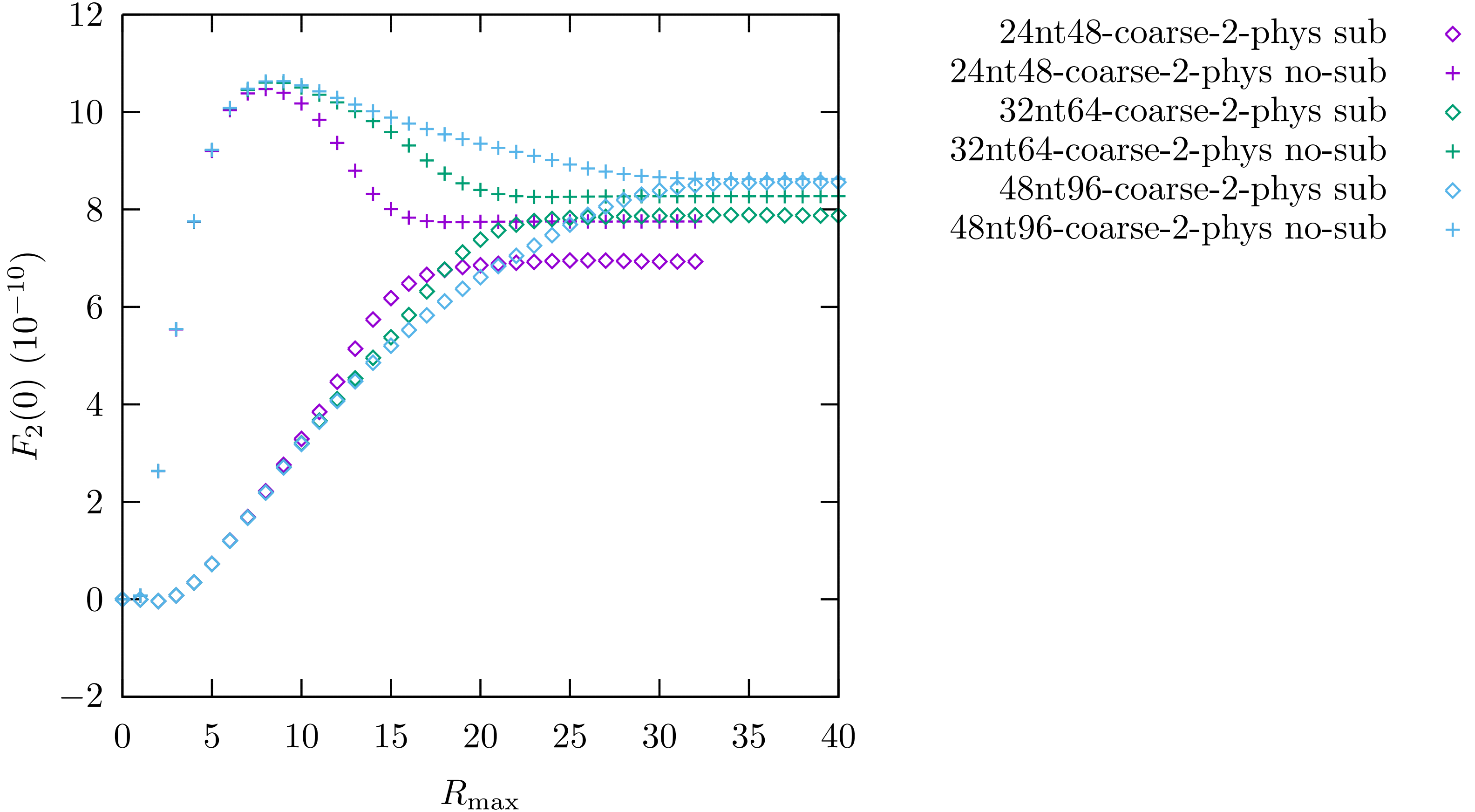
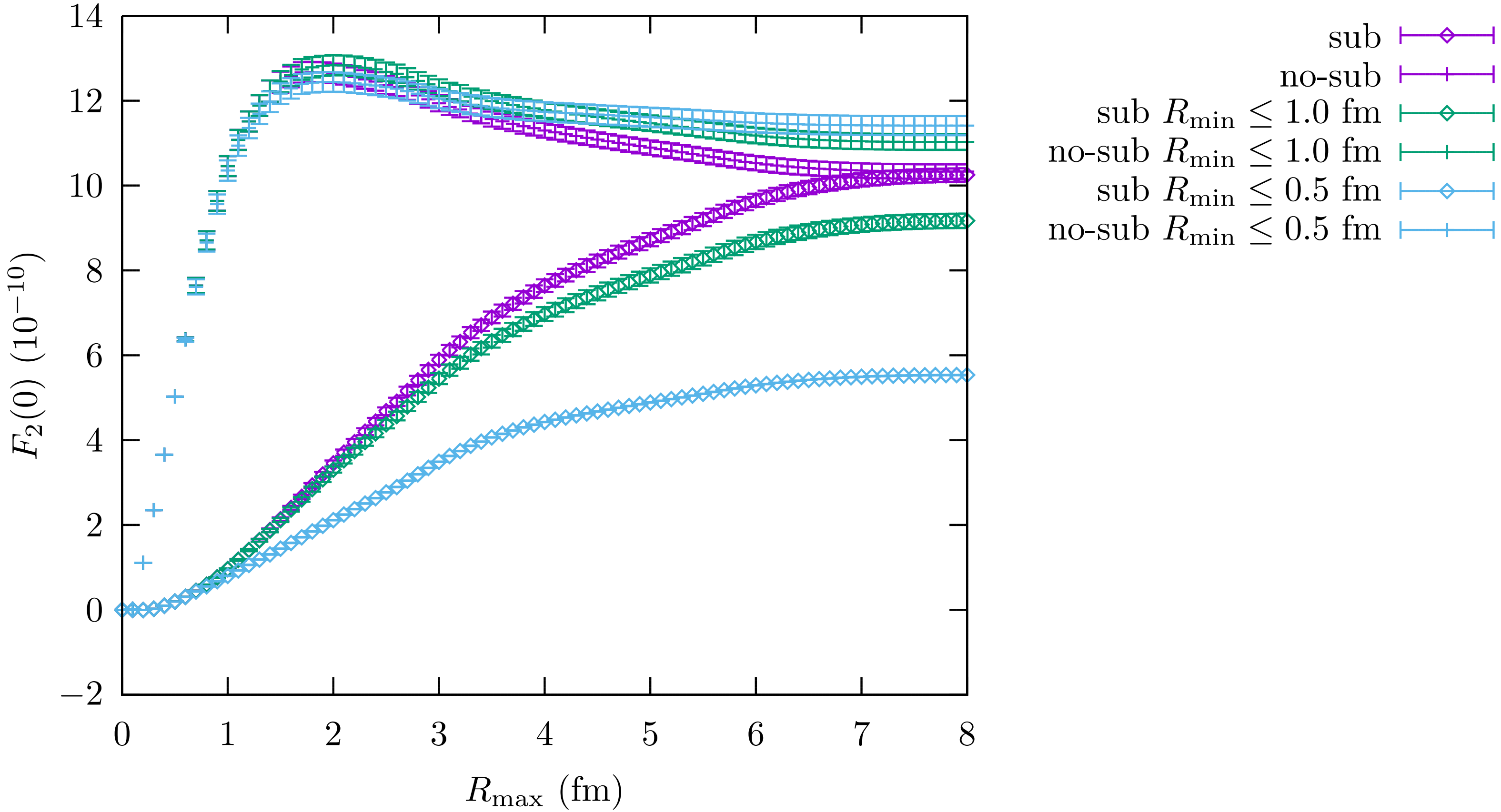


The unit of x-axis is 0.223 fm, the lattice spacing of the lattices.

Contribution to HLbL with pion TFF: TE

Extrapolations:

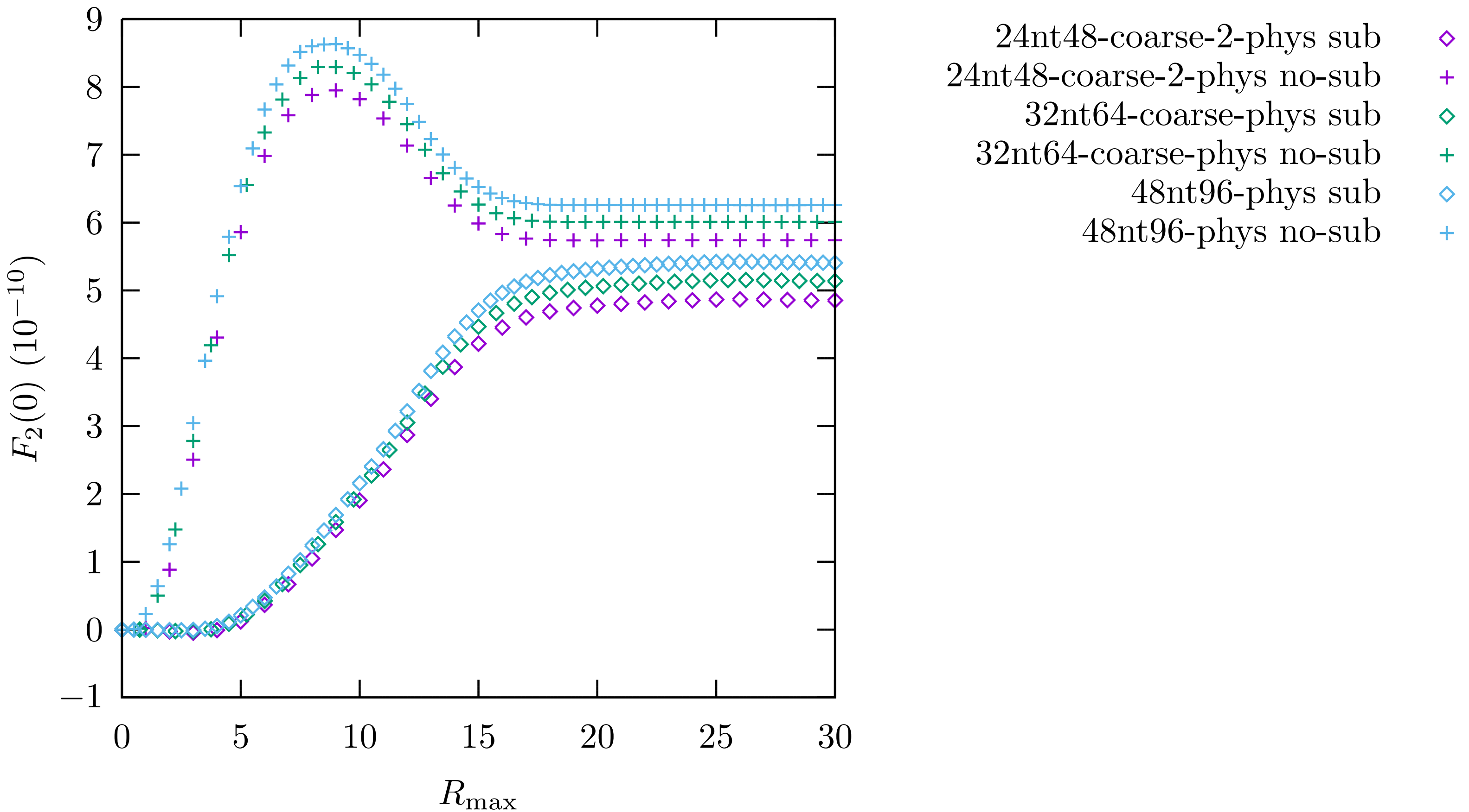
- Infinite volume: use the largest volume $L = 10.9\,\mathrm{fm}$ as approximation.
- Continuum: extrapolate from the two lattice spacing $a = 0.223\,\mathrm{fm}, 0.114\,\mathrm{fm}$, $\mathcal{O}(a^2)$ scaling.



Errorbars are statistical only.

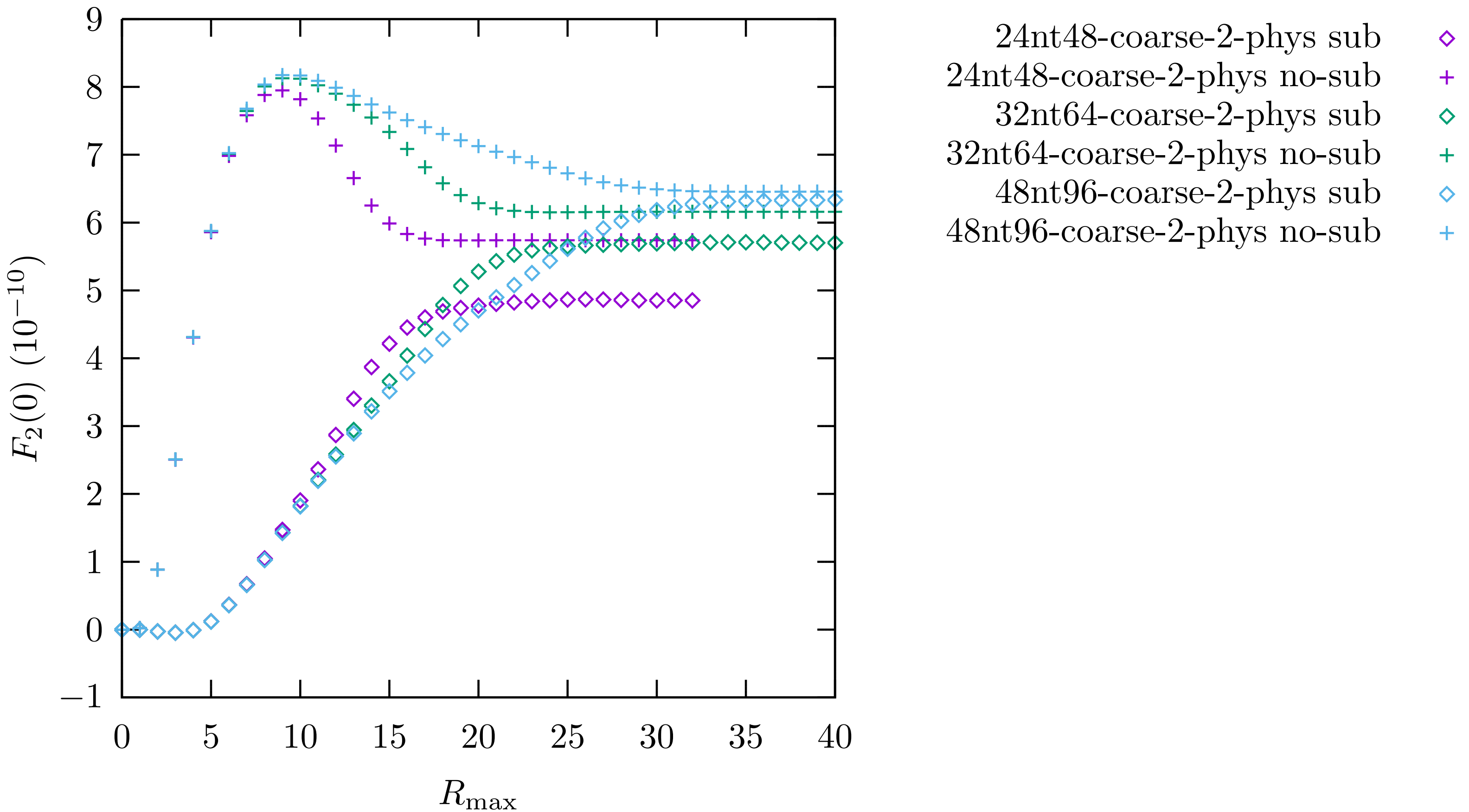
Contribution to HLbL with pion TFF: LMD

Discretization effect: $a = 0.223\,\mathrm{fm}, 0.171\,\mathrm{fm}, 0.114\,\mathrm{fm}$.



The unit of x-axis is 0.223 fm, the lattice spacing of the coarsest lattice.

Finite volume effect: $a = 0.223\,\text{fm}$, $L = 5.5\,\text{fm}, 7.3\,\text{fm}, 10.9\,\text{fm}$.



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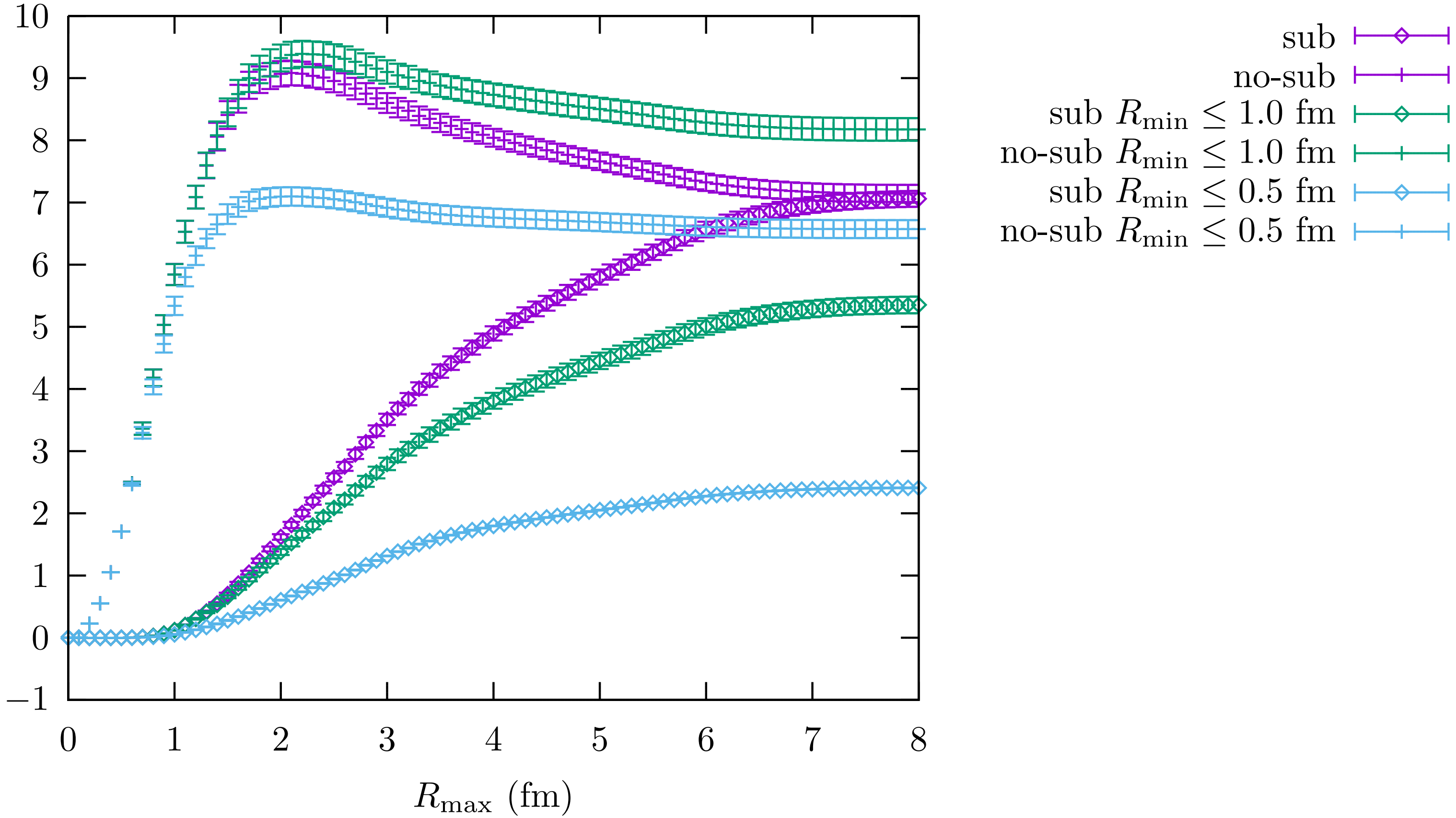

Errorbars are statistical only.

Conclusion

- We developed coordinate space formulation of the pion transition form factor (TFF). It turned out to be pion distribution amplitude at short distance.
- We defined $f(r)$ to describe the r (relative coordinate between the two EM currents) dependence of the coordinate space TFF.
- We calculated $f(r)$ for three models (VMD, TE, LMD) and using lattice data (24D).
- LMD model is designed to satisfy both the Chiral anomaly constraint and the OPE constraint. Among the three models, it also agrees the lattice data the most.
- We computed the π^0 contribution to HLbL using the three models using lattice.
  - The total results agree with the momentum space more analytical evalution.
  - We also obtained the models results in the long distance region, which can be used to correct the QCD finite volume errors of the lattice calculation of HLbL.

Thank You!