# HLbL lattice overview (UConn review)

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Second Workshop of the Muon $g-2$ Theory Initiative
Mainz, Germany, June 18 - 22, 2018

# Talks on HLbL on the lattice at UConn workshop in March 2018

- HLbL contribution to the muon $g-2$ on the lattice: overall strategy (AN, Mainz)
- HLbL contribution to the muon $g-2$ on the lattice: finite volume and discretization effects (Nils Asmussen, Mainz)
- HLbL contribution to the muon $g-2$ on the lattice: overall strategy (Luchang Jin, UConn/RBRC)
- Discretization errors in Light-by-Light scattering calculations (Tom Blum, UConn/RBRC)
- HLbL contribution to $(g-2)_\mu$ on the lattice: finite-volume effects (Christoph Lehner, BNL)
- Models and HLbL: disconnected contributions and first steps towards finite volume corrections (Johan Bijnens, Lund)
- Disconnected quark loop contribution to Hadronic Light-by-light diagram (Taku Izubuchi, BNL/RBRC)
- The neutral pion decay and the chiral anomaly on the lattice (Shoji Hashimoto, KEK, talk cancelled, slides available)
- Pion transition form factor on the lattice, pion-pole contribution to $g-2$ (Antoine Gérardin, Mainz)
- Determining the long-distance contribution to the HLbL portion of $g-2$ in position space from the $\pi^0$ pole (Norman Christ, Columbia)
- Pion transition form factor from lattice QCD in position space (Cheng Tu, UConn)
- HLbL forward scattering sum rules on the lattice (Antoine Gérardin, Mainz)

## Outline

- Brief recap of approaches to HLbL on the lattice and status before the UConn meeting
- New results on HLbL in $g-2$ presented at the UConn meeting
- $\pi^0 \to \gamma^*\gamma^*$ transition form factor on the lattice
  Pion-pole contribution to muon $g-2$ on the lattice
- Conclusions, Tasks for Mainz workshop and Outlook

# Muon $g-2$: current status

| Contribution | $a_\mu \times 10^{11}$ | Reference |
|---|---|---|
| QED (leptons) | $116\,584\,718.853 \pm 0.036$ | Aoyama et al. '12 |
| Electroweak | $153.6 \pm 1.0$ | Gnendiger et al. '13 |
| HVP: LO | $6887.7 \pm 33.8$ | Jegerlehner '17 |
| NLO | $-99.3 \pm 0.7$ | Jegerlehner '17 |
| NNLO | $12.4 \pm 0.1$ | Kurz et al. '14 |
| HLbL | $102 \pm 39$ | Jegerlehner '15 (JN '09) |
| NLO | $3 \pm 2$ | Colangelo et al. '14 |
| Theory (SM) | $116\,591\,778 \pm 52$ | |
| Experiment | $116\,592\,089 \pm 63$ | Bennett et al. '06 |
| Experiment - Theory | $311 \pm 81$ | 3.8 $\sigma$ |

HLbL based on Jegerlehner, AN '09, with downward shift because of smaller axial-vector contribution (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15).
Other frequently used estimate for HLbL: $a_\mu^{\mathrm{HLbL}} = (105 \pm 26) \times 10^{-11}$ (Prades, de Rafael, Vainshtein '09 ("Glasgow consensus")).

Discrepancy a sign of New Physics ?

Goal of Muon $g-2$ Theory Initiative (and this workshop and for the Whitepaper): more precise determination of HVP; more reliable value and error estimate for HLbL that does not rely completely on model calculations.

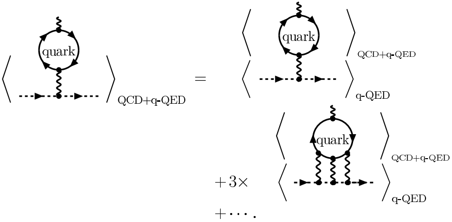
In order to fully profit from future $g-2$ experiments at Fermilab (E989) and J-PARC (E34) with four-fold improvement $\delta a_\mu = 16 \times 10^{-11}$.

Brief recap of approaches and status of HLbL on the lattice before the UConn meeting

# RBC-UKQCD approach to HLbL

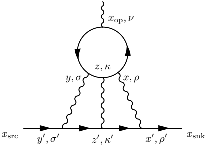
Blum, Hayakawa, et al. '05, . . . , '15:

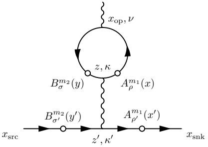
- Put QCD + (quenched) QED on the lattice.
- QED treated non-perturbatively $\Rightarrow$ all orders in $\alpha$
- Need to subtract lower order non-HLbL contribution $\Rightarrow$ very noisy on the lattice. First signal for $F_2(q^2)$ for $q^2 \geq 0.11$ GeV$^2$ only in '15.




Jin et al. '15, '16, '17:

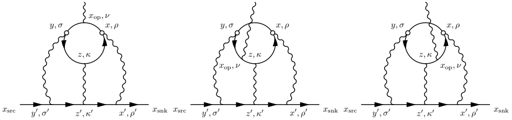
- Step by step improvement of method to reduce statistical error by one or two orders of magnitude and remove some systematic errors.
- Perturbative expansion in QED to deal only with HLbL contribution (no subtraction needed).
- Calculate $a_\mu^{\text{HLbL}} = F_2(q^2 = 0)$ via moment method in position-space (no extrapolation to $q^2 = 0$ needed).
- Exact propagator on lattice between $z, z'$. Stochastic photon propagators between $x, x'$ and $y, y'$ (did not work in practice).



# RBC-UKQCD approach to HLbL (talk by Luchang Jin)

**Point source photon method** **20/36**




$$\mathcal{F}^C_\nu(\vec{q};x,y,z,x_{\rm op}) = (-ie)^6\,\mathcal{G}_{\rho,\sigma,\kappa}(\vec{q};x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \tag{7}$$

$$\begin{aligned} & i^4\,\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \\ = & \sum_{q=u,d,s}\frac{(e_q/e)^4}{6}\left\langle \mathrm{tr}\left[-i\gamma_\rho S_q(x,z)i\gamma_\kappa S_q(z,y)i\gamma_\sigma S_q(y,x_{\rm op})i\gamma_\nu S_q(x_{\rm op},x)\right]\right\rangle_{\rm QCD} \\ + & \text{ other 5 permutations} \end{aligned} \tag{8}$$

$$\begin{aligned} & i^3\,\mathcal{G}_{\rho,\sigma,\kappa}(\vec{q};x,y,z) \\ = & e^{\sqrt{m_\mu^2+\vec{q}^2/4}(t_{\rm snk}-t_{\rm src})}\sum_{x',y',z'} G_{\rho,\rho'}(x,x')G_{\sigma,\sigma'}(y,y')G_{\kappa,\kappa'}(z,z') \\ & \times\sum_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} e^{-i\vec{q}/2\cdot(\vec{x}_{\rm snk}+\vec{x}_{\rm src})}\,S_\mu(x_{\rm snk},x')i\gamma_{\rho'}S_\mu(x',z')i\gamma_{\kappa'}S_\mu(z',y')i\gamma_{\sigma'}S_\mu(y',x_{\rm src}) \\ + & \text{ other 5 permutations} \end{aligned} \tag{9}$$

Master formula:

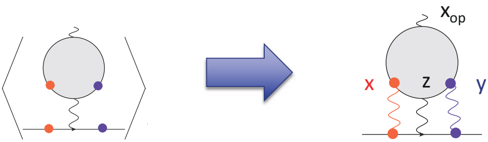
$$\frac{F_2(0)}{m}\overline{u}_{s'}(\vec{0})\,\frac{\vec{\Sigma}}{2}\,u_s(\vec{0}) = \sum_r\left[\sum_{z,x_{\rm op}}\frac{1}{2}\vec{x}_{\rm op}\times\overline{u}_{s'}(\vec{0})\,i\vec{\mathcal{F}}^C\left(\vec{0},x=-\frac{r}{2},y=+\frac{r}{2},z,x_{\rm op}\right)u_s(\vec{0})\right]$$

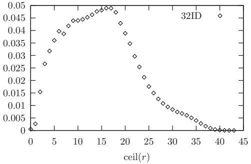
- Use exact photon propagators also between $x, x'$ and $y, y'$ and sample points $x, y$ stochastically.
- Sum all internal points over entire space time, except that one fixes $x + y = 0$.
- Time coordinate of current $(x_{\rm op})_0$ is integrated, not held fixed.

# Coordinate space Point photon method

[ Luchang Jin et al. , PRD93, 014503 (2016) ]

- Treat all 3 photon propagators exactly (3 analytical photons) , which makes the quark loop and the lepton line connected :
  disconnected problem in Lattice QED+QCD -> connected problem with analytic photon
- QED 2-loop in coordinate space. Stochastically sample, two of quark-photon vertex location x,y, z and $x_{op}$ is summed over space-time exactly



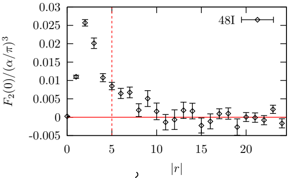



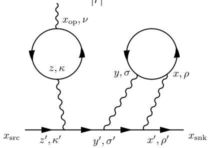

- Short separations, Min[ |x-z|,|y-z|,|x-y| ] < R ~ O(0.5) fm, which has a large contribution due to confinement, are summed for all pairs
- longer separations, Min[ |x-z|,|y-z|,|x-y| ] >= R, are done stochastically with a probability shown above ( Adaptive Monte Carlo sampling )

# RBC-UKQCD approach to HLbL (continued)

Jin et al. '16, '17

- Found empirically: short separations $r = \min\{|x-z|, |y-z|, |x-y|\} < r_{\max} \sim 0.6$ fm (4-6 in lattice units) give large contribution due to confinement. Are summed for all pairs.
- Longer separations $r > r_{\max}$ are done stochastically with empirical probability distribution.
- Test: Reproduce result for QED with muon loop after extrapolation to $a = 0$ and $L = \infty$.
- Calculate leading quark-disconnected diagrams (dHLbL).




Results ($m_{\pi,\text{phys}}$, lattice spacing $a^{-1} = 1.73$ GeV ($a = 0.114$ fm), $L = 5.5$ fm):

$$a_\mu^{\text{cHLbL}} = (116.0 \pm 9.6) \times 10^{-11} \qquad \text{(quark-connected diagrams)}$$
$$a_\mu^{\text{dHLbL}} = (-62.5 \pm 8.0) \times 10^{-11} \qquad \text{(leading quark-disconnected diagrams)}$$
$$a_\mu^{\text{HLbL}} = (53.5 \pm 13.5) \times 10^{-11}$$

Statistical error only ! Missing systematic effects:

- Expect large finite-volume effects from QED $\sim 1/L^2$. Blum et al. '17: use infinite volume, continuum QED$_\infty$ (similar to Mainz approach: Asmussen et al. '16).
- Expect large finite-lattice-spacing effects.
- Omitted subleading quark-disconneced diagrams (10% effect ?).

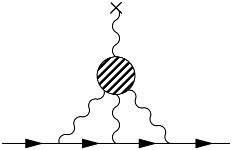
# Mainz approach to HLbL

Developed independently from RBC-UKQCD

Asmussen, Gérardin, Green, Meyer, AN '15 – '17

(Idea by Harvey Meyer after Muon $g-2$ workshop at Mainz in April 2014)

- QCD blob: lattice regularization
- Everything else: position-space perturbation theory in Euclidean formulation



Similarities to approach by RBC-UKQCD '15 –'17:

- Position-space (most natural for lattice QCD)
- Perturbative treatment of the QED part
- Get directly $a_\mu^{\text{HLbL}} = F_2(k^2 = 0)$ as spatial moment

Strengths of our approach:

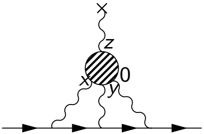
- Semi-analytical calculation of QED kernel
- QED part computed in continuum and in infinite volume (QED$_\infty$)
- Lorentz covariance manifest
- No power law effects $1/L^2$ in the volume

Challenges:

- Need to calculate a QCD four-point function on the lattice
- Numerical efficiency for QCD not yet shown

# HLbL master formula in position-space

$$a_\mu^{\rm HLbL} = \frac{me^6}{3}\int d^4y \int d^4x \, \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED} \, \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm QCD}$$




After contracting the Lorentz indices the integration reduces to a 3-dimensional integral over $x^2, y^2, x\cdot y = |x||y|\cos\beta$:

$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2\int_0^\infty d|y||y|^3} \underbrace{\int d^4x}_{=4\pi\int_0^\infty d|x||x|^3\int_0^\pi d\beta\sin^2(\beta)} \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED} \, \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm QCD}$$

QCD four-point function (spatial moment):

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z \; z_\rho \; \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0)\rangle$$

QED kernel function $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$

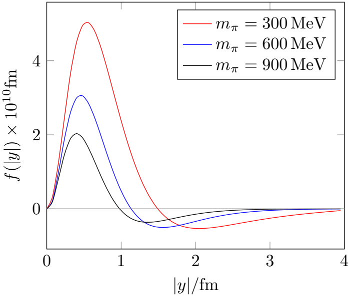
- Weights the QCD four-point function in position-space.
- Tensor decomposition leads to 6 weight functions (and derivatives thereof) that depend on the 3 variables $x^2, y^2, x\cdot y$.
- We have computed these weight functions on a grid to about 5 digits precision, once and for all, and stored on disk.

Tests of QED kernel function: pion-pole contribution with VMD model, lepton-loop in QED, evaluated semi-analytically in position-space in continuum and infinite volume.

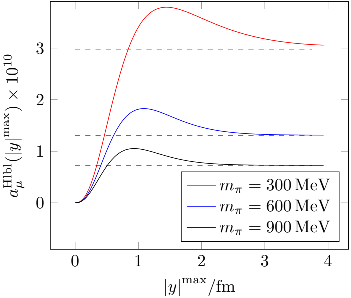
# Numerical tests of QED kernel: (I) Pion-pole contribution to $a_\mu^{\mathrm{HLbL}}$

Result with VMD model for arbitrary pion mass can easily be obtained from 3-dimensional momentum-space representation (Jegerlehner + AN '09).

Integrand after integration over $|x|, \beta$:



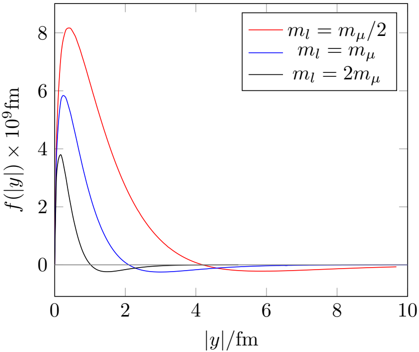

Result for $a_\mu^{\mathrm{HLbL}}(|y|^{\max})$:




- Cutoff for $x$ integration: $|x|^{\max} = 4.05\ \mathrm{fm}$.
- All 6 weight functions contribute to final result, some only at the percent level.
- Agrees at percent level with known results for $m_\pi > 300$ MeV.
- $|x|^{\max}, |y|^{\max} > 4\ \mathrm{fm}$ needed for $m_\pi < 300\ \mathrm{MeV}$.

# Numerical test of QED kernel: (II) Lepton loop contribution $a_\mu^{\rm LbL}$ in QED

Integrand of lepton loop contribution $a_\mu^{\rm LbL}$:



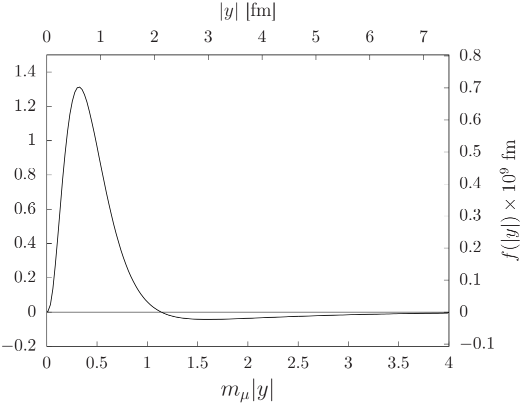

| $m_l/m_\mu$ | $a_\mu^{\rm LbL} \times 10^{11}$ (exact) | $a_\mu^{\rm LbL} \times 10^{11}$ | Precision | Deviation |
|---|---|---|---|---|
| 1/2 | 1229.07 | 1257.5(6.2)(2.4) | 0.5% | 2.3% |
| 1 | 464.97 | 470.6(2.3)(2.1) | 0.7% | 1.2% |
| 2 | 150.31 | 150.4(0.7)(1.7) | 1.2% | 0.06% |

1st uncertainty from 3D integration, 2nd uncertainty from extrapolation to small $|y|$.
Behavior for small $|y|$ compatible with $f(|y|) \propto m_\mu |y| \log^2(m_\mu |y|)$.
Analytical results for $a_\mu^{\rm LbL}$ with $m_l = m_\mu, 2m_\mu$ reproduced at the percent level.
(exact values: Laporta + Remiddi '93, numbers courtesy of Massimo Passera)

New results on HLbL in $g-2$ presented at UConn meeting

## New results by Mainz group at UConn meeting

Integrand of pion-pole contribution in VMD model for physical pion mass (talk by AN):




- Result of integration: $57.9 \times 10^{-11}$. The steep rise to the peak and the long negative tail is not fully captured by the density of points and the extent of the grid where we have evaluated the QED kernel.
- One needs to go to very large values of $|x|$ and $|y|$, i.e. very large lattice volumes $\sim 10$ fm to reproduce known result $57.0 \times 10^{-11}$.

## Challenges in view of the lattice computation (talk by Nils Asmussen)

- Contributions are quite long range $\rightarrow$ finite volume effects
- Integrands peaked at small distances $\rightarrow$ discretization effects

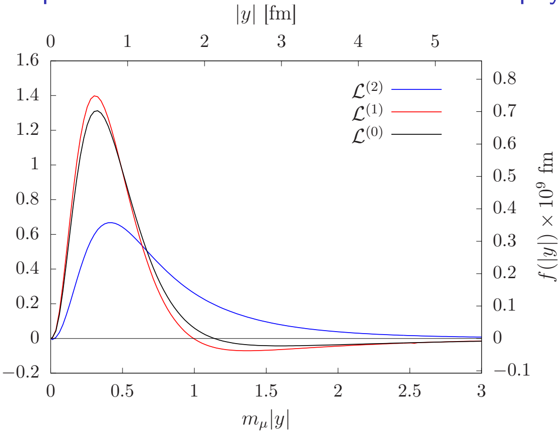
- A way to improve: do subtractions on the QED kernel (first proposed by Blum *et al.* '17).
  Conserved current is a total derivative: $j_\mu(x) = \partial_\nu^{(x)}(x_\mu j_\nu(x))$

$$\rightarrow \int d^4x\, j_\mu(x) = 0 \qquad \text{(in infinite volume and in continuum)}$$

  Therefore we can always subtract terms from QED kernel, that do not depend on both $x$ and $y$ without changing $a_\mu^{\mathrm{HLbL}}$.

- We try (short notation):
  - $\mathcal{L}^{(0)} = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ (standard kernel, $\mathcal{L}^{(0)}(0,0) = 0$)
  - $\mathcal{L}^{(1)} = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) - \frac{1}{2}\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,x) - \frac{1}{2}\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(y,y)$
    $\Rightarrow \mathcal{L}^{(1)}(x,x) = 0$
  - $\mathcal{L}^{(2)} = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) - \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(0,y) - \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,0)$
    $\Rightarrow \mathcal{L}^{(2)}(x,0) = \mathcal{L}^{(2)}(0,y) = 0$

# Integrand of pion-pole contribution with subtractions for physical pion mass




- Integrals for all kernels lead to same $a_\mu^{\text{HLbL}}$ in the continuum and in infinite volume (as they should !).
- Shape of $\mathcal{L}^{(1)}$ similar to standard kernel $\mathcal{L}^{(0)}$, maybe even a bit worse.
- $\mathcal{L}^{(2)}$ better behaved for integration on lattice: peak less pronounced (less steep at short-distances), no negative tail, a bit less long-ranged
  $\rightarrow$ expect reduced discretization effects (lattice artifacts) and reduced finite volume effects in lattice calculation.

# First steps towards the lattice calculation

Finite lattice: $\int_{x,y} \to \sum_{x,y}$

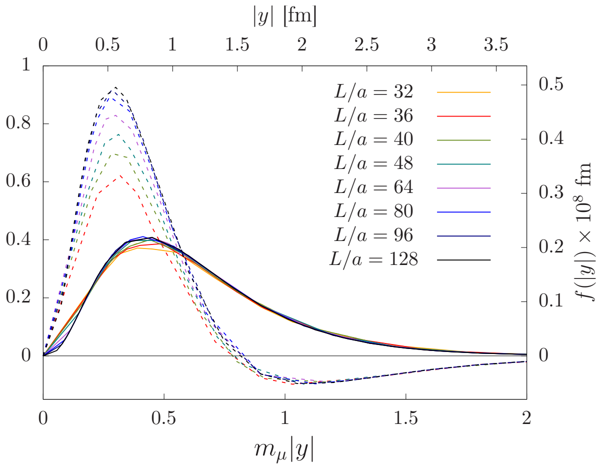
Four-point function $i\hat{\Pi}$: still in continuum, infinite volume for integral over $z$

Master formula:

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} 2\pi^2 \sum_{|y|} a_{|y|} |y|^3 \left[ a^4 \sum_{x\in\Lambda} \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\ i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \right]$$

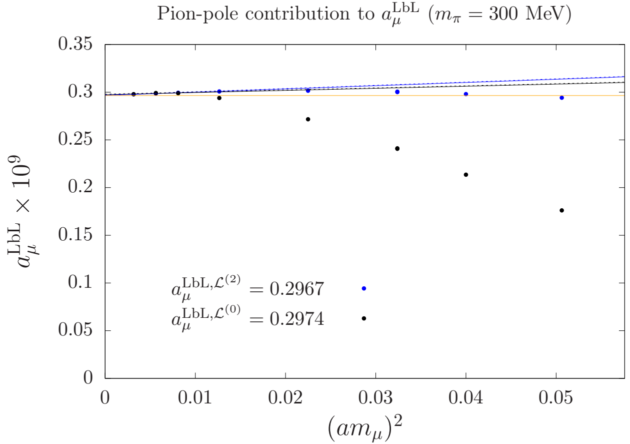
- We can evaluate $\sum_{|y|} a_{|y|}$ on an arbitrary set of $|y|$ and do the integration using e. g. the trapezoidal rule.
- Focus on default kernel $\mathcal{L}^{(0)}$ and kernel $\mathcal{L}^{(2)}$ which seemed to be best.

# Integrand of pion-pole contribution to $a_\mu^{\mathrm{HLbL}}$ for $m_\pi = 300$ MeV




- Dashed line: default kernel $\mathcal{L}^{(0)}$, solid line: subtracted kernel $\mathcal{L}^{(2)}$
- Constant volume $m_\mu L = 7.2$, different lattice spacings $a$
- Less dependence on discretization effects with kernel $\mathcal{L}^{(2)}$

# Continuum extrapolation for pion-pole for $m_\pi = 300$ MeV




- The extrapolation works very well for both kernels.
- The discretization effects with the subtracted kernel $\mathcal{L}^{(2)}$ are smaller.

# Full Lattice QED computation

Master formula:

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} 2\pi^2 \sum_{|y|} a_{|y|} |y|^3 \left[ a^4 \sum_{x\in\Lambda} \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \right]$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -a^4 \sum_{z\in\Lambda} z_\rho \left\langle j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0) \right\rangle$$

- $i\hat{\Pi}$ for lepton loop in lattice QED
- Same code used as for lattice QCD calculation

More details about these tests with Lattice QED and first results for the full Lattice QCD calculation: next talk by Nils Asmussen.

New results on HLbL in $g-2$ by RBC-UKQCD

at UConn meeting

# Discretization effects (talk by Tom Blum)

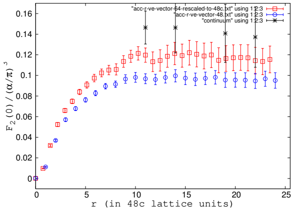
Compared to published result (Blum et al. PRL118 (2017)), consider second ensemble with physical pion mass and finer lattice spacing:

$$a^{-1} = 1.73\ \mathrm{GeV}\ (\mathrm{a} = 0.114\ \mathrm{fm}), \mathrm{L} = 5.47\ \mathrm{fm}\quad (48^3 \times 96\ \mathrm{I})$$

$$a^{-1} = 2.36\ \mathrm{GeV}\ (\mathrm{a} = 0.084\ \mathrm{fm}), \mathrm{L} = 5.38\ \mathrm{fm}\quad (64^3 \times 128\ \mathrm{I})$$

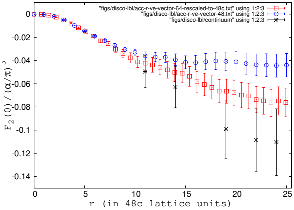
Study continuum limit (connected and leading disconnected):

cHLBL: simple continuum limit (preliminary)




- linear in $a^2 \to 0$
- Collecting more statistics for finer ensemble ($43 \to 65$)

dHLbL: lattice spacing effect (preliminary)




- linear in $a^2 \to 0$
- Effects tend to cancel with connected piece
- Collecting more statistics ($39 \to 65$)

# Discretization effects (talks by Tom Blum and Taku Izubuchi)

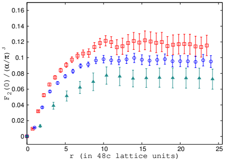
Add new ensembles with physical pion mass, coarser lattice spacing and different gauge action:

$$a^{-1} = 1.0\ \mathrm{GeV}\ (\mathrm{a} = 0.20\ \mathrm{fm}), \mathrm{L} = 4.8\ \mathrm{fm}\quad (24^3 \times 64\ \mathrm{ID})$$

Study lattice spacing effects (connected and leading disconnected):

## cHLbL Different lattice spacings

cHLbL: lattice spacing effect (preliminary)




1/a = 2.37 GeV, 1.73 GeV, 1.0 GeV

- Add new $24^3$, 1 GeV, ID ensemble (green )
- I and ID slightly different, but disc. errors similar
- Collecting more statistics (9 configs)

- Significant increase as $a \to 0$

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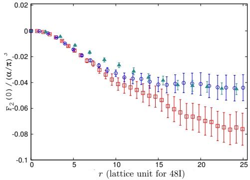
## dHLbL Different lattice spacings

dHLbL contribution: lattice spacing effect (preliminary)




1/a = 2.37 GeV, 1.73 GeV, 1.0 GeV

- Large negative increase tends to cancel connected one
- Collecting more statistics!

16

$\mathcal{O}(a^4)$ effects may be significant.

## Lattice QCD ensembles for the quark loop

**48I** ($48^3 \times 96$), $L = 5.47$ fm, $a^{-1} = 1.730$ GeV, $m_\pi = 139$ MeV
**64I** ($64^3 \times 128$), $L = 5.35$ fm, $a^{-1} = 2.359$ GeV, $m_\pi = 139$ MeV
**24D** ($24^3 \times 64$), $L = 4.67$ fm, $a^{-1} = 1.015$ GeV, $m_\pi = 141$ MeV
**32D** ($32^3 \times 64$), $L = 6.22$ fm, $a^{-1} = 1.015$ GeV, $m_\pi = 141$ MeV
**48D** ($48^3 \times 64$), $L = 9.33$ fm, $a^{-1} = 1.015$ GeV, $m_\pi = 141$ MeV

All of them are chirally symmetric domain-wall configurations with 2+1 flavors.

So far studied: $\mathrm{QED}_L$ on 24D, 48I, and 64I; $\mathrm{QED}_\infty$ on 24D, 48D, and 48I.

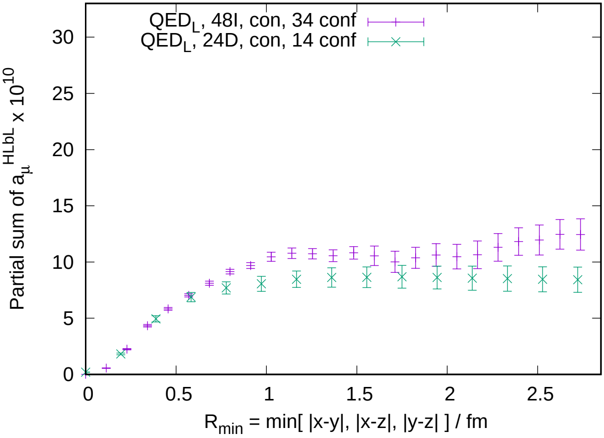
For $\mathrm{QED}_\infty$ for now only show connected diagram since disconnected $\mathrm{QED}_\infty$ analysis is still too premature.

Technical progress to handle large volumes: Multi-Grid Lanczos arXiv:1710.06884

# Finite volume effects (talk by Christoph Lehner) (cont.)

## Study of $\mathrm{QED}_L$ (variable $R_{\min}$ on $x$-axis)

Connected diagram in $\mathrm{QED}_L$ on 24D and 48I lattices:

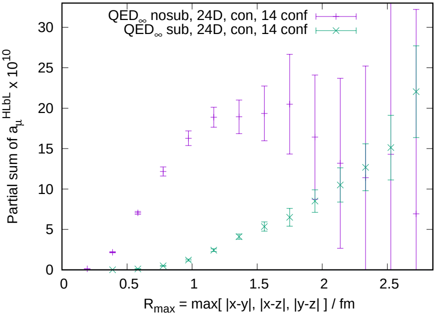



Short-distance: disc errors maybe small, Long-distance: effect of different volume?
($L_{24D} = 4.67$ fm, $L_{48I} = 5.47$ fm)

# Finite volume effects (talk by Christoph Lehner) (cont.)

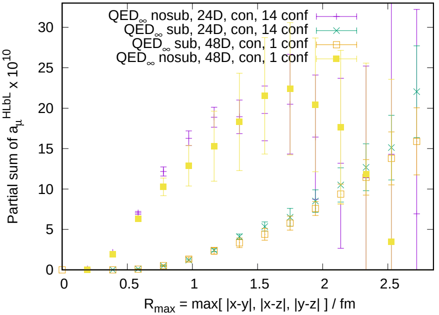
Study of QED$_\infty$: effect of subtraction
(note different variable $R_{\mathrm{max}}$ on $x$-axis compared to QED$_L$ !)

Connected diagram in QED$_\infty$ on 24D:




Subtraction that helped reduce discretization and volume errors in lepton-loop cas
has significant effect on on-set of plateau in this plot. LD noise large.

Connected diagram in QED$_\infty$ on 24D and 48D:

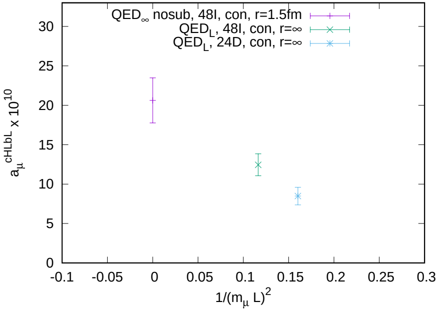



With current poor statistics no sign of QCD FV effect. May be hidden at very long distances? Perform study of long-distance $\pi^0$-pole contribution: talk by N. Christ

# Finite volume effects (talk by Christoph Lehner) (cont.)

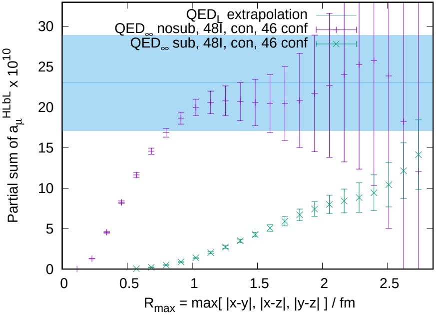
$QED_L$ versus $QED_\infty$: more precise with extrapolation using $QED_L$ !?

Consistency of $QED_L$ and $QED_\infty$:




This plot is preliminary and needs to be refined with a proper continuum limit since 24D and 48I have different lattice cutoff.

Consistency of $QED_L$ and $QED_\infty$:

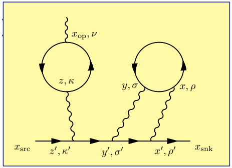


Disconnected quark loop contributions (talk by Taku Izubuchi)

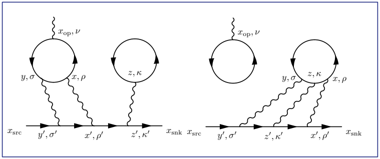
# SU(3) hierarchies for d-HLbL

- At $m_s$=$m_{ud}$ limit, following type of disconnected HLbL diagrams
  $Q_u + Q_d + Q_s = 0$ [Mainz, Lehenr for HVP]
- Other diagrams suppressed by
  O($m_s$-$m_{ud}$) /3, O( $(m_s-m_{ud})^2$ ), and O( $(m_s-m_{ud})^3$ )

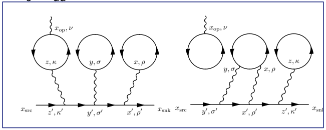
$(m_s-m_{ud})^0$



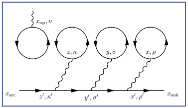

$(m_s-m_{ud})/3$




$(m_s-m_{ud})^2$



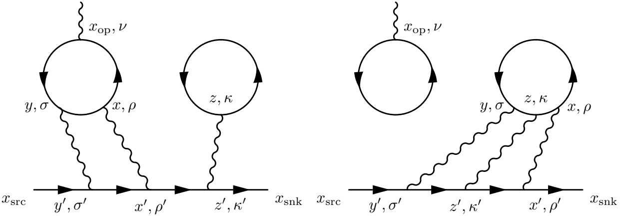

$(m_s-m_{ud})^3$





Results for the leading disconnected contributions have been shown before.

# Remaining dHLbL




- These are the subleading disconnected diagrams in the SU(3) limit.
- The right diagram has a factor of $1/3$ suppression from the multiplicity of the diagram compare with the left diagram, i.e. the external photon is more likely to be on the loop with three photons.
- For the left diagram, the moment method works just like the connected case. With both $\mathrm{QED}_L$ or $\mathrm{QED}_\infty$, we can sample $x, y$ and sum over $z$. We can use the $M^2$ trick for the $x, y$ sampling. Low-modes-averaging for the loop with $z$.
- For the right diagram, The moment method still works, however, we have to use a point on the other loop as the reference point, which may be more noisy. But as mentioned above, the right diagram is more suppressed.

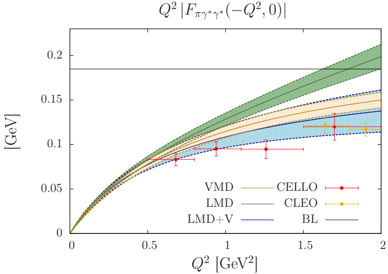
$\pi^0 \to \gamma^*\gamma^*$ transition form factor on the lattice

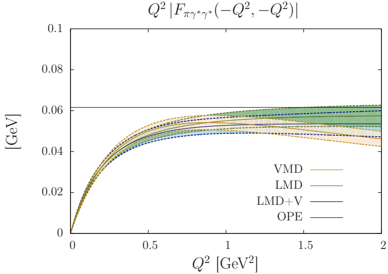
Pion-pole contribution to muon $g-2$ on the lattice

# Mainz: Transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ from Lattice QCD (talk by Antoine Gérardin)

Gérardin, Meyer, AN, PRD 94, 074507 (2016)

Fit lattice data on various $N_f = 2$ CLS ensembles with VMD, LMD, LMD+V models for TFF to perform extrapolation to continuum and physical pion mass.




VMD model: bad fit ($\chi^2/\mathrm{d.o.f.} = 2.9$, uncorrelated global fit), because of wrong high-momentum asymptotics in double-virtual case ($\sim 1/Q^4$).

LMD model ($\chi^2/\mathrm{d.o.f.} = 1.3$, uncorrelated global fit)

$$\alpha^{\mathrm{LMD}} = 0.275(18)(3)\ \mathrm{GeV}^{-1}\,, \beta = -0.028(4)(1)\ \mathrm{GeV}\,, M_V^{\mathrm{LMD}} = 0.705(24)(21)\ \mathrm{GeV}$$

($\alpha^{\mathrm{th}} = 1/(4\pi^2 F_\pi) = 0.274\ \mathrm{GeV}^{-1}$, $\beta^{\mathrm{OPE}} = -F_\pi/3 = -0.0308\ \mathrm{GeV}$)

LMD+V model ($\chi^2/\mathrm{d.o.f.} = 1.4$, uncorrelated global fit)

$$\alpha^{\mathrm{LMD+V}} = 0.273(24)(7)\ \mathrm{GeV}^{-1}, \bar{h}_2 = -11.2(5.4)(2.7)\ \mathrm{GeV}^2, \bar{h}_5 = 8.5(2.9)(1.4)\ \mathrm{GeV}^4$$

($\widetilde{h}_0 = -F_\pi/3 = -0.0308\ \mathrm{GeV}$, $\widetilde{h}_1 = 0$, $M_{V_1} = 0.775\ \mathrm{GeV}$, $M_{V_2} = 1.465\ \mathrm{GeV}$ fixed at physical point)

## Mainz: Pion-pole contribution to $a_\mu^{\mathrm{HLbL};\pi^0}$ from lattice QCD (talk by Antoine Gérardin) (cont.)

Using the LMD+V model from the global fit to the lattice data, we obtain as our preferred estimate:

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11} \quad (\pm 12.8\%)$$

Error from covariance matrix is statistical only.

| Model | $a_\mu^{\mathrm{HLbL};\pi^0} \times 10^{11}$ |
|---|---|
| LMD (lattice fit) | 68.2(7.4) |
| LMD+V (lattice fit) | 65.0(8.3) |
| VMD (theory) | 57.0 |
| LMD (theory) | 73.7 |
| LMD+V (theory + pheno) | 62.9 |

Most model calculations yield results in the range $a_\mu^{\mathrm{HLbL};\pi^0} = (50 - 80) \times 10^{-11}$.

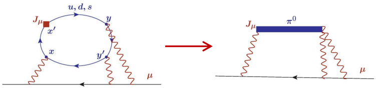
Even though LMD and LMD+V models give almost equally good fits to lattice data, they differ for large momenta, in particular for single-virtual form factor. LMD does not obey Brodsky-Lepage behavior.

Systematic errors: 1. Finite-time extent of the lattice. 2. Finite-size effects (no dedicated study, data suggest rather small effect). 3. Disconnected contributions: below 1% for 1 ensemble ($m_\pi = 440$ MeV).

Preliminary results for $N_f = 2 + 1$ CLS ensembles and with moving pions ($\rightarrow$ access to more values of $q_1^2, q_2^2$) also shown at UConn meeting.
Update will be presented by Antoine Gérardin at this meeting.

# RBC-UKQCD: long-distance contribution to HLbL in $g-2$ in position space from $\pi^0$-pole (talk by Norman Christ)

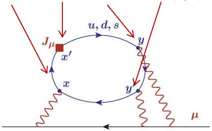
- Long distance contribution to HLbL comes from $\pi^0$ exchange:




- Calculate $\pi^0$ exchange from lattice QCD
  - Direct calculation, without form factor decomposition or parametrization.
  - Position-space based approach.
  - $\pi^0$ contribution well defined at long distances in position-space.
  - Can be applied for large volume. Fixed volume QCD calculation gives $\pi^0$ HLbL contribution in increasing volume.

Compute $\pi^0$ pole contribution by inserting sum over $\pi^0$ states in 4-point function (assuming that $x$ and $y$ are far separated in time direction):

$$\mathcal{A}_{\mu\mu'\nu\nu'}(x, x', y, y') = \langle 0|T(J_\mu(x)J_{\mu'}(x')J_\nu(y)J_{\nu'}(y'))|0\rangle$$




$$\mathcal{A}^{\pi^0}_{\mu\mu'\nu\nu'}(x, x', y, y') = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_\pi(p)} \langle 0|\,T\left(J_\mu(x)J_{\mu'}(x')\right)|\pi^0(\vec{p})\rangle\langle\pi^0(\vec{p})|\,T\left(J_\nu(y)J'_\nu(y')\right)|0\rangle$$

## RBC-UKQCD: long-distance contribution to HLbL in $g-2$ in position space from $\pi^0$-pole (talk by Norman Christ) (cont.)

Using translation invariance one can rewrite vertex function:

$$\begin{aligned} \langle 0|\, T\left(J_\mu(x) J_{\mu'}(x')\right)|\pi^0(\vec{p}) &= \langle 0|\, T\left(J_\mu(0) J_{\mu'}(\tilde{x})\right)|\pi^0(\vec{p})\rangle\, e^{i\vec{p}\cdot\vec{x} - E_p x_0} \\ &= \mathcal{F}_{\mu\mu'}(\tilde{x}, \vec{p})\, e^{i\vec{p}\cdot\vec{x} - E_p x_0} \\ &= \mathcal{F}_{\mu\mu'}(\tilde{x}, -i\vec{\nabla}_x)\, e^{i\vec{p}\cdot\vec{x} - E_p x_0} \end{aligned}$$

One can then perform integral over $\vec{p}$ in sum over $\pi^0$ states and with a further long-distance approximation one obtains:

$$\mathcal{A}^{\pi^0}_{\mu\mu'\nu\nu'}(x, x', y, y') = \mathcal{F}_{\mu\mu'}(\tilde{x}, iM_\pi \hat{n})\, \mathcal{F}_{\nu\nu'}(\tilde{y}, -iM_\pi \hat{n})\, \Delta_F(x-y, M_\pi)$$

where $\hat{n} = (\vec{x} - \vec{y})/|x-y|$ and $\Delta_F$ is Euclidean pion propagator in position-space.

On the other hand, the amplitude $\mathcal{F}_{\mu\mu'}(\tilde{x}, iM_\pi \hat{n})$ also appears in simpler 3-point function that involves the same $\gamma\gamma - \pi^0$ vertex as the 4-point function:

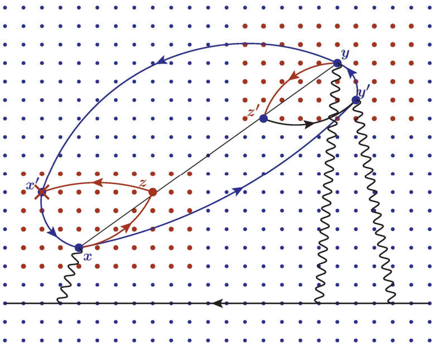
$$\begin{aligned} \mathcal{B}_{\mu\mu'}(x, x', z) &= \langle 0|\, T\left(J_\mu(x) J_{\mu'}(x') \pi^0(z)\right)|0\rangle \\ \mathcal{B}^{\pi^0}_{\mu\mu'}(x, x', z) &= \mathcal{F}_{\mu\mu'}(\tilde{x}, iM_\pi \hat{n})\, Z^{1/2}_{\pi^0}\, \Delta_F(x-z, M_\pi) \\ Z^{1/2}_{\pi^0} &= \langle \pi^0(\vec{p}=0)|\pi^0(0)|0\rangle \end{aligned}$$

Combining the results:

$$\mathcal{A}^{\pi^0}_{\mu\mu'\nu\nu'}(x, x', y, y') = \mathcal{B}^{\pi^0}_{\mu\mu'}(x, x', z)\, \mathcal{B}^{\pi^0}_{\nu\nu'}(y, y', z)\, \frac{1}{Z_{\pi^0}}\, \frac{\Delta_F(x-y, M_\pi)}{\Delta_F(x-z, M_\pi)\Delta_F(z'-y, M_\pi)}$$

# RBC-UKQCD: long-distance contribution to HLbL in $g-2$ in position space from $\pi^0$-pole (talk by Norman Christ) (cont.)

## Lattice implementation




HLbL-UConn - 3/13/2018 (8)

- For $|x-y| \geq R_{\min}$ replace $\mathcal{A}_{\mu\mu'\nu\nu'}(x, x', y, y')$ with $\mathcal{A}^{\pi^0}_{\mu\mu'\nu\nu'}(x, x', y, y')$ to use the $\pi^0$ contribution at long distances.
- Should allow the large volume systematic error to be reduced.
- No explicit implementation by time of UConn meeting.

# RBC-UKQCD: Pion transition form factor from lattice QCD in position space (talk by Cheng Tu)

TFF in momentum space:

$$\int d^4u\, e^{-iq_1\cdot u - iq_2\cdot v}\langle 0|T\{iJ_\mu(u)\, iJ_\nu(v)\}|\pi^0(p)\rangle = \frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}q_{1,\rho}q_{2,\sigma}F(q_1^2, q_2^2)$$

In position space:

$$\begin{aligned}\langle 0|T\{iJ_\mu(u)\, iJ_\nu(v)\}|\pi^0(p)\rangle &= \frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}\int_0^1 dx\left[-\partial_\rho^u F_c(x,(u-v)^2)\right] ip_\sigma e^{ip\cdot(xu+(1-x)v)} \\ F_c(x,(u-v)^2) &= 0 \quad \text{if } x<0 \text{ or } x>1\end{aligned}$$

For $v=0, p=q_1+q_2$ we get the following mapping:

$$F(q_1^2, q_2^2) = \int d^4u\, e^{-iq_1\cdot u}\int_0^\infty dx\, F_c(x,u^2)\, e^{ixp\cdot u}$$

In chiral limit and for $\vec{p}=0, q_1=0$ and using the normalization $F(q_1^2\to 0, q_2^2\to 0)=1$ we have:

$$\int d^4u\int_0^1 dx\, F_c(x,u^2) = 1$$

$F_c(x,(u-v)^2)$: extract leading singular behavior from OPE when $u\to v$ (Gérardin et al. '16) and write remaining regular dependence on $|u-v|$ in Fourier series in $0\le x\le 1$ (maybe better to use expansion in Gegenbauer polynomials, cf. pion distribution amplitude (Luchang Jin (private communication), see Bali et al. '17)):

$$-\partial_\rho^u F_c(x,(u-v)^2) = 2(u-v)_\rho\left[\frac{F_\pi^2}{3}\frac{1}{((u-v)^2)^2}\right]\sum_{n=0}^\infty f_n(|u-v|)\frac{(2n+1)\pi}{2}\sin((2n+1)\pi x)$$

# RBC-UKQCD: Pion transition form factor from lattice QCD in position space (talk by Cheng Tu)

Study r dependence of position-space function, define $f(|r|)$:

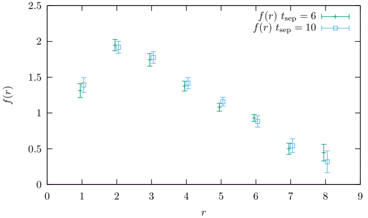
$$\int_0^1 dx\,\left[-\partial_\rho^u F_c(x,r^2)\right] = 2r_\rho\left[\frac{F_\pi^2}{3}\frac{1}{(r^2)^2}\right] f(|r|),\quad f(|r|)=\sum_{n=0}^{\infty} f_n(|r|)$$

$$\langle 0|T\{\,iJ_\mu(0,\vec{r}/2)\,iJ_\nu(0,-\vec{r}/2)\}\,|\pi^0(\vec{p}=0)\rangle = \frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}2r_\rho ip_\sigma\left[\frac{F_\pi^2}{3}\frac{1}{(r^2)^2}\right] f(|r|)$$



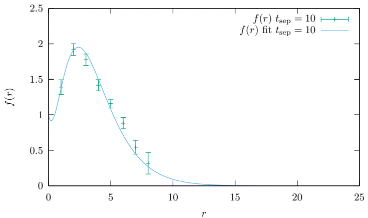
## Plots

24c64 Lattice, 1024 Point Source Propagator








Fitting Formular: $f(r)=(c_0+c_2r^2)e^{-0.77r}$, and the fitted parameters are:
$c_0 = 0.9871 \pm 0.2258$
$c_2 = 1.9816 \pm 0.0430$
based on 24c lattice, 16 configurations.



Result for integral with fitted function:

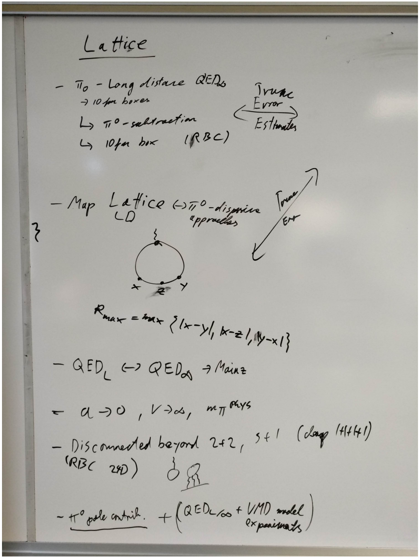
$$\frac{\pi^2}{2}\int_0^\infty \frac{F_\pi^2}{3} f_{\rm fit}(r)2rdr = 0.9965\pm 0.017 \qquad \text{(normalization consistent with 1)}$$

# Conclusions, Tasks for Mainz workshop

## Conclusions

- Lot of progress on conceptual and technical side achieved for HLbL on the lattice in last 3-4 years by RBC-UKQCD and Mainz group.
- RBC-UKQCD: Many results already obtained at physical pion mass, finite lattice spacing, finite volume, fully connected and leading disconnected diagrams.
- Mainz: semi-analytical approach for QED kernel in continuum and infinite volume ($QED_\infty$) to have full control over non-QCD part of HLbL. First results for QCD calculation of fully connected diagram (next talk by Nils Asmussen).
- Observed by both groups: Importance to better understand pion-pole contribution by direct lattice calculation of transition form factor or directly in 4-point function to control finite-volume and long-distance effects.

## Tasks for Mainz workshop




(From final discussion at UConn, Christoph Lehner)

# Outlook

Timeframes: End of 2018 (Whitepaper) and final result by Fermilab $g-2$ experiment in 2-3 years

- RBC-UKQCD: first number for $a_\mu^{\mathrm{HLbL}}$ with physical pion mass, extrapolated to continuum and infinite volume, with some control over systematics seems very likely by end of 2018 (Whitepaper).

- Mainz: try to cross-check these numbers, at least for fully connected contribution and extrapolated to physical pion mass, with completely different approach and different lattice action.

- More consolidated number of $a_\mu^{\mathrm{HLbL}}$ with 10% uncertainty (with controlled systematics by both collaborations and hopefully other lattice groups !) seems within reach by the time the Fermilab experiment publishes its final result.