HLbL contribution to the muon g-2 on the lattice
Mainz results

Nils Asmussen

in Collaboration with
Antoine Gérardin, Harvey Meyer, Andreas Nyffeler

Institut für Kernphysik, Johannes Gutenberg-Universität Mainz

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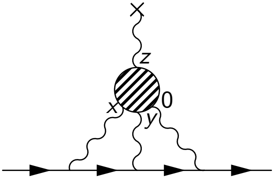
Euclidean position-space approach to a_μ^{HLbL}



master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int d^4y \Big[\int d^4x \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\mathrm{QED}} \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\mathrm{QCD}} \Big].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no power-law finite-volume effects from the photons
- manifest Lorentz covariance

Stages of the Computation

tests of the QED kernel

- continuum and infinite volume
- π^0 pole and lepton loop
- test different choices for the QED kernel

tests of the lattice gauge theory code

- Lattice QED
- compare to lepton loop results

Lattice QCD

- first results for the fully connected contribution
- study pion mass dependence and discretisation/finite volume effects

Outline

1. Steps towards the lattice computation
2. Tests using Lattice QED
3. Lattice QCD
4. Conclusion

Continuum, Infinite Volume

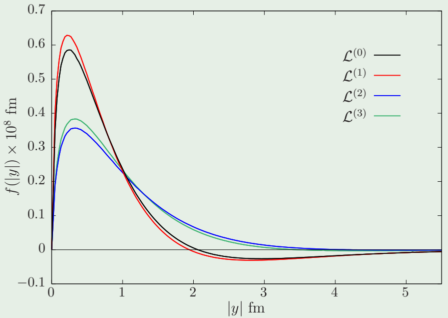
master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} 8\pi^3 \int_0^\infty d|y||y|^3 \left[\int_0^\infty d|x||x|^3 \int_0^\pi d\beta \sin^2\beta \, \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \right].$$

subtractions on the kernel

- first proposed by Blum *et al.* '17
- we try (short notation):
  $\mathcal{L}^{(0)} = \bar{\mathcal{L}}(x,y)$ (standard kernel)
  $\mathcal{L}^{(1)} = \bar{\mathcal{L}}(x,y) - \frac{1}{2}\bar{\mathcal{L}}(x,x) - \frac{1}{2}\bar{\mathcal{L}}(y,y)$
  $\mathcal{L}^{(2)} = \bar{\mathcal{L}}(x,y) - \bar{\mathcal{L}}(0,y) - \bar{\mathcal{L}}(x,0)$
  $\mathcal{L}^{(3)} = \bar{\mathcal{L}}(x,y) - \bar{\mathcal{L}}(0,y) - \bar{\mathcal{L}}(x,x) + \bar{\mathcal{L}}(0,x)$
- $\mathcal{L}^{(0)}(0,0) = 0$
  $\mathcal{L}^{(1)}(x,x) = 0$
  $\mathcal{L}^{(2)}(0,y) = \mathcal{L}^{(2)}(x,0) = 0$
  $\mathcal{L}^{(3)}(x,x) = \mathcal{L}^{(3)}(0,y) = 0$

y integrand lepton loop $m_l = m_\mu$



- with all kernels $\mathcal{L}^{(0,1,2,3)}$ we can reproduce the known result
- we expect $\mathcal{L}^{(2,3)}$ to be advantageous on the Lattice

Integrand: Continuum/Infinite Volume
Integration: Summation over Hypercubic Lattice

discretizing the integration

$$\int \mathrm{d}x\, f(x) \to a \sum_{-(N/2-1)}^{N/2} f(x)$$

- a: lattice spacing
- N: number of lattice points
- $L = aN$: lattice extent

goal

- resemble the situation on the Lattice
- study artifacts due to the discrete integration

Integrand: Continuum/Infinite Volume
Integration: Summation over Hypercubic Lattice

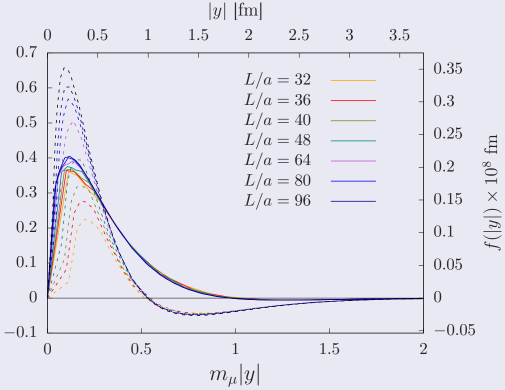
master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} 2\pi^2 \sum_{|y|} a_{|y|} |y|^3 \left[a^4 \sum_{x\in\Lambda} \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \right].$$

- we can evaluate $\sum_{|y|} a_{|y|}$ on an arbitrary set of $|y|$ and do the integration using e. g. the trapezoidal rule
- integrand is still in the continuum
- focus on standard kernel $\mathcal{L}^{(0)}$ and subtracted kernel $\mathcal{L}^{(2)}$

Integrand: Continuum/Infinite Volume
Integration: Summation over Hypercubic Lattice

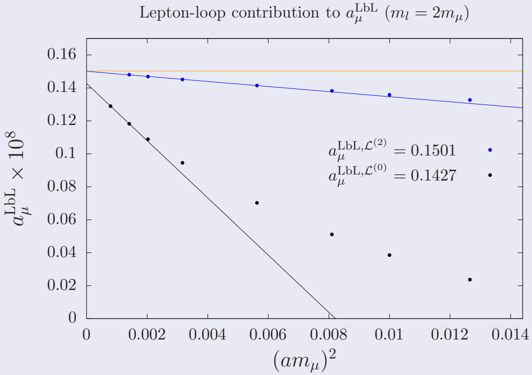
y integrand lepton loop $m_l = 2m_\mu$



- dashed line: standard kernel $\mathcal{L}^{(0)}$, solid line: $\mathcal{L}^{(2)}$
- constant volume $m_\mu L = 7.2$, different lattice spacings a
- less dependence on discretisation effects with kernel $\mathcal{L}^{(2)}$

Integrand: Continuum/Infinite Volume
Integration: Summation over Hypercubic Lattice

continuum extrapolation lepton loop



- standard kernel $\mathcal{L}^{(0)}$ (black curve), subtracted kernel $\mathcal{L}^{(2)}$ (blue curve)
- the extrapolation is easier for $\mathcal{L}^{(2)}$

Lattice QED Computation

master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} 2\pi^2 \sum_{|y|} a_{|y|} |y|^3 \left[a^4 \sum_{x\in\Lambda} \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \right].$$

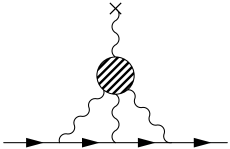
$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -a^4 \sum_{z\in\Lambda} z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle.$$

- $i\hat{\Pi}$ in Lattice QED

goal

- reproduce known lepton loop result
- validate Lattice QCD code

Lattice QED Computation with Wilson Fermions



lattice gauge theory

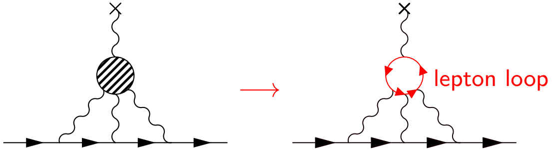
$$\langle O\rangle = \frac{1}{Z}\int \mathcal{D}[U]e^{-S_G[U]}\mathcal{D}[\psi,\bar\psi]e^{-S_F[\psi,\bar\psi,U]}O[\psi,\bar\psi,U]$$

$$S_G[U] = \beta\sum_{n\in\Lambda}\sum_{\mu<\nu}\mathrm{Re}\,\mathrm{Tr}[1-U_{\mu\nu}(n)]$$

$$S_F[\psi,\bar\psi,U] = \text{Wilson fermions}$$

- local vector currents: $j^l_\lambda(x) = \bar q_x \gamma_\lambda q_x$
- conserved vector currents: $j^c_\lambda(x) = \frac{1}{2}\left(\bar q_{x+\hat\lambda}(\gamma_\lambda+1)U^\dagger_{\lambda,x}q_x + \bar q_x(\gamma_\lambda-1)U_{\lambda,x}q_{x+\hat\lambda}\right)$

Lattice QED Computation with Wilson Fermions



lattice gauge theory

$$\langle O \rangle = \frac{1}{Z} \int \cancel{\mathcal{D}[U] e^{-S_G[U]}} \mathcal{D}[\psi, \bar{\psi}] e^{-S_F[\psi, \bar{\psi}, U]} O[\psi, \bar{\psi}, U]$$

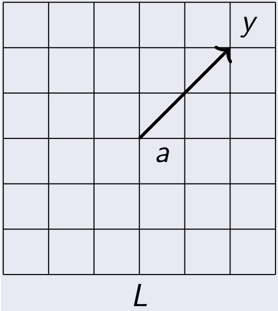
$$S_G[U] = \beta \sum_{n \in \Lambda} \sum_{\mu < \nu} \mathrm{Re}\,\mathrm{Tr}[1 - \cancelto{1}{U_{\mu\nu}(n)}] = 0$$

$$S_F[\psi, \bar{\psi}, U] = \text{Wilson fermions}$$

- QED leading order
- local vector currents: $j^l_\lambda(x) = \bar{q}_x \gamma_\lambda q_x$
- conserved vector currents:
  $j^c_\lambda(x) = \frac{1}{2} \left(\bar{q}_{x+\hat{\lambda}} (\gamma_\lambda + 1) U^\dagger_{\lambda,x} q_x + \bar{q}_x (\gamma_\lambda - 1) U_{\lambda,x} q_{x+\hat{\lambda}} \right)$

Lattice QED Computation with Wilson Fermions

Lorentz covariance allows to choose the direction of y freely

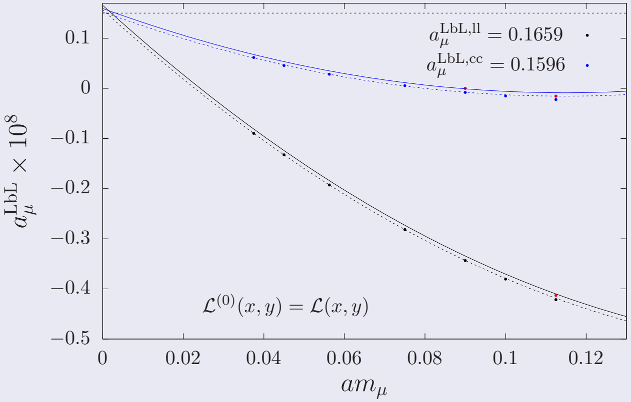


preferred choice $y = (i, i, i, i)$ (lattice diagonal)

- reduced volume effects (distance from border larger than for other choices)
- reduced discretization effects

Lattice QED Computation with Wilson Fermions

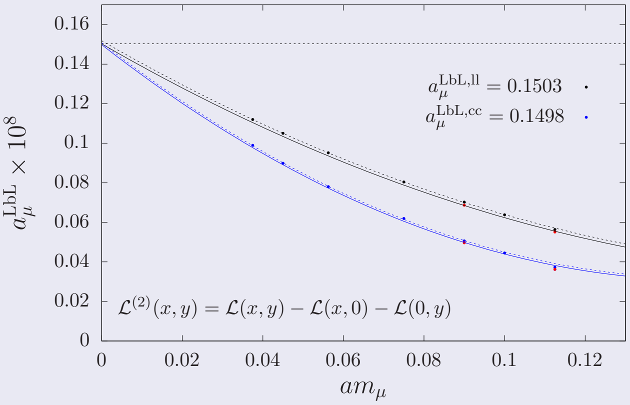
continuum extrapolation lepton loop ($m_l = 2m_\mu$) $\mathcal{L}^{(0)}$



- dashed line: continuum extrapolation for $m_\mu = 7.2$ using a quadratic fit
- solid line: volume extrapolation: curve shifted by the difference between the results for lattice extents $m_\mu L = 7.2$ and 14.4 at fixed a

Lattice QED Computation with Wilson Fermions

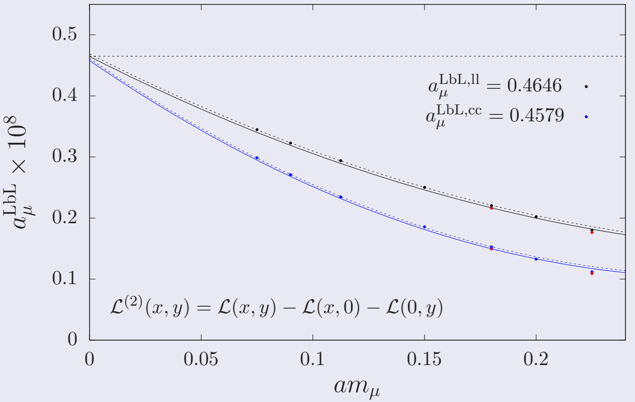
continuum extrapolation lepton loop ($m_l = 2m_\mu$) $\mathcal{L}^{(2)}$



- less discretisation effects
- it is advantageous to use the subtracted kernel $\mathcal{L}^{(2)}$

Lattice QED Computation with Wilson Fermions

continuum extrapolation lepton loop ($m_l = m_\mu$) $\mathcal{L}^{(2)}$



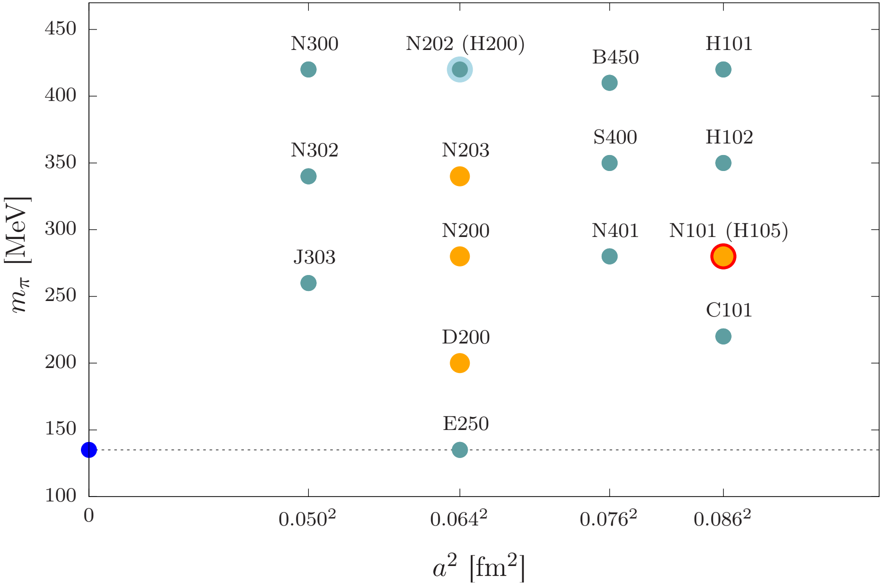
Lattice Setup

CLS $N_f = 2 + 1$ ensembles

CLS	$L^3 \times T$	a [fm]	m_π [MeV]	$m_\pi L$	L [fm]	#confs
H105	$32^3 \times 96$	0.086	285	3.9	2.7	1000
N101	$48^3 \times 128$		285	5.9	4.1	400
N203	$48^3 \times 128$	0.064	340	5.4	3.1	750
N200	$48^3 \times 128$		285	4.4	3.1	800
D200	$64^3 \times 128$		200	4.2	4.2	1100

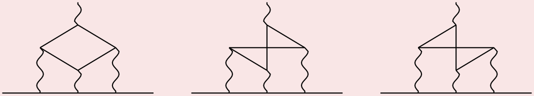
- $\mathcal{O}(a)$ improved Wilson fermions

$N_f = 2 + 1$ CLS Ensembles



Lattice Setup

Old

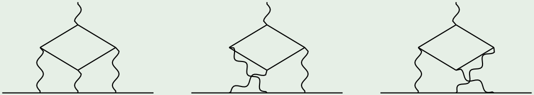


computational cost

- (1+N) forward propagators
- 6(1+N) sequential prop.

$$\int_{y,x,z} \mathcal{L}(x,y) z [\Pi(x,y,z) + \Pi(y,x,z) + \Pi(x-y,-y,z-y)]$$

New



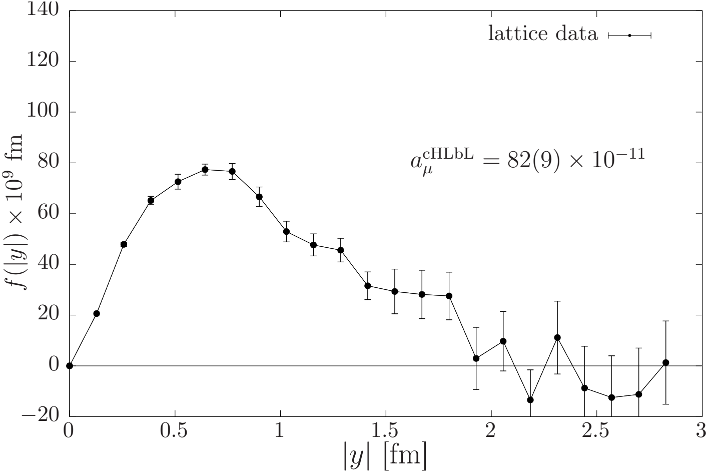
computational cost

- (1+N) forward propagators

$$\int_{y,x,z} \Big([\mathcal{L}(x,y) + \mathcal{L}(y,x) + \mathcal{L}(x-y,-y)] z \Pi(x,y,z) \;-\; \mathcal{L}(x-y,-y) y \Pi(x,y,z) \Big)$$

- If the 1-dim. integral over $|y|$ is done with N evaluations of the integrand.
- we sum over x and z explicitly over the whole lattice

Integrand of a_μ^{cHLbL} with $\mathcal{L}^{(2)}$, $m_\pi = 340\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$



- fully connected contribution only
- we already observe a good signal
- integrand non-zero up to 2 fm

Comparison with Pseudoscalar Pole Contribution

- we computed the fully connected contribution
- $N_f = 2+1$, up, down mass degenerate + strange

comparison with pseudoscalar pole contribution

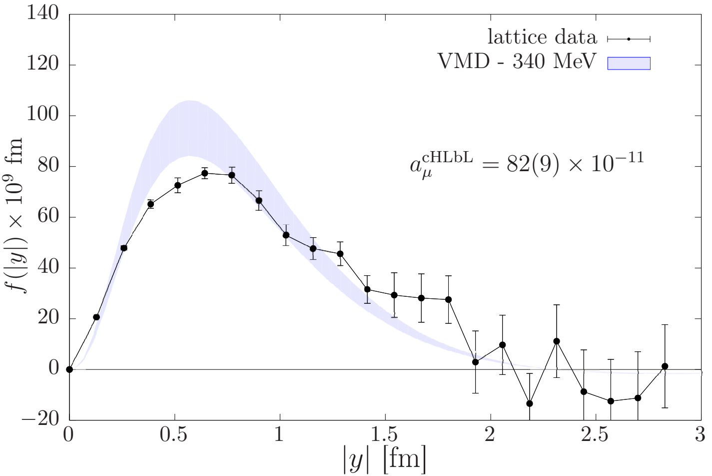
$N_f = 2:$ fully connected contr. $\approx \frac{34}{9}\Pi^{\text{HLbL,isovector}}$ Bijnens '16

$N_f = 3:$ fully connected contr. $\approx 3\Pi^{\text{HLbL,octet}}$ Gérardin *et al.* '17

The π^0 pole in the VMD model provides an estimate of Π^{HLbL} for the non-singlet contribution

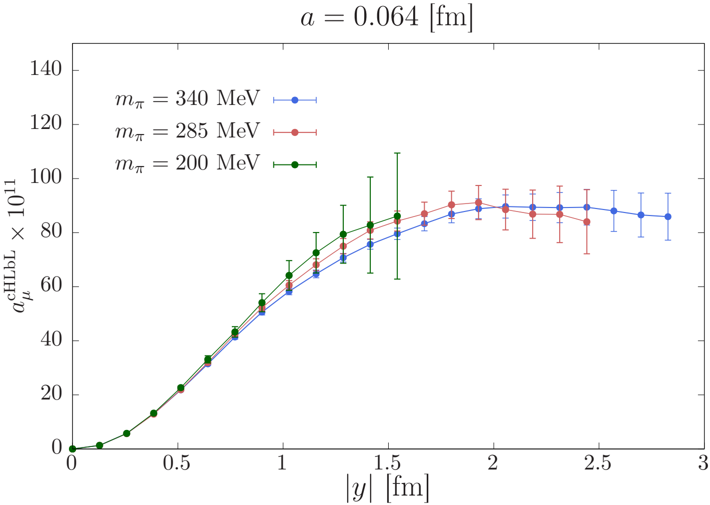
- $N_f = 2+1$ value is expected to lie between the $N_f = 2$ and $N_f = 3$ estimates

Integrand of a_μ^{cHLbL} with $\mathcal{L}^{(2)}$, $m_\pi = 340\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$



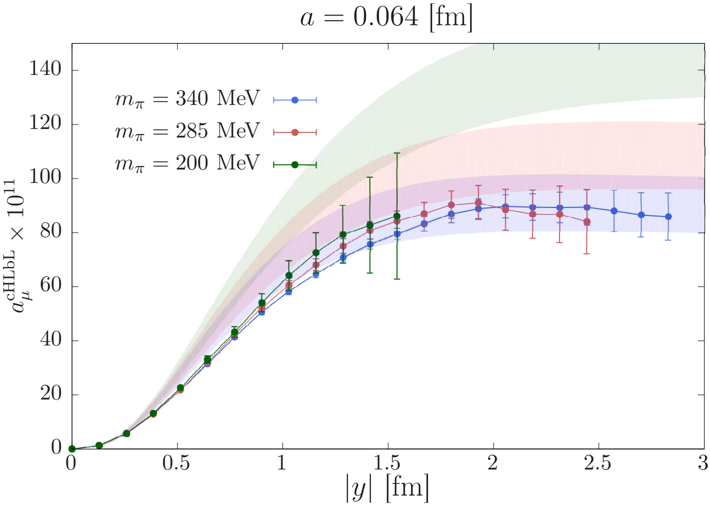
- for long distances the simple VMD Model seems to provide a good approximation to the full QCD computation
- the size of the box $L = 3.1\,\mathrm{fm}$ is large enough to capture the HLbL contribution for this pion mass

Pion Mass Dependence of a_μ^{cHLbL}



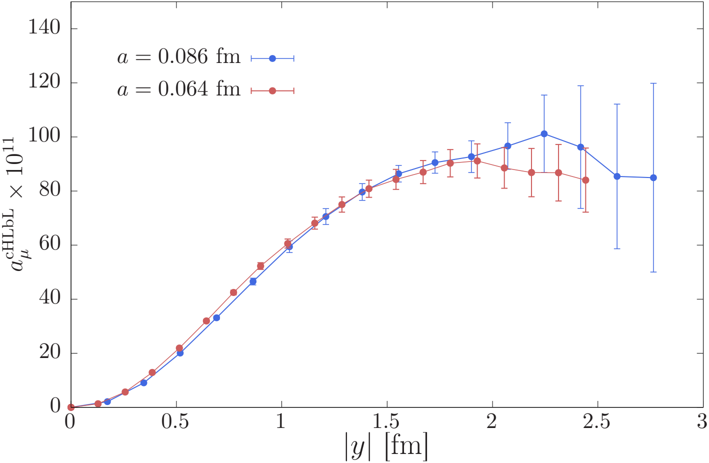
- the results show an upward trend for decreasing pion mass
- currently collecting more statistics in long distance regime

Pion Mass Dependence of a_μ^{cHLbL}



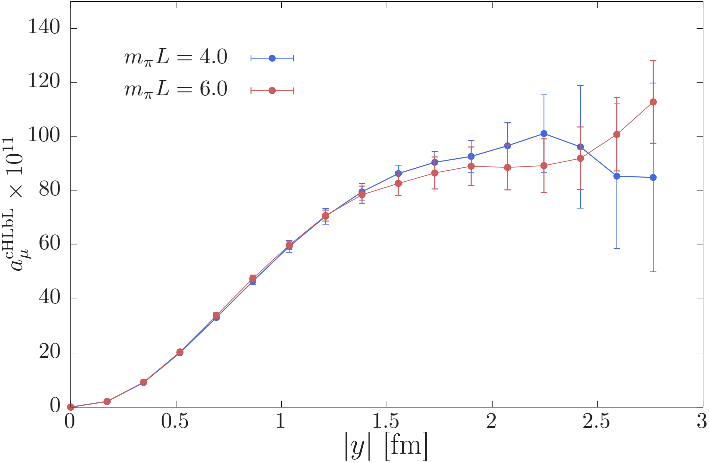
- the results show an upward trend for decreasing pion mass
- currently collecting more statistics in long distance regime

Discretisation Effects, $m_\pi = 285\,\mathrm{MeV}$



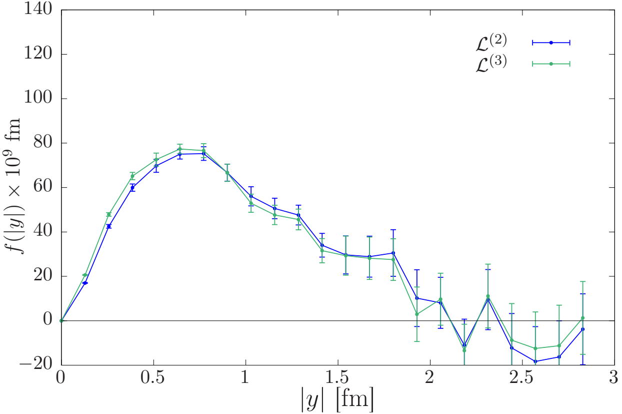
- discretisation effects seem to be small (we are increasing statistics)

Finite Size Effects, $a = 0.086\,\mathrm{fm}$



- finite size effects seem to be small (we are increasing statistics)

Integrand: Effect of Different Subtractions



- $m_\pi = 340\,\mathrm{MeV}$
- $a = 0.064\,\mathrm{fm}$
- $L = 3.1\,\mathrm{fm}$

- The subtracted kernels $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(3)}$ both work well

Conclusions

- Explicit formula for a_μ^{HLbL}
  - QED kernel function multiplying the position-space QCD correlation function
- Tests
  - QED kernel: reproduce known results for π^0 pole and lepton loop in the continuum for the standard kernel $\mathcal{L}^{(0)}$ and subtracted kernels $\mathcal{L}^{(1,2,3)}$
  - Lattice implementation: Reproduce lepton loop result in Lattice QED
- Lattice QCD
  - First Mainz results for the fully connected contribution (in QED_∞)
  - Subtractions are needed to obtain a signal at long distances
  - The discretisation and finite-size effects seem to be small
- Future
  - We are collecting more statistics
  - Study finite-volume effects, see talks by Harvey Meyer and Antoine Gérardin
  - Perform chiral and continuum extrapolations
  - Implement disconnected contribution

Explicit form of the QED kernel

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \langle \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon},x,y)\rangle_{\hat{\epsilon}} = \sum_{A=\mathrm{I,II,III}} \mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda}\, T^{(A)}_{\alpha\beta\delta}(x,y),$$

with e. g. $\mathcal{G}^{\mathrm{I}}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} \equiv \frac{1}{8}\mathrm{Tr}\Big\{\Big(\gamma_\delta[\gamma_\rho,\gamma_\sigma] + 2(\delta_{\delta\sigma}\gamma_\rho - \delta_{\delta\rho}\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\lambda\Big\},$

$$\begin{aligned}
T^{(\mathrm{I})}_{\alpha\beta\delta}(x,y) &= \partial^{(x)}_\alpha(\partial^{(x)}_\beta + \partial^{(y)}_\beta)V_\delta(x,y),\\
T^{(\mathrm{II})}_{\alpha\beta\delta}(x,y) &= m\partial^{(x)}_\alpha\Big(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y)\Big)\\
T^{(\mathrm{III})}_{\alpha\beta\delta}(x,y) &= m(\partial^{(x)}_\beta + \partial^{(y)}_\beta)\Big(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y)\Big),
\end{aligned}$$

$$\begin{aligned}
S(x,y) &= \langle\mathcal{I}\rangle_{\hat{\epsilon}} = \bar{\mathfrak{g}}^{(0)}(|x|,\hat{x}\cdot\hat{y},|y|),\\
V_\delta(x,y) &= \langle\hat{\epsilon}_\delta\mathcal{I}\rangle_{\hat{\epsilon}} = x_\delta\bar{\mathfrak{g}}^{(1)}(|x|,\hat{x}\cdot\hat{y},|y|) + y_\delta\bar{\mathfrak{g}}^{(2)}(|x|,\hat{x}\cdot\hat{y},|y|),\\
T_{\alpha\beta}(x,y) &= \langle(\hat{\epsilon}_\delta\hat{\epsilon}_\beta - \frac{1}{4}\delta_{\delta\beta})\mathcal{I}\rangle_{\hat{\epsilon}}\\
&= (x_\alpha x_\beta - \frac{x^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(1)} + (y_\alpha y_\beta - \frac{y^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(2)} + (x_\alpha y_\beta + y_\alpha x_\beta - \frac{x\cdot y}{2}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(3)}.
\end{aligned}$$

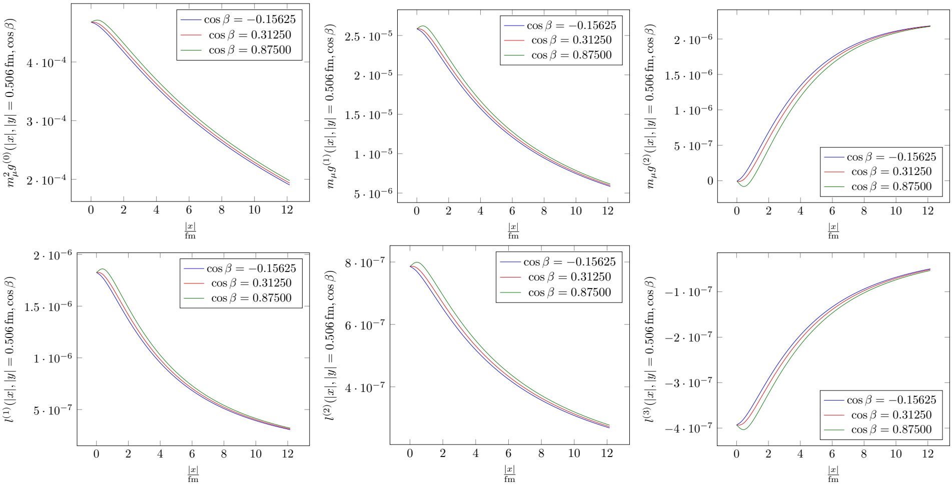
The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is parametrized by six weight functions.

Example: Weight Function $g^{(2)}$

$$g^{(2)}(x^2, x\cdot y, y^2) = \frac{1}{8\pi y^2 |x| \sin^3\beta} \int_0^\infty du\, u^2 \int_0^\pi d\phi_1$$
$$\left\{ 2\sin\beta + \left(\frac{y^2+u^2}{2|u||y|} - \cos\beta\cos\phi_1 \right) \frac{\log\chi}{\sin\phi_1} \right\} \sum_{n=0}^{\infty}$$
$$\Big\{ z_n(|u|) z_{n+1}(|x-u|) \left[|x-u|\cos\phi_1 \frac{U_n}{n+1} + (|u|\cos\phi_1 - |x|) \frac{U_{n+1}}{n+2} \right]$$
$$+ z_{n+1}(|u|) z_n(|x-u|) \left[(|u|\cos\phi_1 - |x|) \frac{U_n}{n+1} + |x-u|\cos\phi_1 \frac{U_{n+1}}{n+2} \right] \Big\}$$

where

$$x\cdot y = |x||y|\cos\beta\,, \quad |x-u| = \sqrt{|x|^2 + |u|^2 - 2|x||u|\cos\phi_1}$$
$$\chi = \frac{y^2+u^2-2|u||y|\cos(\beta-\phi)}{y^2+u^2-2|u||y|\cos(\beta+\phi)}\,, \quad U_n = U_n\left(\frac{|x|\cos\phi_1 - |u|}{|u-x|} \right)$$
$$z_n = \text{linear combination of products of two modified Bessel functions.}$$



$\bar{\mathfrak{g}}^{(0)}(|x|, \hat{x} \cdot \hat{y}, |y|)$ contains an arbitrary additive constant (due to the IR divergence in $I(\hat{\epsilon}, x, y)$), which does not contribute to $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y)$.

The π^0 pole contribution

Assume a vector-meson-dominance transition form factor (parameters: m_V, m_π and overall normalization)

$$\mathcal{F}(-q_1^2, -q_2^2) = \frac{c}{(q_1^2 + m_V^2)(q_2^2 + m_V^2)}, \qquad c = -\frac{N_c m_V^4}{12\pi^2 F_\pi}.$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = \frac{c^2}{m_V^2(m_V^2 - m_\pi^2)} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial y_\beta} \Big\{ \epsilon_{\mu\nu\alpha\beta}\epsilon_{\sigma\lambda\rho\gamma}\Big(\frac{\partial}{\partial x_\gamma} + \frac{\partial}{\partial y_\gamma}\Big) K_\pi(x,y)$$
$$+\epsilon_{\mu\lambda\alpha\beta}\epsilon_{\nu\sigma\gamma\rho}\frac{\partial}{\partial y_\gamma} K_\pi(y-x,y) + \epsilon_{\mu\sigma\alpha\rho}\epsilon_{\nu\lambda\beta\gamma}\frac{\partial}{\partial x_\gamma} K_\pi(x,x-y)\Big\}.$$

where

$$K_\pi(x,y) \equiv \int d^4u \Big(G_{m_\pi}(u) - G_{m_V}(u) \Big) G_{m_V}(x-u) G_{m_V}(y-u) = K_\pi(y,x).$$

The lepton loop: fully analytic result for $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$

$$
\begin{aligned}
i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \quad = \quad & \widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) \\
& +\widehat{\Pi}^{(1)}_{\rho;\nu\lambda\mu\sigma}(y-x,-x) + x_\rho\, \Pi^{(r,1)}_{\nu\lambda\mu\sigma}(y-x,-x) \\
& +\widehat{\Pi}^{(1)}_{\rho;\lambda\nu\mu\sigma}(-x,y-x) + x_\rho\, \Pi^{(r,1)}_{\lambda\nu\mu\sigma}(-x,y-x).
\end{aligned}
$$

$$
\begin{aligned}
&\Pi^{(r,1)}_{\mu\nu\lambda\sigma}(x,y) \\
&= 2\Big(\frac{m}{2\pi}\Big)^8 \Big[\frac{(-x_\alpha)(x-y)_\beta\ K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2} \cdot I_{\gamma\delta}(y) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+ \frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|} \cdot p(|y|) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(-x_\alpha)(x-y)_\beta\ K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2} \cdot p(|y|) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|} \cdot q_\gamma(y) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2} \cdot q_\gamma(y) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|} \cdot q_\delta(y) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+ \frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2} \cdot q_\delta(y) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+ \frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|} \cdot I_{\gamma\delta}(y) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\}\Big]
\end{aligned}
$$

The lepton loop (continued)

$$
\begin{aligned}
\widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) \quad &= 2\Big(\frac{m}{2\pi}\Big)^8 \Big[\frac{(-x_\alpha)(x-y)_\beta K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2} \cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+ \frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|} \cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+ \frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\, g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(-x_\alpha)(x-y)_\beta\; K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2} g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(-x_\alpha)\; K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|} h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(x-y)_\beta\; K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2} h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(-x_\alpha)\; K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|} \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+ \frac{(x-y)_\beta\; K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2} \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\}\Big]
\end{aligned}
$$

$$
l_{\gamma\delta}(y) = \frac{2\pi^2}{m^2}\Big(\hat{y}_\gamma\hat{y}_\delta\; K_2(m|y|) - \delta_{\gamma\delta}\;\frac{K_1(m|y|)}{m|y|}\Big), \quad h_{\rho\gamma}(y) = \frac{\pi^2}{m^3}\Big(\hat{y}_\gamma\hat{y}_\rho\; m|y|\; K_1(m|y|) - \delta_{\gamma\rho}K_0(m|y|)\Big),
$$

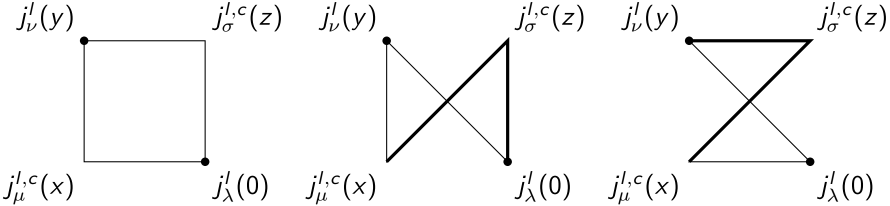
$$
\hat{f}_{\rho\delta}(y) = \frac{\pi^2}{m^3}\Big\{\hat{y}_\rho\hat{y}_\delta\;\; m|y|K_1(m|y|) + \delta_{\rho\delta}K_0(m|y|)\Big\} \qquad q_\gamma(y) = \frac{2\pi^2}{m^2}\;\hat{y}_\gamma\; K_1(m|y|),
$$

$$
f_{\rho\delta\gamma}(y) = \frac{\pi^2}{m^3}\Big\{\hat{y}_\gamma\hat{y}_\delta\hat{y}_\rho\; m|y|K_2(m|y|) + (\delta_{\rho\delta}\hat{y}_\gamma - \delta_{\gamma\rho}\hat{y}_\delta - \delta_{\gamma\delta}\hat{y}_\rho)\,K_1(m|y|)\Big\}, \quad p(|y|) = \frac{2\pi^2}{m^2}\;K_0(m|y|).
$$

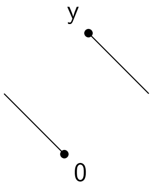
Lattice QED Computation with Wilson Fermions

four-point correlation function

$$\left\langle\, v_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0) \right\rangle.$$



point sources:



sequential propagators: