Interplay between Lattice and { Model and/or Dispersive Representation } for g-2 HLbL

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Introduction

- How to compare and check a_μ^{LbL} from lattice QCD and { Model, Dispersive Representation, ... } ?

[HVP: Bernecker Meyer 2011]

- How to *safely* maximize precision of a_μ^{LbL} from LQCD and DR/Model studies.

- Simplest way : compare two numbers, $a_\mu^{\mathrm{LbL,LQCD}}$ and $a_\mu^{\mathrm{LbL,Model/DR}}$, and average with the error weight.

- We could find more convinient rendez-vouz point ?

- Exaxtly same question discussed for HVP one year ago at Q-Center, but within less strict error budget this time : $\sim 10\%$ error.

Sweat spots of Lattice vs DR/Model

- Lattice, after take continuum/infinite volume limits with all disconnected,

  short distance (high energy) : less noisy
  long distance (low energy) : very noisy

- DR / Model (or experiments)

heavy particle / multiple hadron : less control
light particle, pi0 pole or pion-loop : well controlled

-> Could cover sweat spots complementarily ?

- For HVP, a good comparison/interplay is done in Euclidean coordinate space [Christoph Lehner's talk]

First try

[Luchang Jin's talk]

- LMD model in coordinate space
- Fixed min $\{|x-y|, |x-z|, |y-z|\} < R(\min)$
- Plot as function of
  $\max\{|x-y|, |x-z|, |y-z|\} = R(\max)$
- L = 9.6 fm, a=0.1fm, Nf=2+1 physical pion mass

- Subtracted lepton part (to isolate the long-distant part in this exercise)
- Connected only. Model is multiplied by 34/9 according to conn:disconn = 34:(-25) from charge factors

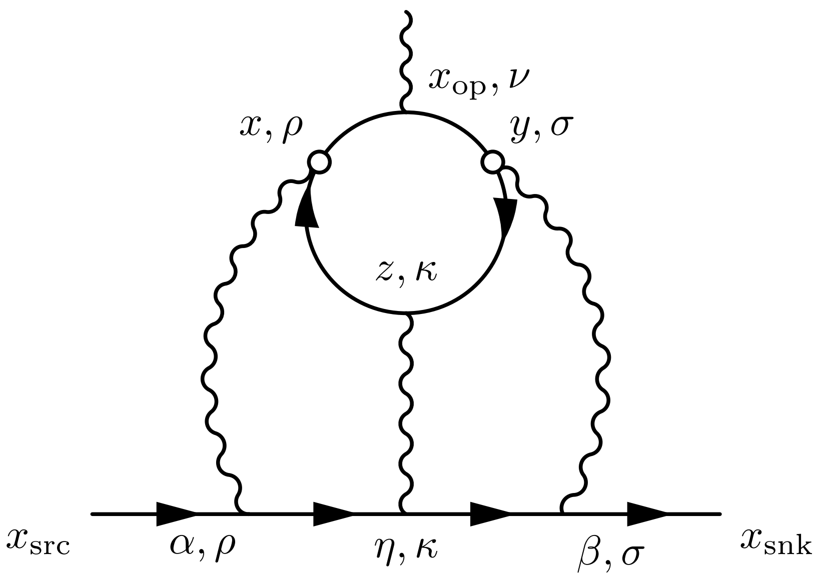
HLbL point source method [L. Jin et al. 1510.07100]

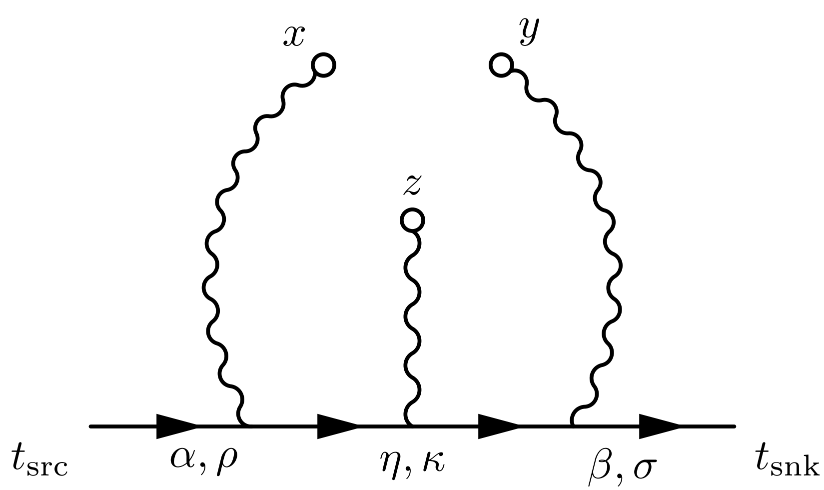
- Anomalous magnetic moment, $F_2(q^2)$ at $q^2 \to 0$ limit

$$\frac{F_2^{\mathrm{cHLbL}}(q^2=0)}{m}\frac{(\sigma_{s',s})_i}{2} = \frac{\sum_{x,y,z,x_{\mathsf{op}}}}{2VT}\epsilon_{i,j,k}\left(x_{\mathsf{op}}-x_{\mathrm{ref}}\right)_j \cdot i\bar{u}_{s'}(\vec{0})\mathcal{F}_k^C\left(x,y,z,x_{\mathsf{op}}\right)u_s(\vec{0}$$

- Stochastic sampling of x and y point pairs. Sum over x and z.

$$\mathcal{F}_\nu^C\left(x,y,z,x_{\mathsf{op}}\right) = (-ie)^6\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}),$$





cHLbL Subtraction using current conservation

- From current conservation, $\partial_\rho V_\rho(x) = 0$, and mass gap, $\langle x V_\rho(x) \mathcal{O}(0)\rangle \sim |x|^n \exp(-m_\pi |x|)$

$$\sum_x \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}) = \sum_x \langle V_\rho(x) V_\sigma(y) V_\kappa(z) V_\nu(x_{\mathsf{op}})\rangle = 0$$

$$\sum_z \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}) = 0$$

at $V \to \infty$ and $a \to 0$ limit (we use local currents).

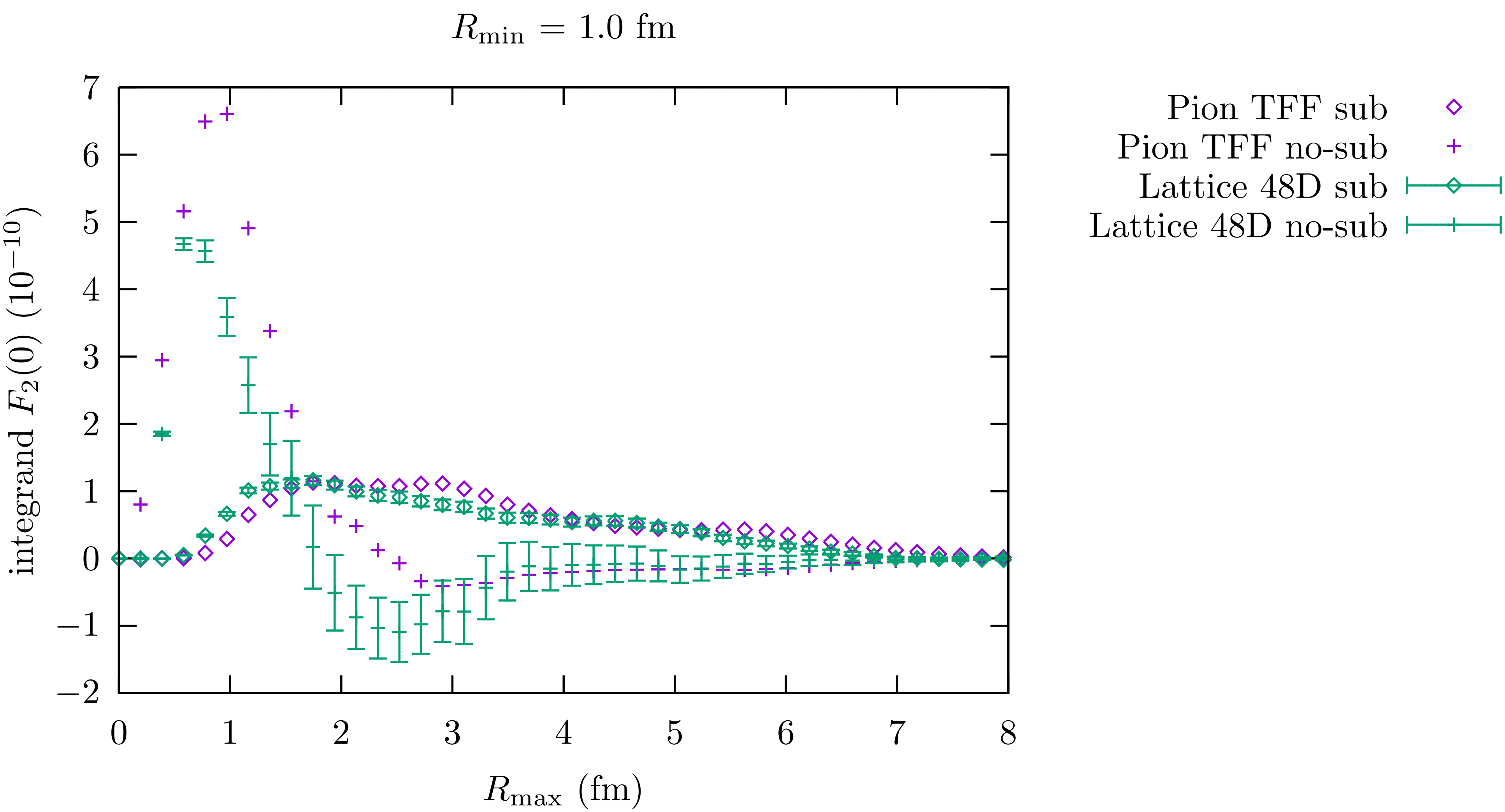
- We could further change QED weight

$$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z) = \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,y) + \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,y)$$

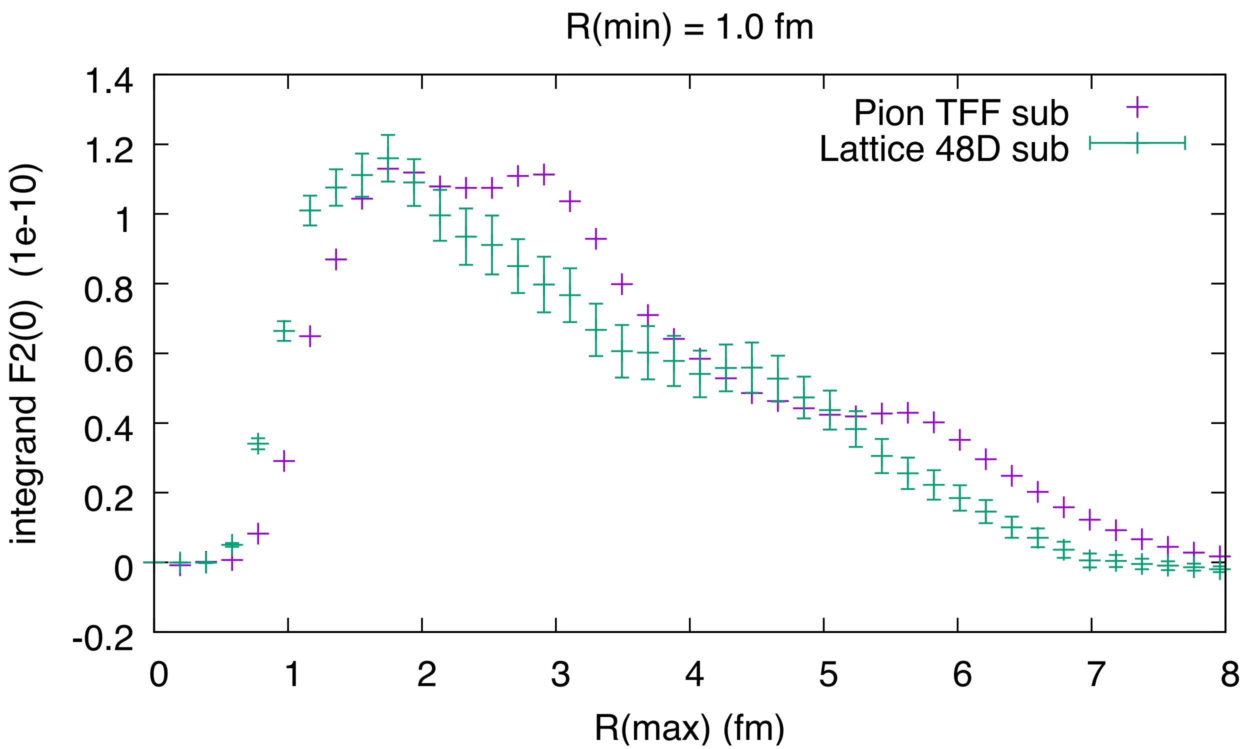
without changing sum $\sum_{x,y,z} \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}})$.

- Subtraction changes discretization error and finite volume error.
- Similar subtraction is used for HVP case in TMR kernel, which makes FV error smaller.
- Also now $\mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(z,z,x) = \mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(y,z,z) = 0$, so short distance $\mathcal{O}(a^2)$ is suppressed.
- The 4 dimensional integral is calculated numerically with the CUBA library cubature rules. (x,y,z) is represented by 5 parameters, compute on N^5 grid points and interpolates. ($|x-y| < 11$ fm).

Integrand : Lattice vs LVD
(preliminary)

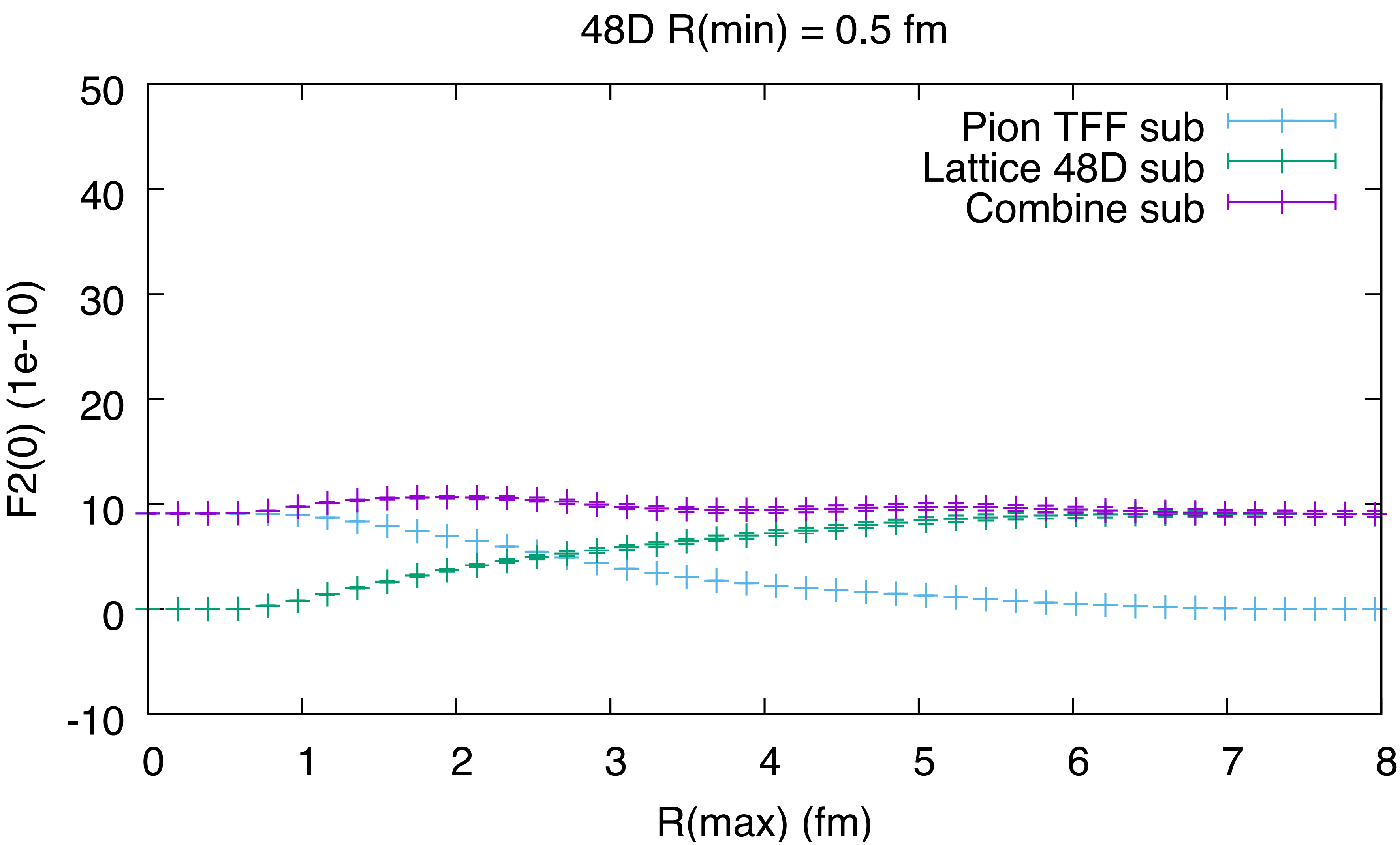


Integrand (preliminary)



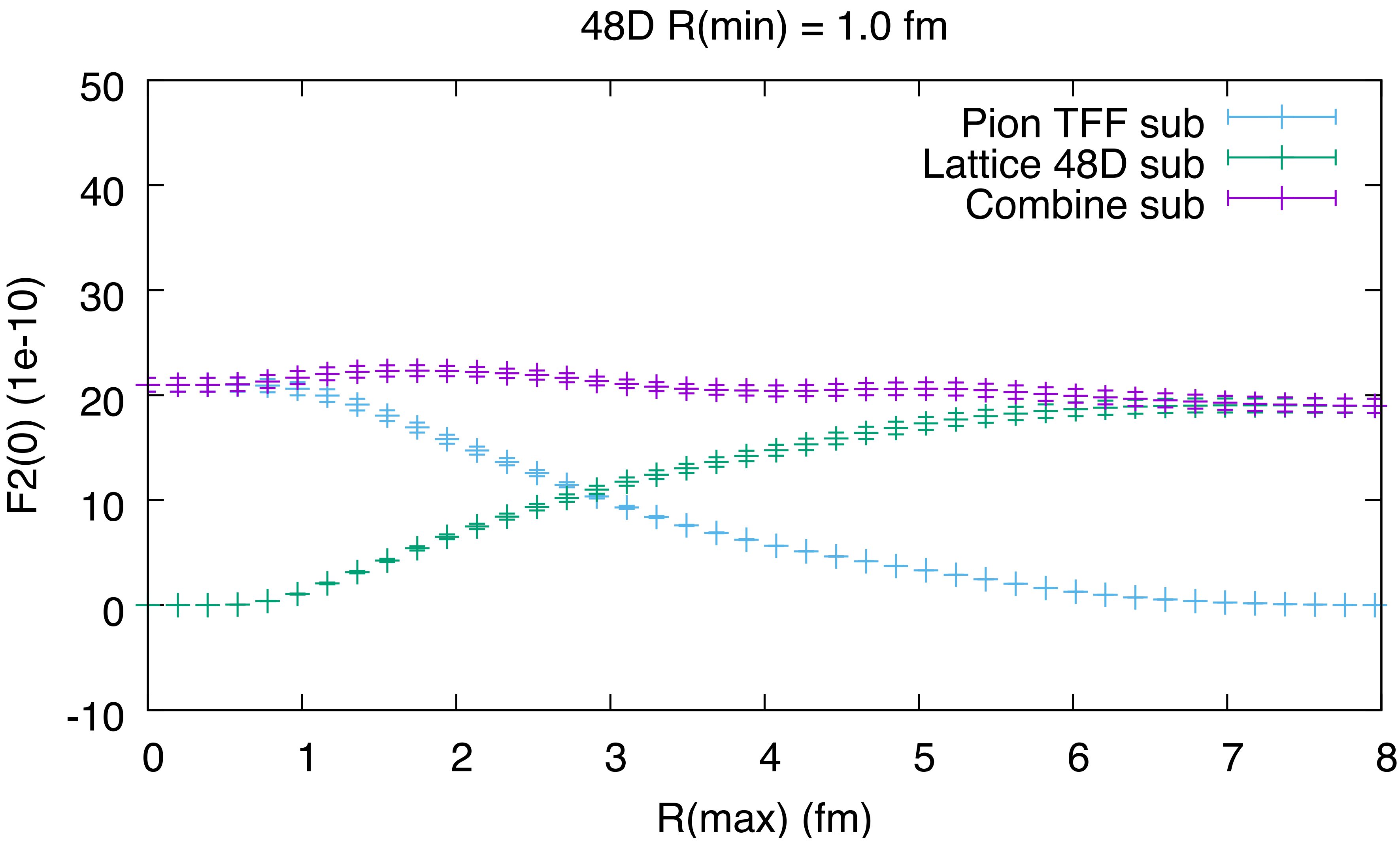
model integral is extrapolated to continuum/infinite volume limits
extrapolations to be scrutinized

Patch-up example
Preliminary

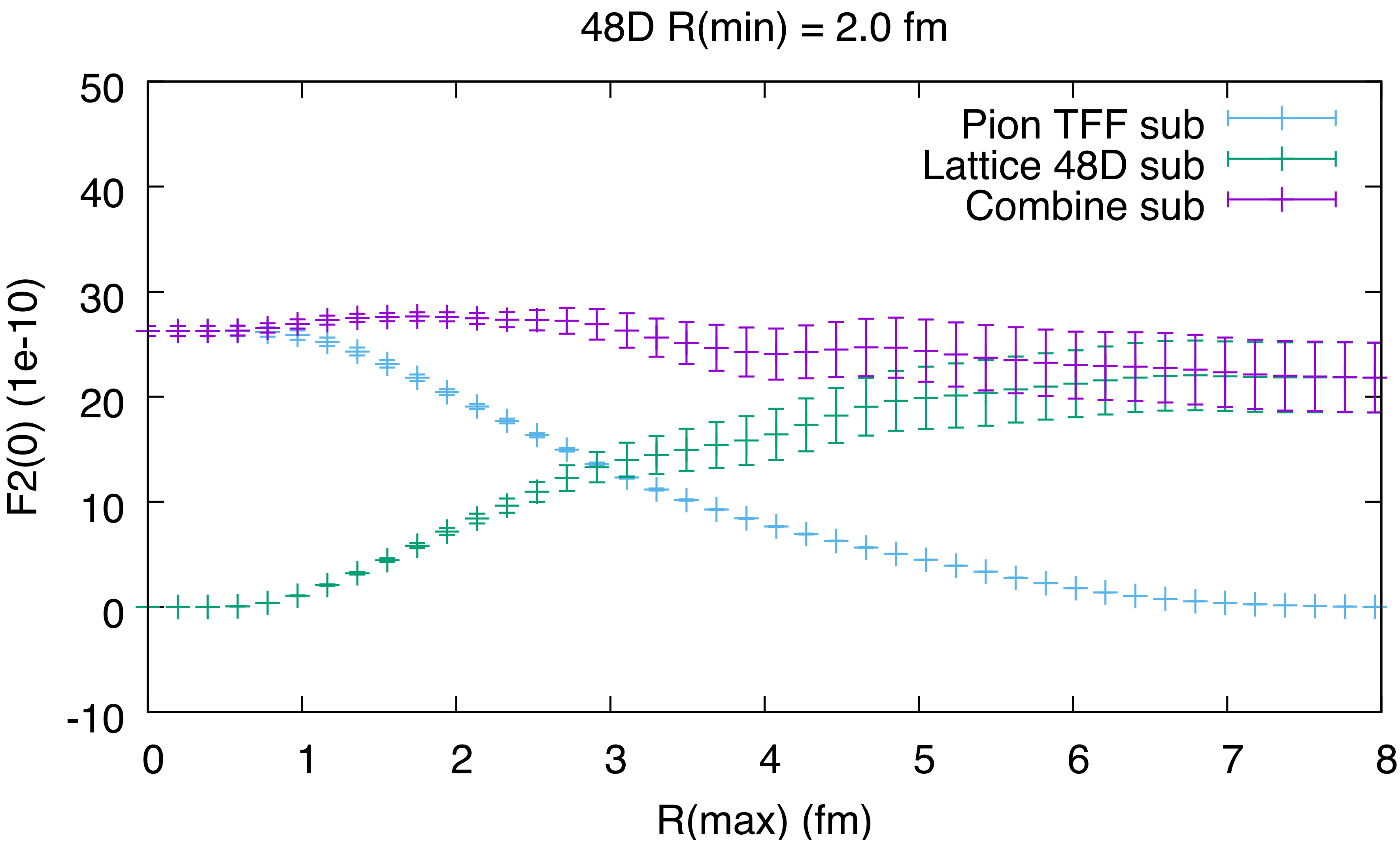


Switching point (long: model, short: lattice)

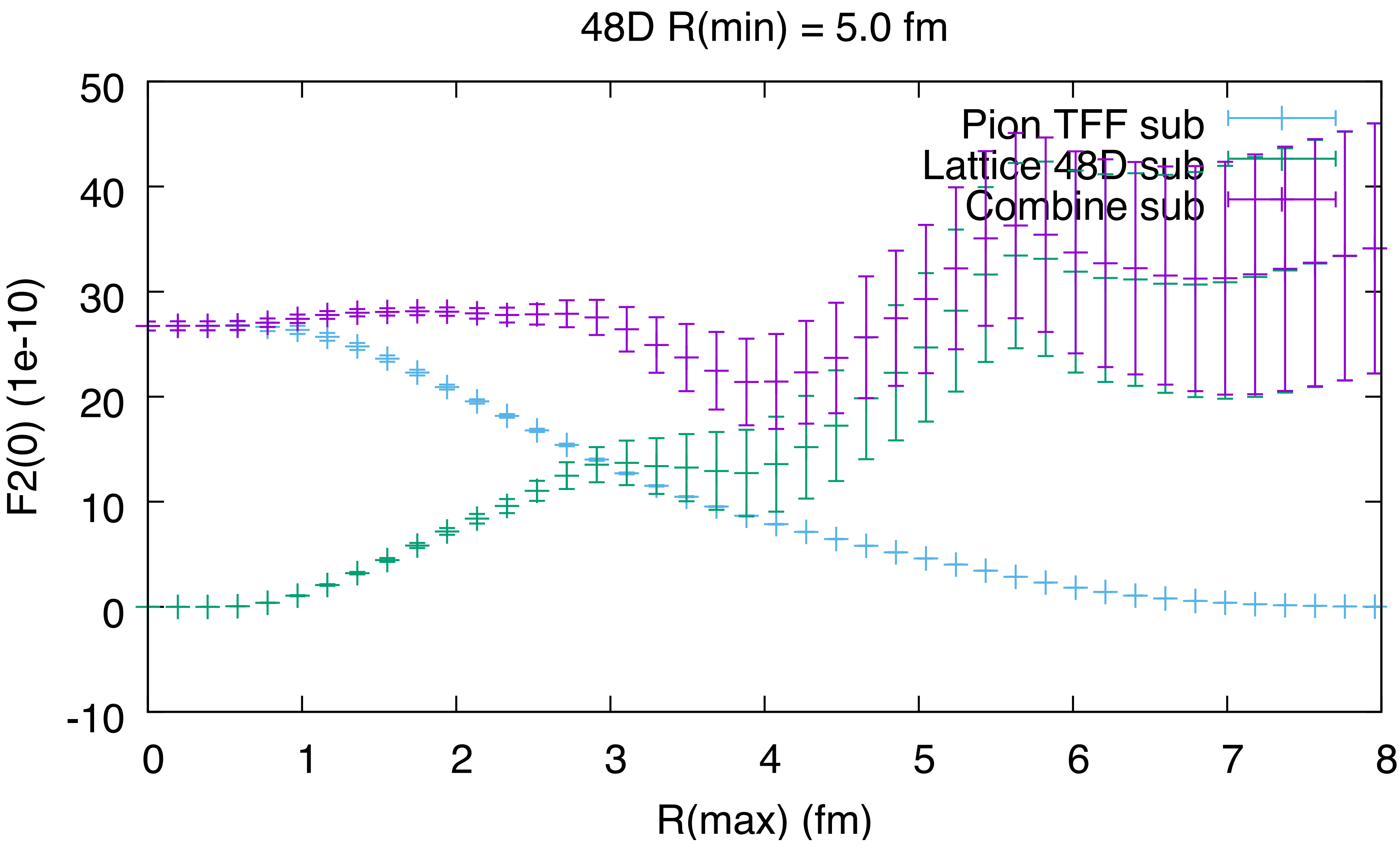
Preliminary



Preliminary



Preliminary



Is this safe ?

- At given distance, there are other than pi0 contribution in DR and models [truncation]
- Probably not large for appropriate choice
- To be safer, we could try to consider subtracting pi0 contribution from Lattice

GH = GH(Lat; all) - GH(Lat; pi0) + GH(DR; pi0)

- How to compute GH(Lat; pi0) is non-trivial

Similar problem in tau HVP

[Hiroshi Ohki et al. arXiv:1803.07228]

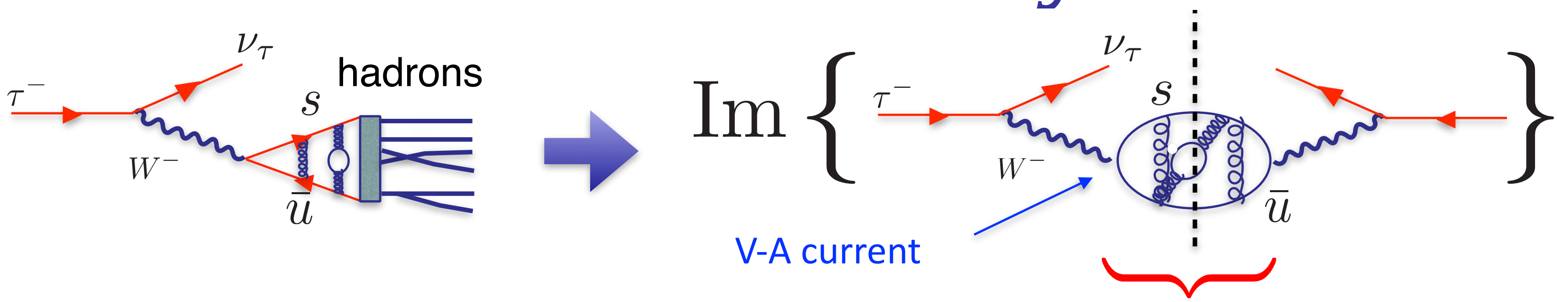
- In case of Vus analysis of tau -> up-strange inclusive hadronic decay
- We subtract K-pole contribution from lattice by fitting HVP in the on-shell long-distance, and evaluate the rest:

$$C(t) = A \exp(- mK\, t) + \text{rest}(t)$$

[A, mK is from fit]

(also tau-input for g-2 : [Mattia Brunno's talk])

Tau decay



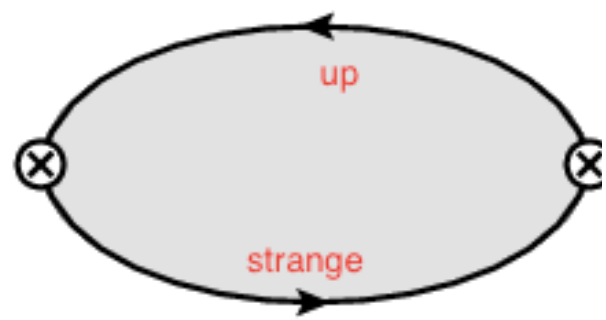
(Hadronic) vacuum polarization function $\Pi(Q^2)$

- Experiment side : $\tau \to \nu + had$ through V-A vertex. EW correction S_{EW}

$$
\begin{aligned}
R_{ij} &= \frac{\Gamma(\tau^- \to \text{hadrons}_{ij}\ \nu_\tau)}{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau)} \\
&= \frac{12\pi |V_{ij}|^2 S_{EW}}{m_\tau^2} \int_0^{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right) \underbrace{\left[\left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\Pi^{(1)}(s) + \mathrm{Im}\Pi^{(0)}(s)\right]}_{\equiv\ \mathrm{Im}\,\Pi(s)}
\end{aligned}
$$

- Lattice side : The Spin=0 and 1, vacuum polarization, Vector(V) or Axial (A) current-current two point

$$
\begin{aligned}
\Pi^{\mu\nu}_{ij;V/A}(q^2) &= i \int d^4x e^{iqx} \left\langle 0 | T J^{\mu}_{ij;V/A}(x) J^{\dagger\mu}_{ij;V/A}(0) | 0 \right\rangle \\
&= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi^{(1)}_{ij;V/A}(q^2) + q^\mu q^\nu \Pi^{(0)}_{ij;V/A}
\end{aligned}
$$



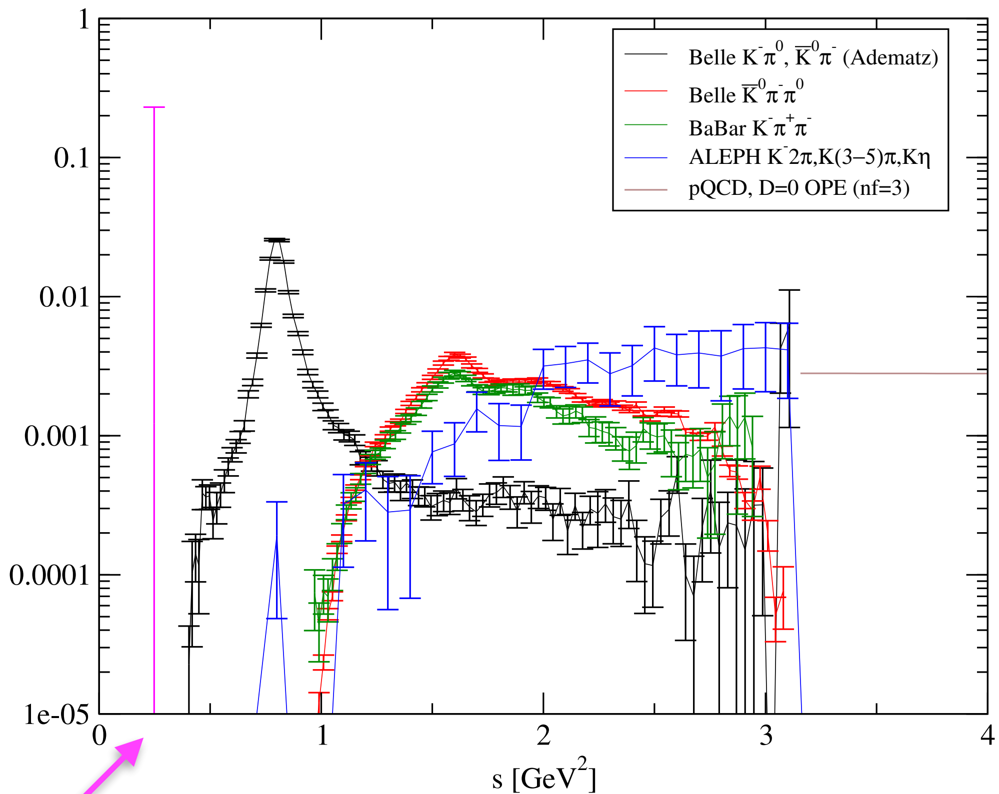
τ inclusive decay experiments

$$\tilde{\rho}(s) \equiv |V_{us}|^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\Pi^1(s) + \mathrm{Im}\Pi^0(s) \right]$$

To compare with experiments,

a conventional value of |Vus|=0.2253 is used

$\tilde{\rho}(s)$



[K. Maltman]

For K pole, we assume a delta function form $\gamma_K \omega(m_K^2)$

$\gamma_K \sim 2|V_{us}|^2 f_K^2$ obtained from either experimental value of K→μ or τ→k decay width.

$$\gamma_K[\tau \to K\nu_\tau] = 0.0012061(167)_{exp}(13)_{IB} \ \text{[HFAG16]}$$

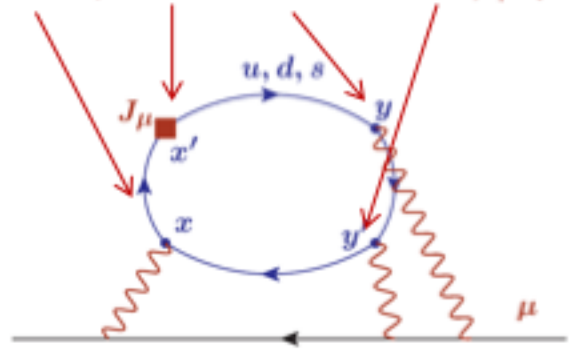
$$\gamma_K[K_{\mu 2}] = 0.0012347(\ \ 29)_{exp}(22)_{IB} \ \text{[PDG16]}$$

Pi0 subtraction on Lattice

[N. Christ et al @ UConn]

- Compute the π^0 pole contribution to:

$$\mathcal{A}_{\mu\mu'\nu\nu'}(x,x',y,y') = \langle 0 | T(J_\mu(x) J_{\mu'}(x') J_\nu(y) J_{\nu'}(y')) | 0 \rangle$$



- Assume x and y are far separated in the time direction and insert sum over π^0 states:

$$\mathcal{A}^{\pi^0}_{\mu_1\mu_2\nu_1\nu_2}(x,x',y,y') = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_\pi(p)} \langle 0 | T(J_\mu(x) J_{\mu'}(x')) | \pi^0(\vec{p}) \rangle \langle \pi^0(\vec{p}) | T(J_\nu(y) J_{\nu'}(y')) | 0 \rangle \quad (3)$$

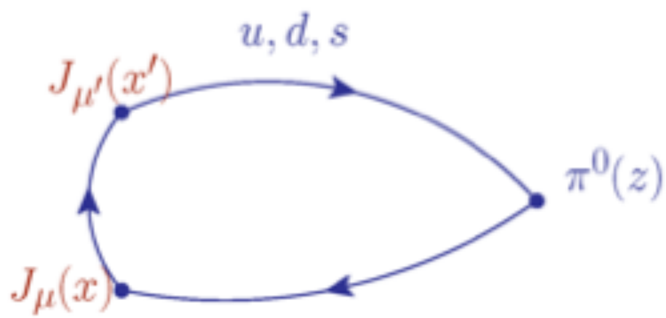
- Dominant contribution for x_0-y_0 large.

$$\mathcal{A}^{\pi^0}_{\mu\mu'\nu\nu'}(x,x',y,y') = \mathcal{F}_{\mu\mu'}\left(\tilde{x}, iM_\pi\hat{n}\right) \mathcal{F}_{\nu\nu'}\left(\tilde{y}, -iM_\pi\hat{n}\right) \Delta_F(x-y, M_\pi)$$

where $\hat{n} = \dfrac{\vec{x}-\vec{y}}{|x-y|}$, a unit Euclidean four-vector.

- The amplitude $\mathcal{F}_{\mu\mu'}\left(\tilde{x}, iM_\pi\hat{n}\right)$ also appears in a simpler Green's function:

$$\mathcal{B}_{\mu\mu'}(x,x',z) = \langle 0|T\left(J_\mu(x)J_{\mu'}(x')\pi^0(z)\right)|0\rangle$$

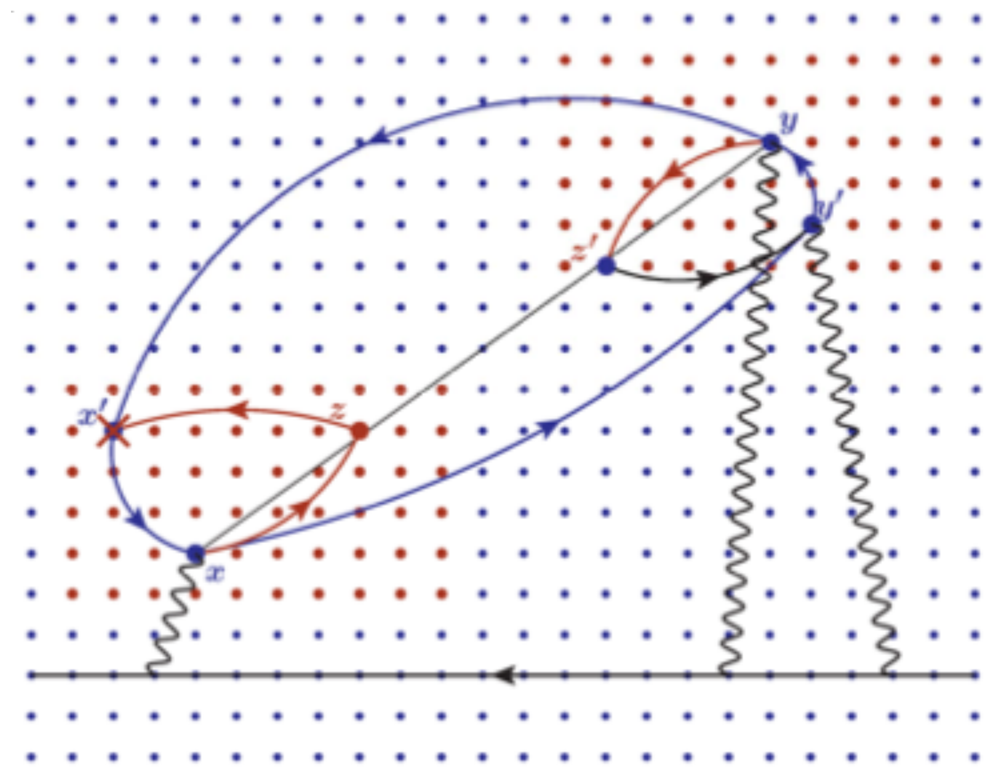


$$\mathcal{B}^{\pi^0}_{\mu\mu'}(x,x',z) = \mathcal{F}_{\mu\mu'}\left(\tilde{x}, iM_\pi\hat{n}\right) Z^{1/2}_{\pi^0} \Delta_F(x-z, M_\pi)$$

[also Luchang Jin's talk]

Lattice implementation

- lattice pi0-gamma-gamma FF could be computed separately, and if it's accurately determined, we could replace for long-distance of the full HLbL
- Or compute pi0-pole contribution simultaneously with the full HLbL on the same ensemble and subtract under the jack-knife



Discussion

- Interplay b/w Lattice and DR/model is a useful "plan-B" for HVP. Could we apply to HLbL ?
- Lattice : disconnected, continuum/infinite V limit
- Another interplay for HLbL possible ?
- How about the box diagram in DR ?
- Sum-rule for the full HLbL from Lattice to constraint DR or model ?

  Int[pole, cuts in DR] = Int[Euclidean Amp]

- Use of GEVP in subtracting pi0 or other specific contribution ? [A. Meyer's talk]

Finite Energy Sum Rule (FESR)

[Shifman, Vainshtein, and Zakharov '79]

The finite energy sum rule (FESR)

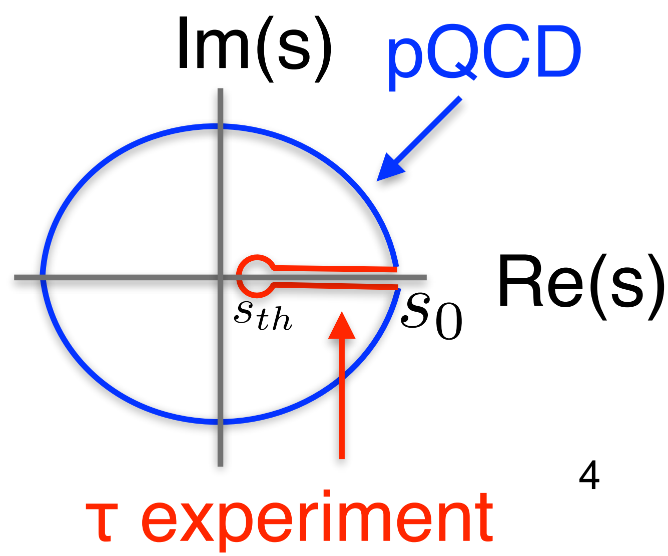
$$\int_0^{s_0} \omega(s)\rho(s)ds = -\frac{1}{2\pi i}\oint_{|s|=s_0} \omega(s)\Pi(s)ds, \quad (s_0 : \text{ finite energy})$$

w(s) is an arbitrary regular function such as polynomial in s.

- LHS : spectral function ρ(s) is related to the experimental τ inclusive decays

$$\frac{dR_{us;V/A}}{ds} = \frac{12\pi^2|V_{us}|^2 S_{EW}}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2 \left[\left(1+2\frac{s}{m_\tau^2}\right)\mathrm{Im}\Pi^1(s) + \mathrm{Im}\Pi^0(s)\right]$$

$$\tilde{\rho}(s) \equiv |V_{us}|^2 \left[\left(1+2\frac{s}{m_\tau^2}\right)\mathrm{Im}\Pi^1(s) + \mathrm{Im}\Pi^0(s)\right]$$



- RHS ⋯ Analytic calculation
  with perturbative QCD (pQCD) and OPE

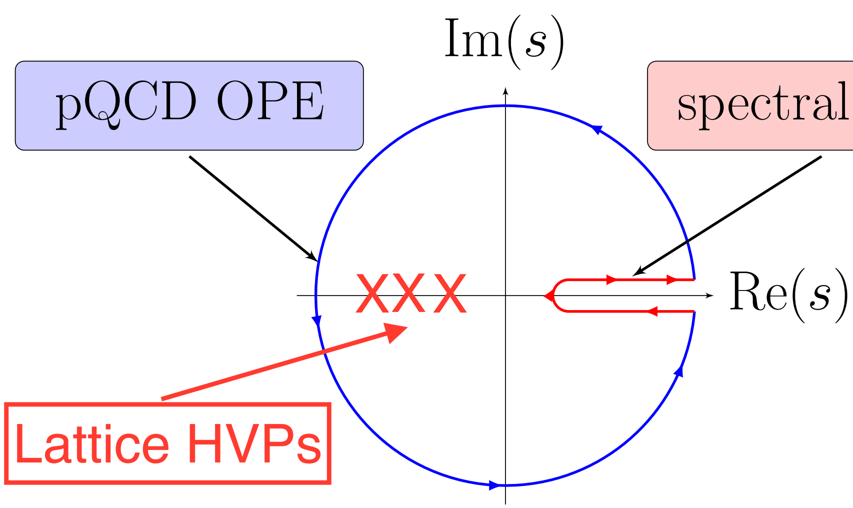
Our new method : Combining FESR and Lattice

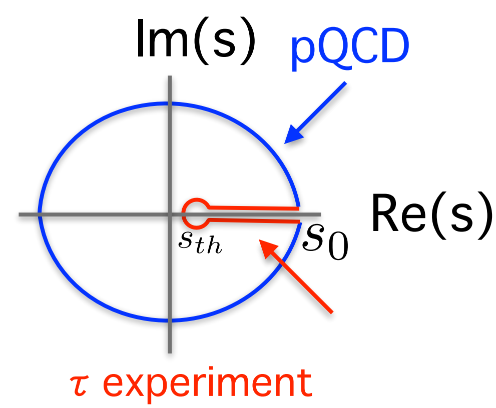
- If we have a reliable estimate for $\Pi(s)$ in Euclidean (space-like) points, $s = -Q_k^2 < 0$, we could extend the FESR with weight function $w(s)$ to have poles there,

$$\int_{s_{th}}^{\infty} w(s)\mathrm{Im}\Pi(s) = \pi \sum_{k}^{N_p} \mathrm{Res}_k[w(s)\Pi(s)]_{s=-Q_k^2}$$

$$\Pi(s) = \left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\Pi^{(1)}(s) + \mathrm{Im}\Pi^{(0)}(s) \propto s \quad (|s| \to \infty)$$

- For $N_p \geq 3$, the $|s| \to \infty$ circle integral vanishes.





Collaborators / Machines

g-2 DWF HVP & HLbL

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Norman Christ (Columbia)
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James Harrison (Southampton)
Taku Izubuchi (BNL/RBRC)
Christoph Lehner (BNL)
Kim Maltman (York)
Chulwoo Jung (BNL)
Andreas Jüttner (Southampton)
Luchang Jin (BNL)
Antonin Portelli (Edinburgh)

HVP Clover on $(8.5\ \mathrm{fm})^3$

Taku Izubuchi (BNL/RBRC)
Christoph Lehner (BNL)
Yoshinobu Kuramashi (Tsukuba/ AICS)
Eigo Shintani (RIKEN AICS)

tau decay

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Kim Maltman (York)
Antonin Portelli (Edinburgh)
Renwick James Hudspith (York)
Andreas Jüttner(Southampton)
Randy Lewis (Southampton)
Hiroshi Ohki (RBRC/Nara Women)
Matthew Spraggs (Edinburgh)

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