Padé approach to pseudoscalar poles in HLbL

based on P. Masjuan, PS: Phys.Rev. D95 (2017)

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Outline

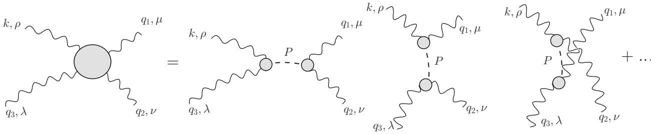
1. A (very) brief reminder
2. Our proposal: Padé approximants
3. Application to HLbL and results
4. Summary

Section 1

A (very) brief reminder

Our aim: pseudoscalar poles in HLbL

- We want the pseudoscalar (π^0, η, η') pole contributions to a_μ^{HLbL}



unambiguosly identified (see eg. tomorrow's talks)

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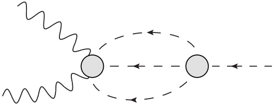

off-shellness in χPT?

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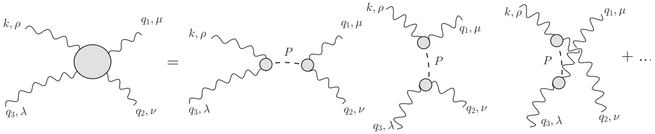

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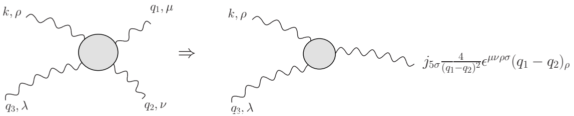
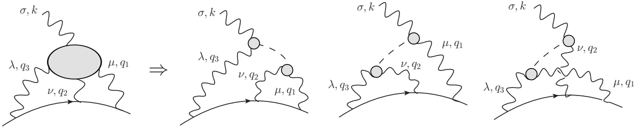
- Commonly *off-shellness* is coined for high-energy link



To my point of view one of the next obstacles ahead (tomorrow talks?)

The pseudoscalar poles in brief

- It amounts to calculate



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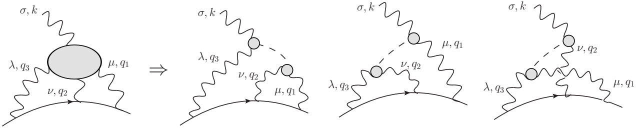

- Result expressed as weighted integral over space-like *on-shell* form factors

$$a_\ell^{\mathrm{HLbL};P} = \frac{-2\pi}{3}\left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt \sqrt{1-t^2}\, Q_1^3 Q_2^3$$
$$\times \left[\frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_3^2) F_{P\gamma^*\gamma}(Q_2^2, 0) I_1(Q_1, Q_2, t)}{Q_2^2 + m_P^2} + \frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) F_{P\gamma^*\gamma}(Q_3^2, 0) I_2(Q_1, Q_2, t)}{Q_3^2 + m_P^2} \right]$$

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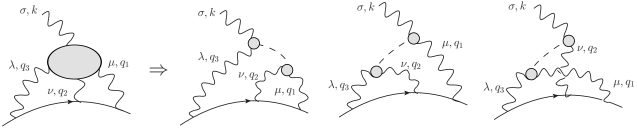

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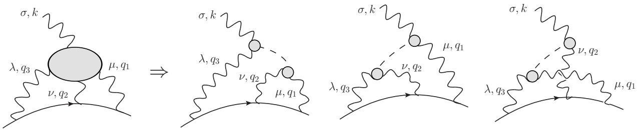

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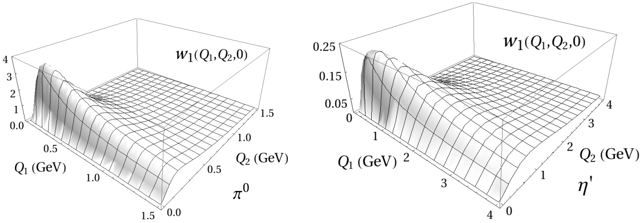

notice the peaks at the relevant low energies

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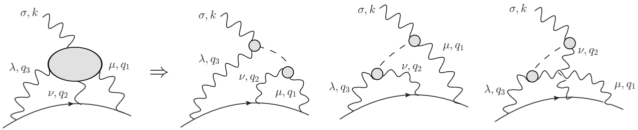

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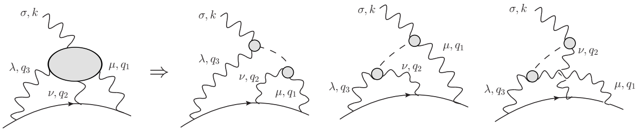

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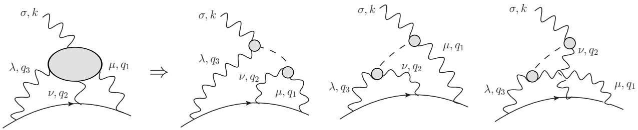

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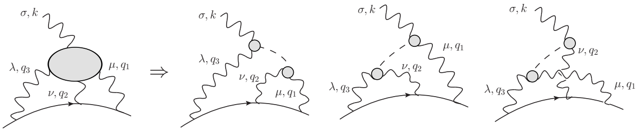

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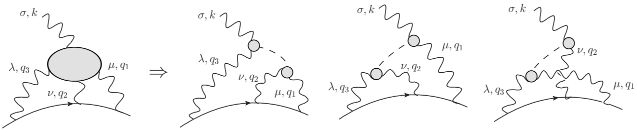

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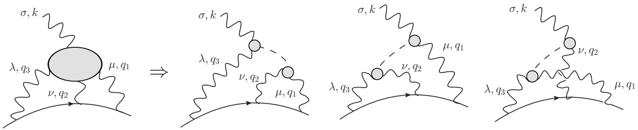

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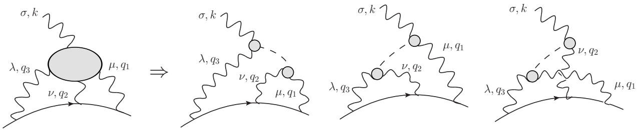

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  - Framework: avoid model-building (as model-independent as possible)
  - Keep track of systematic errors
  - Emphasize the low-energy region
  - Incorporate theoretical high-energy constraints

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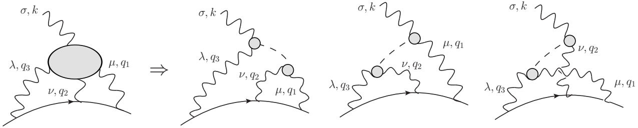

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Our Proposal: use of Padé approximants

Section 2

Our proposal: Padé approximants

Padé approximants: singly virtual

- How to approximate (not model) non-perturbative hadronic functions?

  Taylor series: $F_{\pi\gamma\gamma^*}(q^2) = F_{\pi\gamma\gamma}(1 + b_P q^2 + \ldots)$ ✗ poles(cuts)

Padé approximants: singly virtual

- How to approximate (not model) non-perturbative hadronic functions?

Laurent exp.: $F_{\pi\gamma\gamma^*}(q^2) = \sum_{n=-1} c_n (q^2 - M^2)^n$ ✗ next pole(cuts)

Padé approximants: singly virtual

- How to approximate (not model) non-perturbative hadronic functions?

$$\text{PAs: } F_{\pi\gamma\gamma^*}(q^2) = P_M^N = \frac{Q_N(q^2)}{R_M(q^2)} = F_{\pi\gamma\gamma}\left(1 + b_P q^2 + ... + \mathcal{O}(q^{N+M+1})\right)$$

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- Convergence not only to meromorphic but Stieltjes → beyond large-N_c

  Convergence for sequences: $P_1^N, P_N^N, P_{N+1}^N, \ldots$ *analytic related*

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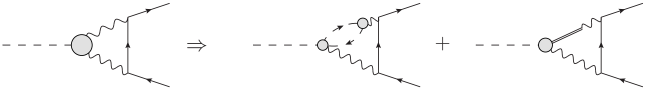

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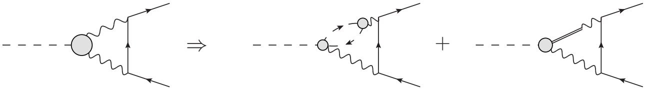

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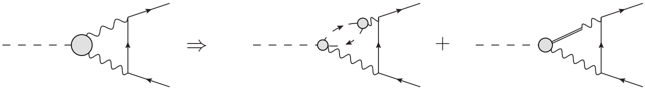

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- Obtain the derivatives from data (later)

Padé approximants: double virtual

- Commonly referred to as Canterbury approximants

$$F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) = C_M^N = \frac{Q_N(q_1^2, q_2^2)}{R_M(q_1^2, q_2^2)}; \quad Q_N(R_M) = \sum_{i,j}^{N(M)} c_{n,m}^{Q(R)} q_1^{2i} q_2^{2j}$$

Again, match derivatives to get $c_{ij}^{Q,R}$*'s*

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$$F_{P\gamma^*\gamma^*}^{\log}(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma} M^2}{Q_1^2 - Q_2^2} \ln\left(\frac{1 + Q_1^2/M^2}{1 + Q_2^2/M^2}\right)$$

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$$F_{P\gamma^*\gamma^*}^{\log}(Q_1^2,Q_2^2) = F_{P\gamma\gamma}M^2 \int_0^1 dx \frac{1}{xQ_1^2 + (1-x)Q_2^2 + M^2}$$

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Again, match derivatives to get $c_{ij}^{Q,R}$*'s*

- Again nice convergence properties, as for instance, pQCD models

$$F_{P\gamma^*\gamma^*}^{\mathrm{pQCD;as}}(Q_1^2, Q_2^2) = 2F_P^a \,\mathrm{tr}\, \mathcal{Q}^2 \lambda^a \int_0^1 dx \frac{6x(1-x)}{xQ_1^2 + (1-x)Q_2^2}$$

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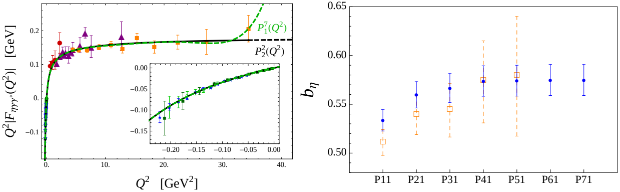
- Simple example with appropriate power-like behavior

$$C_1^0 = \frac{1}{1 + c_{0,1}^R(q_1^2 + q_2^2) + c_{1,1}^R q_1^2 q_2^2} \to \frac{1}{1 + c_{0,1}^R(q_1^2 + q_2^2)}$$

Again, benefits of not doing spectroscopy!

Inputs: data fitting

- Inputs: sequences data fitting (not just fitting models)



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$$\begin{array}{l|lllllll} \text{Reduced} & P^N_{N+1}: & P^{0;\text{fit}}_1 \to b_P & , & \ldots & , & P^{1;\text{fit}}_2 & ✗ \\ \text{Data Set} & P^N_1: & P^{0;\text{fit}}_1 \to b_P & , & \ldots & , & P^{4;\text{fit}}_1 \to^{b_P}_{c_P}, \; P^{5;\text{fit}}_1 & ✗ \end{array}$$

- Let's see impact on $a_\mu^{\mathrm{HLbL};\eta}$ (fact)

	Fit
P^0_1	14.5
P^1_2	—

Inputs: data fitting

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$$\begin{array}{r|lll}
\text{Reduced} & P^N_{N+1}: & P_1^{0;\mathrm{der}} \to b_P \ , \ \ldots \ , \ P_2^{1;\mathrm{der}} \ ✓ & \\
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\end{array}$$

- Let's see impact on $a_\mu^{\mathrm{HLbL};\eta}$ (fact)

	Fit	Der
P^0_1	14.5	13.2
P^1_2	—	13.3

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$$\begin{array}{c|lllll} \text{New larger} & P^N_{N+1}: & P_1^{0;\text{fit}} \to b_P, & \ldots, & P_2^{1;\text{fit}}, & P_3^{2;\text{fit}} ✗ \\ \text{Data Set} & P^N_1: & P_1^{0;\text{fit}} \to b_P, & \ldots, & P_1^{7;\text{fit}} \to^{b_P}_{c_P}, & P_1^{8;\text{fit}} ✗ \end{array}$$

- Let's see impact on $a_\mu^{\text{HLbL};\eta}$ (fact)

	Fit	Der	New Fit
P_1^0	14.5	13.2	14.0
P_2^1	—	13.3	13.4

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$$\begin{array}{r|ll} \text{New larger} & P^N_{N+1}: & P_1^{0;\mathrm{der}} \to b_P \ , \ \dots \ , \ P_2^{1;\mathrm{der}} \checkmark \\ \text{Data Set} & P^N_1: & P_1^{0;\mathrm{fit}} \to b_P \ , \ \dots \ , \ P_1^{7;\mathrm{fit}} \to^{b_P}_{c_P} \ ; \ P_1^{8;\mathrm{fit}} \text{✗} \end{array}$$

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P^1_2	—	13.3	13.4	13.3

Advantage of PAs vs. resonance fits: interrelation, convergence, systematics

Section 3

Application to HLbL and results

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

- Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}$$

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$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}$$

- Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1}Q_1^2Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1}Q_1^2Q_2^2 + \beta_{2,1}Q_1^2Q_2^2(Q_1^2 + Q_2^2) + \beta_{2,2}Q_1^4Q_2^4}$$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

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$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1}Q_1^2Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1}Q_1^2Q_2^2 + \beta_{2,1}Q_1^2Q_2^2(Q_1^2 + Q_2^2) + \beta_{2,2}Q_1^4Q_2^4}$$

Reconstruction

1.Reduce to Padé Approximants
$F_{P\gamma\gamma}(0,0), \alpha_1, \beta_1, \beta_2 \rightarrow$ from PAs

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

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$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1}Q_1^2Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1}Q_1^2Q_2^2 + \beta_{2,1}Q_1^2Q_2^2(Q_1^2 + Q_2^2) + \beta_{2,2}Q_1^4Q_2^4}$$

Reconstruction

1.Reduce to Padé Approximants
$F_{P\gamma\gamma}(0,0), \alpha_1, \beta_1, \beta_2 \rightarrow$ from PAs

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

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$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2) + \beta_{2,2} Q_1^4 Q_2^4}$$

Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

$$F_{\pi\gamma^*\gamma^*} = \frac{1}{3Q^2}(2F_\pi)\left(1 - \frac{8}{9}\frac{\delta^2}{Q^2} + \mathcal{O}(\alpha_s(Q^2))\right) \Rightarrow \beta_{2,2} = 0, \alpha_{1,1}, \beta_{2,1}$$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

- Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}$$

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$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2(Q_1^2 + Q_2^2)}$$

Reconstruction

1. Reduce to Padé Approximants
2. Reproduce the OPE behavior (high energies)
3. Reproduce the low energies ($a_{P;1,1}$)

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

- Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}$$

- Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1}Q_1^2Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1}Q_1^2Q_2^2 + \beta_{2,1}Q_1^2Q_2^2(Q_1^2 + Q_2^2)}$$

Reconstruction

1. Reduce to Padé Approximants
2. Reproduce the OPE behavior (high energies)
3. Reproduce the low energies ($a_{P;1,1}$)
   Be generous: all configurations with no poles $\Rightarrow a_{P;1,1}^{\min} < a_{P;1,1} < a_{P;1,1}^{\max}$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

- Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}$$

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Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

3.Reproduce the low energies ($a_{P;1,1}^{\min} < a_{P;1,1} < a_{P;1,1}^{\max}$)

Pseudoscalar-pole contribution: Final results

$$—C_1^0(Q_1^2, Q_2^2)—$$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	Fact ($a_{P;1,1} = b_P^2$)
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$54.3(1.5)_F(2.2)_{b_\pi}[2.5]_t$
η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$13.0(0.4)_F(0.2)_{b_\eta}[0.5]_t$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

Pseudoscalar-pole contribution: Final results

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Total	$98.4[2.9]_t$	$79.3[2.6]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t$	$63.0(1.1)_L(0.5)_\delta[1.2]_t$
η	$16.3(0.8)_L(0)_\delta[0.8]_t$	$16.2(0.8)_L(0.6)_\delta[1.0]_t$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
Total	$95.1[1.7]_t$	$93.5[1.7]_t$

Pseudoscalar-pole contribution: Final results

$$—C_1^0(Q_1^2, Q_2^2)—$$

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η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

$$—C_2^1(Q_1^2, Q_2^2)—$$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\mathrm{sys}}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\mathrm{sys}}$
η	$16.3(0.8)_L(0)_\delta[0.8]_t$	$16.2(0.8)_L(0.6)_\delta[1.0]_t$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
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Pseudoscalar-pole contribution: Final results

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η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\mathrm{sys}}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\mathrm{sys}}$
η	$16.3(0.8)_L(0)_\delta[0.8]_t\{0.8\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.6)_\delta[1.0]_t\{0.9\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.1[1.7]_t\{3.3\}_{\mathrm{sys}}$	$93.5[1.7]_t\{4.9\}_{\mathrm{sys}}$

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	Fact ($a_{P;1,1} = b_P^2$)
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η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$13.0(0.4)_F(0.2)_{b_\eta}[0.5]_t$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\mathrm{sys}}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\mathrm{sys}}$
η	$16.3(0.8)_L(0)_\delta[0.8]_t\{0.8\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.6)_\delta[1.0]_t\{0.9\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.1[1.7]_t\{3.3\}_{\mathrm{sys}}$	$93.5[1.7]_t\{4.9\}_{\mathrm{sys}}$

—Final Result (combining errors just for clarity)

$$a_\mu^{\pi,\eta,\eta'} = (63.6(2.7) + 16.3(1.3) + 14.5(1.8)) \times 10^{-11} = 94.3(5.3) \times 10^{-11}$$

Pseudoscalar-pole contribution: Final results

$$—C_1^0(Q_1^2, Q_2^2)—$$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	Fact ($a_{P;1,1} = b_P^2$)
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$54.3(1.5)_F(2.2)_{b_\pi}[2.5]_t$
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η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

$$—C_2^1(Q_1^2, Q_2^2)—$$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\mathrm{min}}$	$a_{P;1,1}^{\mathrm{max}}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\mathrm{sys}}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\mathrm{sys}}$
η	$16.3(0.8)_L(0)_\delta[0.8]_t\{0.8\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.6)_\delta[1.0]_t\{0.9\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.1[1.7]_t\{3.3\}_{\mathrm{sys}}$	$93.5[1.7]_t\{4.9\}_{\mathrm{sys}}$

—Final Result (combining errors just for clarity)

$$a_\mu^{\pi,\eta,\eta'} = (63.6(2.7) + 16.3(1.3) + 14.5(1.8)) \times 10^{-11} = 94.3(5.3) \times 10^{-11}$$

- Low-energy emphasis but high-energies too
- η and η' fulfill high-energies (5×10^{-11} effect: 1/3 of exp error)
- Systematic from sequence results

Summary

- Padé approximants to reconstruct form factors
- Full use of data and theory in a systematic approach; not modelling
- New value $a_\mu^{HLbL;\pi,\eta,\eta'} = 94.3(5.3) \times 10^{-11}$ including systematics
- OPE for all the pseudoscalars implemented
- Bypass $\eta - \eta'$ mixing (output): non-trivial if fully theory-driven approach

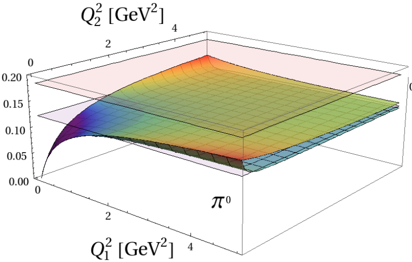
Related projects

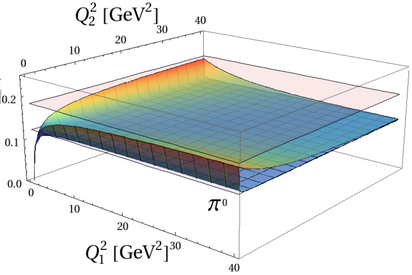
- Radiative corrections for $P \to \bar{\ell}\ell\bar{\ell}'\ell'$: Phys.Rev. D97 (2018) 056010
- In contact with H. Czyz for $e^+e^- \to e^+e^- P$

Section 4

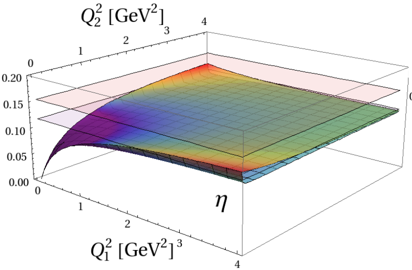
Backup

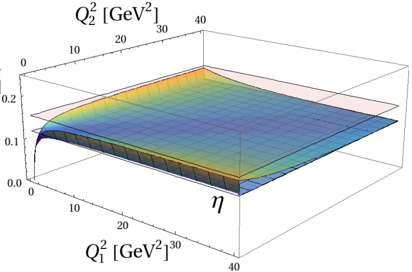
$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$



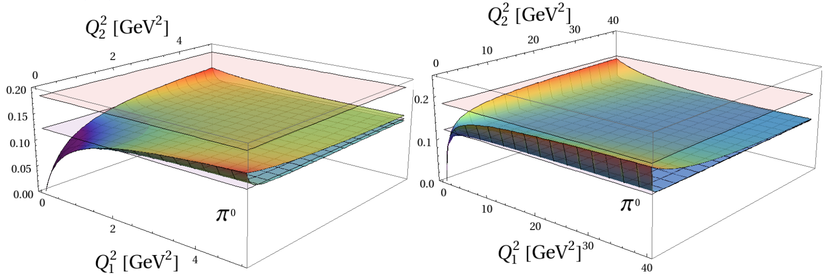


The two planes: boundaries for the $a_{P;1,1}$ region

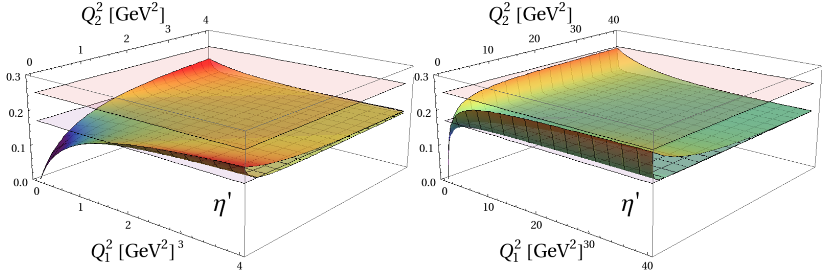




$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$



The two planes: boundaries for the $a_{P;1,1}$ region



Seeing is believing: toy models and systematics

—a_μ^π: Regge Model—

$$F^{\text{Regge}}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)=\frac{aF_{P\gamma\gamma}}{Q_1^2-Q_2^2}\frac{\left[\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right)-\psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right]}{\psi^{(1)}\left(\frac{M^2}{a}\right)}$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	55.2	59.7	60.4	60.6
OPE	65.7	60.8	60.7	60.7
Fit^{OPE}	66.3	62.7	61.1	60.8
Exact		60.7		

—a_μ^π: Logarithmic Model—

$$F^{\log}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)=\frac{F_{P\gamma\gamma}M^2}{Q_1^2-Q_2^2}\ln\left(\frac{1+Q_1^2/M^2}{1+Q_2^2/M^2}\right)$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	56.7	64.4	66.1	66.8
OPE	65.7	67.3	67.5	67.6
Fit^{OPE}	79.6	71.9	69.3	68.4
Exact		67.6		

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Seeing is believing: toy models and systematics

—a_μ^π: Regge Model—

$$F^{\rm Regge}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)=\frac{aF_{P\gamma\gamma}}{Q_1^2-Q_2^2}\frac{\left[\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right)-\psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right]}{\psi^{(1)}\left(\frac{M^2}{a}\right)}$$

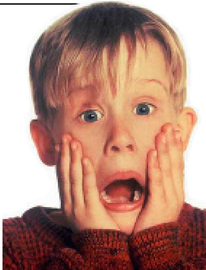
	C_1^0	C_2^1	C_3^2	C_4^3
LE	55.2	59.7	60.4	60.6
OPE	65.7	60.8	60.7	60.7
Fit^{OPE}	66.3	62.7	61.1	60.8
Exact		60.7		

—a_μ^π: Logarithmic Model—

$$F^{\log}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)=\frac{F_{P\gamma\gamma}M^2}{Q_1^2-Q_2^2}\ln\left(\frac{1+Q_1^2/M^2}{1+Q_2^2/M^2}\right)$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	56.7	64.4	66.1	66.8
OPE	65.7	67.3	67.5	67.6
Fit^{OPE}	79.6	71.9	69.3	68.4
Exact		67.6		

- The convergence result is excellent!
- The OPE choice seems the best $\to$ high energy matters
- Still, low energies provide a good performance
- Error $\sim$ difference among elements $\to$ Systematics!



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