Hadronic light-by-light contribution to the muon anomalous magnetic moment from lattice QCD

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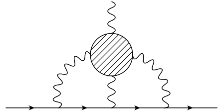
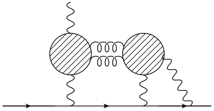
Outline I

Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

The desired amplitude  +  + $\cdots$ is obtained from a Euclidean space lattice calculation

$$\mathcal{M}_\nu(\vec{q}) = \lim_{\substack{t_{\rm src}\to-\infty \\ t_{\rm snk}\to\infty}} e^{E_{q/2}(t_{\rm snk}-t_{\rm src})} \sum_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} e^{-i\frac{\vec{q}}{2}\cdot(\vec{x}_{\rm snk}+\vec{x}_{\rm src})} e^{i\vec{q}\cdot\vec{x}_{\rm op}} \mathcal{M}_\nu(x_{\rm snk}, x_{\rm op}, x_{\rm src}),$$

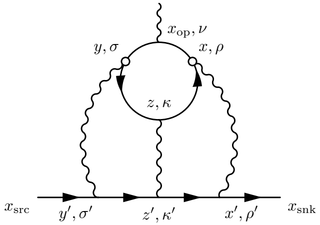
where

$$\begin{aligned} -e\mathcal{M}_\nu(x_{\rm src}, x_{\rm op}, x_{\rm snk}) &= \left\langle \mu(x_{\rm snk}) J_\nu(x_{\rm op}) \overline{\mu}(x_{\rm src}) \right\rangle \\ &= -e \sum_{x,y,z} \sum_{x',y',z'} \mathcal{F}_\nu(x, y, z, x', y', z', x_{\rm op}, x_{\rm snk}, x_{\rm src}). \end{aligned}$$

and

$$\left[\left(\frac{-i\not{q}^+ + m_\mu}{2E_{q/2}}\right)\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu, \gamma_\rho] q_\rho\right)\left(\frac{-i\not{q}^- + m_\mu}{2E_{q/2}}\right)\right]_{\alpha\beta} = \left(\mathcal{M}_\nu(\vec{q})\right)_{\alpha\beta},$$

Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]



$$\mathcal{F}^C_\nu\left(\vec{q};x,y,z,x_{\rm op}\right)=(-ie)^6\mathcal{G}_{\rho,\sigma,\kappa}(\vec{q};x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$$

$$\begin{aligned} & i^4\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \\ = \quad & \sum_{q=u,d,s}\frac{(e_q/e)^4}{6}\left\langle \mathrm{tr}\left[-i\quad \gamma_\rho S_q(x,z)i\gamma_\kappa S_q(z,y)i\gamma_\sigma S_q\left(y,x_{\rm op}\right)i\gamma_\nu S_q\left(x_{\rm op},x\right)\right]\right\rangle_{\rm QCD}+5\text{ permutations}\end{aligned}$$

$$\begin{aligned} & i^3\mathcal{G}_{\rho,\sigma,\kappa}(\vec{q};x,y,z) \\ = \quad & e^{\sqrt{m^2+\vec{q}^2/4}(t_{\rm snk}-t_{\rm src})}\sum_{x',y',z'}G_{\rho,\rho'}(x,x')G_{\sigma,\sigma'}(y,y')G_{\kappa,\kappa'}(z,z') \\ & \times\sum_{\vec{x}_{\rm snk},\vec{x}_{\rm src}}e^{-i\vec{q}/2\cdot(\vec{x}_{\rm snk}+\vec{x}_{\rm src})}S\left(x_{\rm snk},x'\right)i\gamma_{\rho'}S(x',z')i\gamma_{\kappa'}S(z',y')i\gamma_{\sigma'}S\left(y',x_{\rm src}\right)+5\text{ permutations}\end{aligned}$$

Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

- Do all sums in the QED part exactly (using FFT's),
- QCD part done stochastically
- Key idea: contribution exponentially suppressed with $r = |x - y|$, so importance sample, concentrate on $r \lesssim \lambda_\pi^{\mathrm{compton}}$
- space-time translational invariance allows coordinates relative to the hadronic loop

$$\mathcal{M}_\nu(\vec{q}) = \sum_r \left\{ \sum_{z, x_{\mathrm{op}}} \mathcal{F}_\nu \left(\vec{q}, \frac{r}{2}, \frac{-r}{2}, z, x_{\mathrm{op}} \right) e^{i\vec{q}\cdot\vec{x}_{\mathrm{op}}} \right\}$$

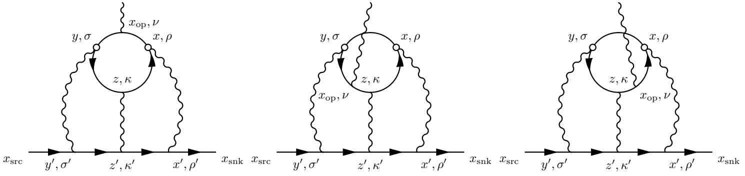
where $r = x - y$, $z \to z - w$, $x_{\mathrm{op}} \to x_{\mathrm{op}} - w$ and $w = (x + y)/2$

- We sum all the internal points over the entire space-time except we fix $x + y = 0$.
- (x, y) pairs stochastically sampled, z and x_{op} sums exact

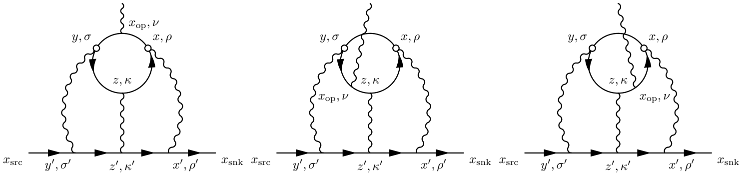
Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

$$\langle \mu(\vec{p}')|J_\nu(0)|\mu(\vec{p})\rangle = -e\bar{u}(\vec{p}')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\right)u(\vec{p})$$

- implies $F_2(0)$ only accessible by extrapolation $q \to 0$.
- Form is due to Ward Identity, or charge conservation
- need WI to be exact on each config, or error blows up as $\vec{q} \to 0$
- To enforce WI compute average of diagrams with all possible insertions of $J_\nu(x_{\rm op})$



Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]



- WI allows a moment method that projects directly to $q = 0$

$$\begin{aligned} \mathcal{M}_\nu(\vec{q}) &= \sum_{r,z,x_{\rm op}} \mathcal{F}^C_\nu\left(\vec{q}, \frac{r}{2}, -\frac{r}{2}, z, x_{\rm op}\right)\left(e^{i\vec{q}\cdot\vec{x}_{\rm op}} - 1\right) \\ &\approx \sum_{r,z,x_{\rm op}} \mathcal{F}^C_\nu\left(\vec{q}, \frac{r}{2}, -\frac{r}{2}, z, x_{\rm op}\right)(i\vec{q}\cdot\vec{x}_{\rm op}) \\ \frac{\partial}{\partial q_i}\mathcal{M}_\nu(\vec{q})|_{\vec{q}=0} &= i\sum_{r,z,x_{\rm op}} \mathcal{F}^C_\nu\left(\vec{q}=0, r, -r, z, x_{\rm op}\right)(x_{\rm op})_i \end{aligned}$$

Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

Sandwich $\mathcal{M}_\nu(\vec{q})$ between positive energy Dirac spinors $u(\vec{0},s)$, $\bar{u}(\vec{0},s)$

$$\overline{u}(\vec{0},s')\left(\frac{F_2(q^2=0)}{2m_\mu}\frac{i}{2}[\gamma_i,\gamma_j]\right)u(\vec{0},s) = \overline{u}(\vec{0},s')\frac{\partial}{\partial q_j}\mathcal{M}_i(\vec{q})|_{\vec{q}=\vec{0}}u(\vec{0},s)$$

multiply both sides by $\frac{1}{2}\epsilon_{ijk}$, sum over i and j,

$$\frac{F_2(0)}{m}\bar{u}_{s'}(\vec{0})\frac{\vec{\Sigma}}{2}u_s(\vec{0}) \;=\; \sum_r\left[\sum_{z,x_{\text{op}}}\frac{1}{2}\vec{x}_{\text{op}}\times\bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C\left(\vec{0};x=-\frac{r}{2},y=+\frac{r}{2},z,x_{\text{op}}\right)u_s(\vec{0})\right]$$

where $\Sigma_i = \frac{1}{4i}\epsilon_{ijk}[\gamma_j,\gamma_k]$.

Lattice setup

- Photons: Feynman gauge, QED_L [Hayakawa and Uno, 2008] (omit all modes with $\vec{q} = 0$)
- Gluons: Iwasaki (I) gauge action (RG improved, plaquette+rectangle)
- muons: $L_s = \infty$ free domain-wall fermions (DWF)
- quarks: Möbius-DWF

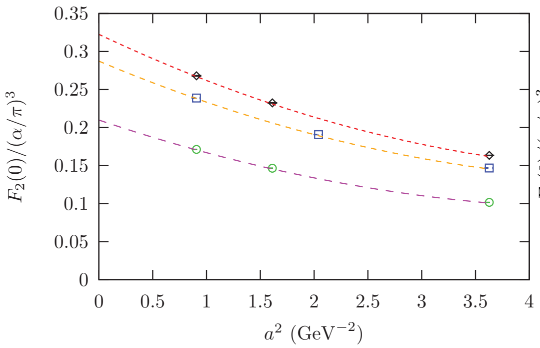
2+1f Möbius-DWF, I and I-DSDR physical point QCD ensembles (RBC/UKQCD) [Blum et al., 2014]

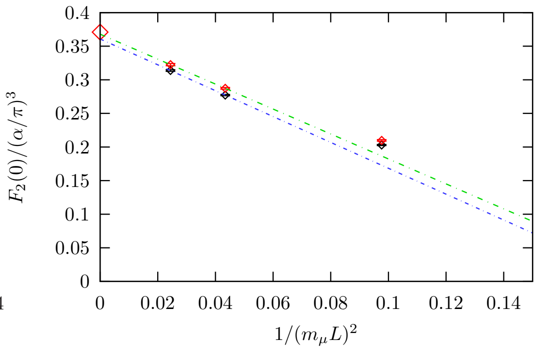
	48I	64I	24D	32D	32D fine	48D
a^{-1} (GeV)	1.73	2.36	1.0	1.0	1.38	1.0
a (fm)	0.114	0.084	0.2	0.2	0.14	0.2
L (fm)	5.47	5.38	4.8	6.4	4.6	9.6
L_s	48	64	24	24	24	24
m_π (MeV)	139	135	140	140	140	140
m_μ (MeV)	106	106	106	106	106	106
meas (con,disco)	65,65	43,44	33,32	42,20	8,7	62,0

Continuum and ∞ volume limits in QED [Blum et al., 2016]

Test method in pure QED

QED systematics large, $O(a^4)$, $O(1/L^2)$, but under control

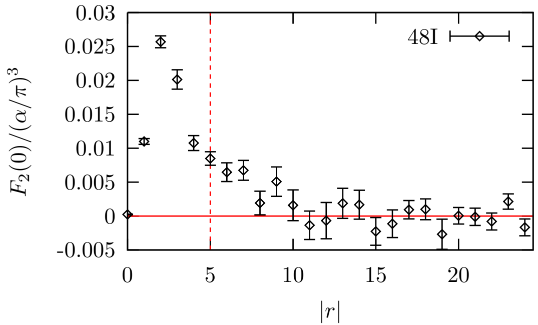


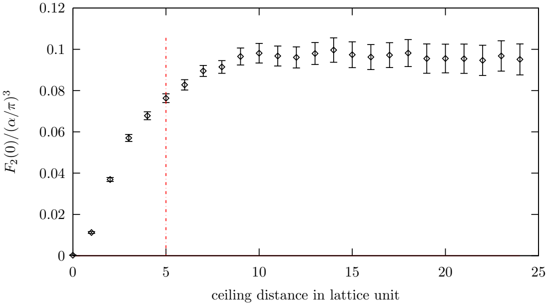


Limits quite consistent with well known PT result

Physical point cHLbL contribution, 48^3, 1.73 GeV lattice [Blum et al., 2017a]

- Measurements on 65 configurations, separated by 20 trajectories
- ignore strange quark contribution (down by 1/17 plus mass suppressed)
- exponentially suppressed with distance
- most of contribution by about 1 fm

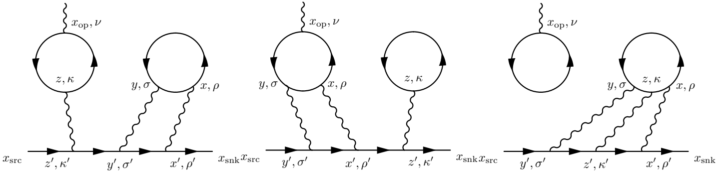




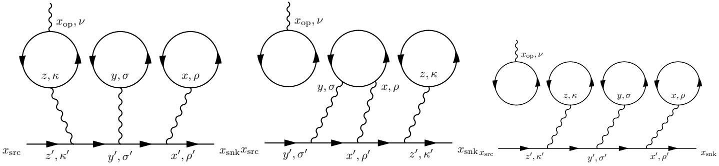
$$a_\mu^{\mathrm{cHLbL}} = 11.60 \pm 0.96 \times 10^{-10}$$

Disconnected contributions

SU(3) flavor:



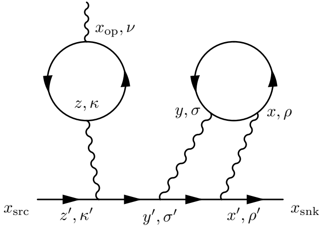
Leading $\qquad\qquad$ $O(m_s - m_{u,d})$



$O(m_s - m_{u,d})^2$ $\qquad\qquad$ and higher

- Gluons within and connecting quark loops have not been drawn
- To ensure loops are connected by gluons, explicit "vacuum" subtraction is required

Leading disconnected contribution



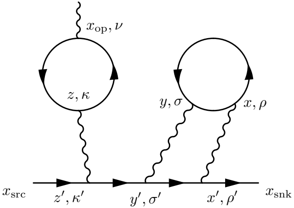
- We use two point sources at y and z, chosen randomly. The points sinks x_{op} and x are summed over exactly on lattice.
- Only point source quark propagators are needed. We compute M point source propagators and all M^2 combinations are used to perform the stochastic sum over $r = z - y$ (M^2 trick).

$$\mathcal{F}^D_\nu(x, y, z, x_{op}) = (-ie)^6 \mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{op})$$

$$\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{op}) = \left\langle \frac{1}{2}\Pi_{\nu,\kappa}(x_{op}, z)\left[\Pi_{\rho,\sigma}(x, y) - \Pi^{\rm avg}_{\rho,\sigma}(x - y)\right]\right\rangle_{\rm QCD}$$

$$\Pi_{\rho,\sigma}(x, y) = -\sum_q (e_q/e)^2\, \mathrm{Tr}[\gamma_\rho S_q(x, y)\gamma_\sigma S_q(y, x)].$$

Leading disconnected contribution



$$\frac{F_2^{\rm dHLbL}(0)}{m} \frac{(\sigma_{s',s})_i}{2} = \sum_{r,x} \sum_{x_{\rm op}} \frac{1}{2} \epsilon_{i,j,k} (x_{\rm op})_j \cdot i \bar{u}_{s'}(\vec{0}) \mathcal{F}_k^D (x, y = r, z = 0, x_{\rm op}) \, u_s(\vec{0})$$

$$\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\rm op}) = \left\langle \frac{1}{2} \Pi_{\nu,\kappa} (x_{\rm op}, z) \left[\Pi_{\rho,\sigma}(x, y) - \Pi^{\rm avg}_{\rho,\sigma}(x - y) \right] \right\rangle_{\rm QCD}$$

$$\sum_{x_{\rm op}} \frac{1}{2} \epsilon_{i,j,k} (x_{\rm op})_j \langle \Pi_{\rho,\sigma} (x_{\rm op}, 0) \rangle_{\rm QCD} = \sum_{x_{\rm op}} \frac{1}{2} \epsilon_{i,j,k} (-x_{\rm op})_j \langle \Pi_{\rho,\sigma} (-x_{\rm op}, 0) \rangle_{\rm QCD} = 0$$

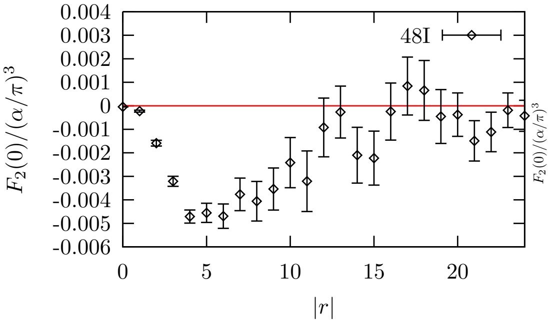
- Because of parity, the expectation value for the (moment of) left loop averages to zero.
- $\left[\Pi_{\rho,\sigma}(x, y) - \Pi^{\rm avg}_{\rho,\sigma}(x - y) \right]$ is only a noise reduction technique. $\Pi^{\rm avg}_{\rho,\sigma}(x - y)$ should remain constant through out the entire calculation.

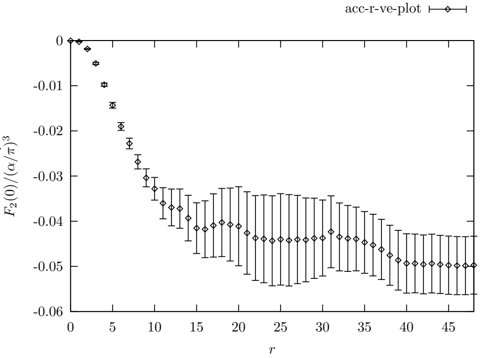
Physical point dHLbL contribution [Blum et al., 2017a]

- Use AMA with 2000 low-modes of the Dirac operator and
- randomly choose 256 “spheres” of radius 6 lattice units
- Uniformly sample 4 (unique) points in each
- do half as many strange quark props
- Construct $(1024 + 512)^2$ point-pairs per configuration

Physical point dHLbL contribution, 48^3, 1.73 GeV lattice [Blum et al., 2017a]

- strange contributes less than 5 %

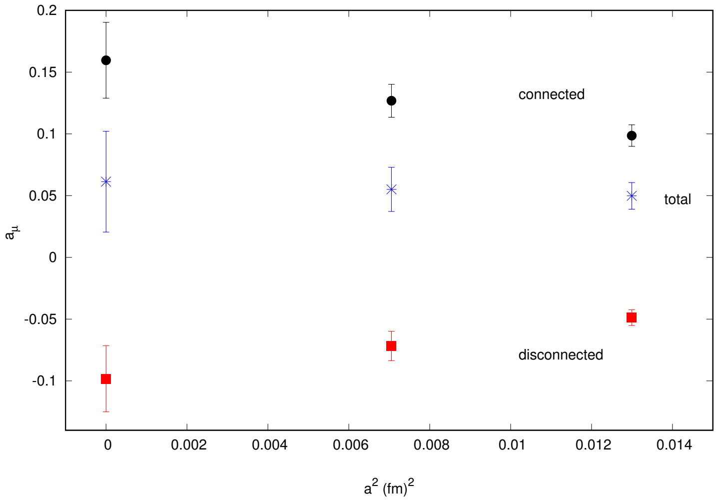




$$a_\mu^{\mathrm{dHLbL}} = -6.25 \pm 0.80 \times 10^{-10}$$

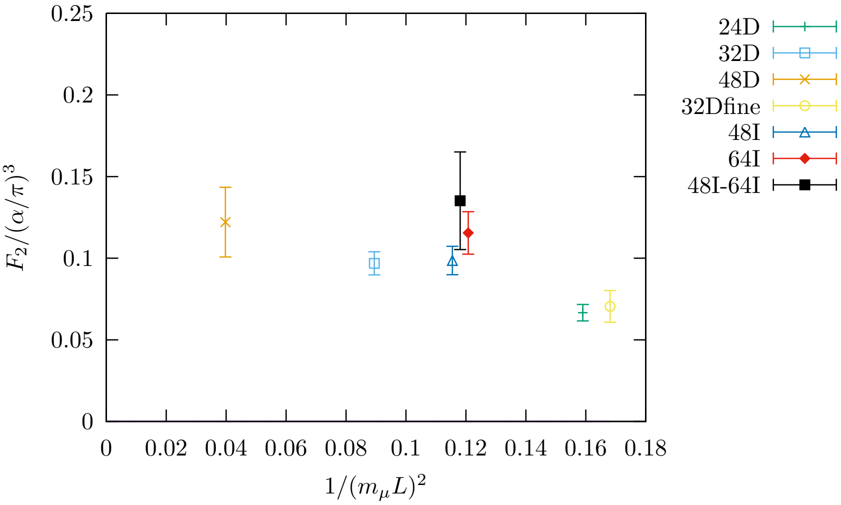
$$a_\mu^{\mathrm{cHLbL}} + a_\mu^{\mathrm{dHLbL}} = 5.35 \pm 1.35 \times 10^{-10}$$

Continuum extrapolation, Iwasaki ensembles (preliminary)



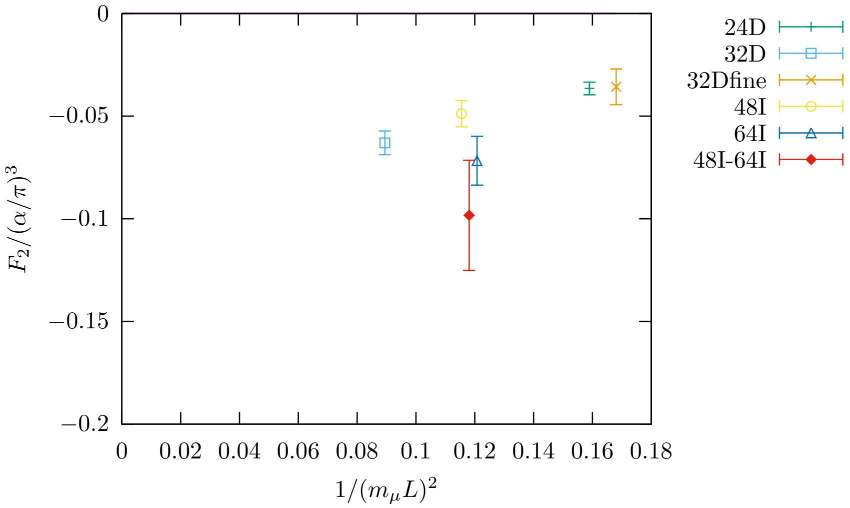
- linear in $a^2 \to 0$ extrapolation
- Effects tend to cancel between cHLbL and dHLbL contributions
- Collecting more statistics

QED$_L$, connected diagram



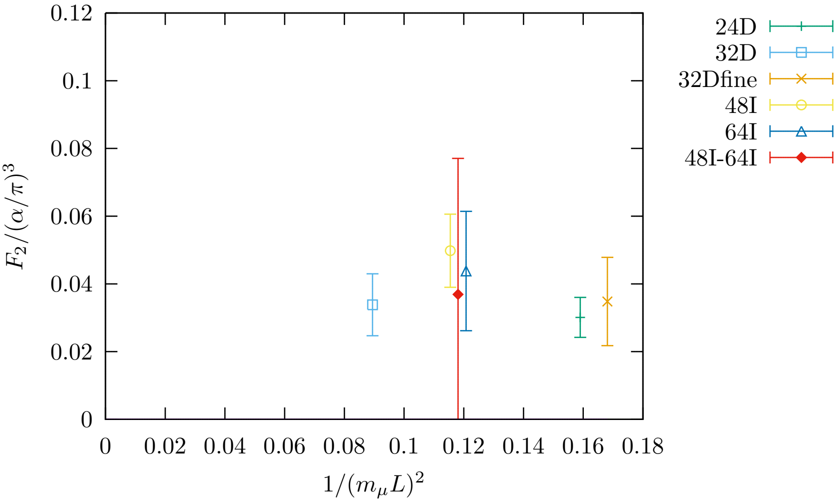
(all particles with physical masses)

QED$_L$, leading disconnected diagram



(all particles with physical masses)

QED$_L$, connected + leading disconnected

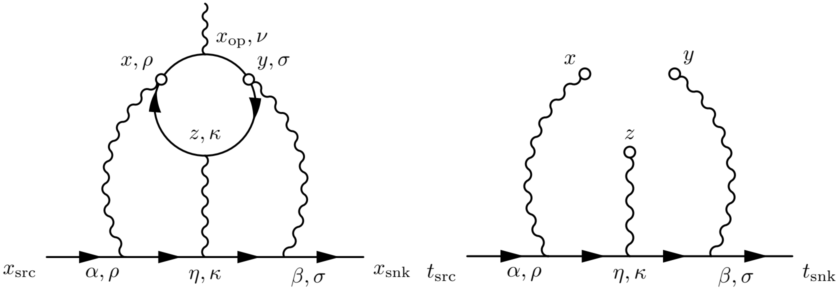


(all particles with physical masses)

QED_L, $a \to 0$ and $L \to \infty$ limits (PRELIMINARY)

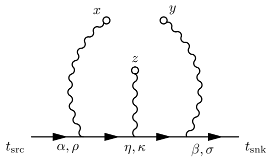
- linear in lattice spacing-squared and $1/L^2$
- ignore correlations between connected and disconnected
  $a_\mu(a, L) = a_\mu(0, \infty) + a_I a^2 + a_{ID} a^2 + b_0 1/L^2$
- connected: $0.171 \pm 0.027 (\alpha/\pi)^3$
- disconnected: $-0.122 \pm 0.023 (\alpha/\pi)^3$
- sum: $0.049 \pm 0.035 (\alpha/\pi)^3 = 6.1 \pm 4.4 \times 10^{-10}$
- Glasgow Consensus is $10.5 \pm 2.6 \times 10^{-10}$
- warning: need sub-leading disconnected contributions

QED_∞ [Green et al., 2015, Asmussen et al., 2016, Lehner and Izubuchi, 2015, Jin et al., 2015, Blum et al., 2017b]



- Mainz group made first concrete proposal for QED_∞
- QED_∞: muon, photons computed in infinite volume (*c.f.* HVP)
- QCD mass gap: $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\rm op}) \sim \exp -m_\pi \times \mathrm{dist}(x, y, z, x_{\rm op})$
- QED weight function does not grow exponentially
- So leading FV error is exponentially suppressed (*c.f.* HVP) instead of $O(1/L^2)$

QED$_\infty$ weighting function [Blum et al., 2017b]



$$= \quad \mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z) + 5 \text{ perms.}$$

- Note Hermitian part gives same F_2 but is infrared finite,

$$\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x, y, z) = \frac{1}{2}\mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z) + \frac{1}{2}\mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z)^\dagger$$

- In units of the muon mass m_μ,

$$\begin{aligned}\mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y, z, x) \quad = \quad & \frac{\gamma_0 + 1}{2} i\gamma_\sigma(-\partial\!\!\!/_y + \gamma_0 + 1) i\gamma_\kappa(\partial\!\!\!/_x + \gamma_0 + 1) i\gamma_\rho \frac{\gamma_0 + 1}{2} \\ & \times \frac{1}{4\pi^2} \int d^4\eta \frac{1}{(\eta - z)^2} f(\eta - y) f(x - \eta)\end{aligned}$$

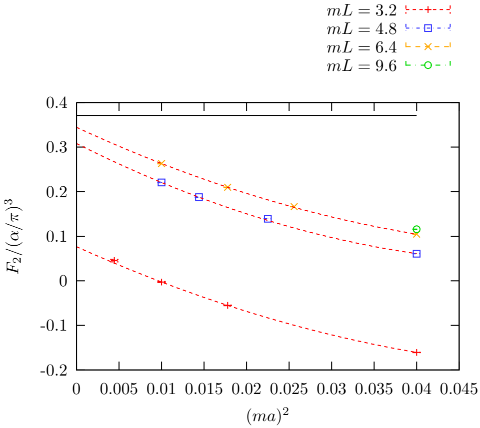
QED_∞ subtraction [Blum et al., 2017b]

- Current conservation implies $\sum_x \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\text{op}}) = 0$ ($V \to \infty$ and $a \to 0$)
- Subtract terms that vanish as $a, V \to 0$
  $\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z) = \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,y) + \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,y)$
- subtraction changes (may reduce) a and V systematic errors ($c.f.$ HVP)
- Further, $\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(z,z,x) = 0$ so short distance $O(a^2)$ effects suppressed.

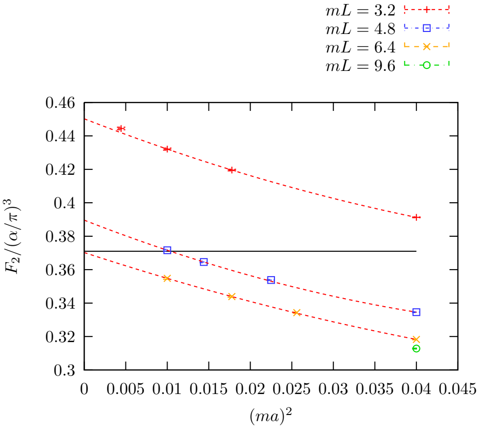
- The 4-dim integral is (pre-)calculated numerically with CUBA library (cubature rules).
- Translation/rotation symmetry: parametrize (x,y,z) by 5 parameters on N^5 grid points (Mainz uses 3 params by averaging over muon time direction).
- (linearly) Interpolate grid in stochastic integral over (x,y)

QED$_\infty$ results- pure QED, lattice-spacing error [Blum et al., 2017b]

- lattice spacing error ≈const for $mL \gtrsim 4.8$
- FV effect $\lesssim 1\%$ for $mL = 9.6$
- fit: $F_2(L,a) = F_2(L) + k_1 a^2 + k_2 a^4$



$\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z)$

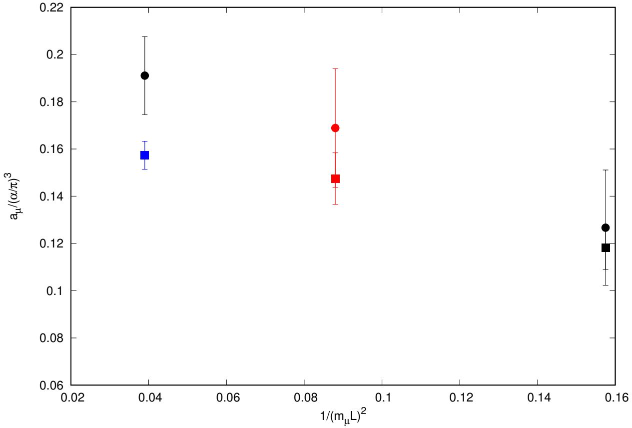


$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z)$

QED_∞ results- pure QED, finite volume error [Blum et al., 2017b]

- Take $F_2(\infty) \approx F_2(mL = 9.6)$
- results for $m_{\text{loop}} = m_{\text{line}}$ (a_e) and $m_{\text{loop}} = 2m_{\text{line}}$
- $F_2/(\alpha/\pi)^3 =$ 0.3686(37)(35) and 0.1232(30)(28) compared to
- QED perturbation theory results : 0.371 and 0.120

QED_∞, connected diagram, $a = 0.2$ fm (preliminary)



(all particles with physical masses)

- QED_∞ noisier than QED_L
- make distance cuts to enhance signal, suppress noise
  - Upper: 'short' cut = 0.16 fm
  - Lower: 'short' cut = 0.10 fm
- Collecting more statistics

Outline I

Hadronic light-by-light summary and outlook

- Lattice QCD(+QED) calculations done with physical masses, large boxes + improved measurement algorithms
- Physical point calculations published at $a = 0.114$ fm, 5.5 fm box [Blum et al., 2017a]
- Preliminary $a \to 0$, $L \to \infty$ limits taken in QED_L,s
  - connected, disconnected significant corrections, but total has mild dependence
  - improving statistics
  - need non-leading disconnected diagrams (see talk by Hayakawa)
  - consistent with model, dispersive results (somewhat smaller CV).
- QED_∞ noisier, $a \to 0$, $L \to \infty$ limits not yet available
- unlikely that HLbL contribution will rescue standard model

On track for solid result in time for E989

Acknowledgments

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Outline I

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