Hadronic vacuum polarization: $\pi\pi$ channel and pion form factor

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G. Colangelo, MH, M. Procura, P. Stoffer, work in progress

C. Hanhart, MH, B. Kubis, work in progress

Motivation

- How to estimate uncertainty in the $\pi\pi$ channel?

  $\hookrightarrow$ **local error inflation** wherever tensions between data sets arise

- In QCD: **analyticity** and **unitarity** imply strong relation between pion form factor and $\pi\pi$ scattering

  $\hookrightarrow$ defines **global fit function**

- Main motivation: Can one use these constraints to corroborate the uncertainty estimate for the $\pi\pi$ channel?

- Idea not new de Trocóniz, Ynduráin 2001, 2004, Leutwyler, Colangelo 2002, 2003, Ananthanarayan et al. 2013, 2016

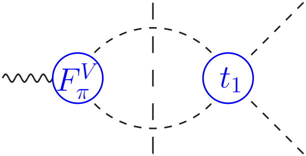
- Here: towards practical implementation, first numerical results

  see talk at Tsukuba meeting for more details on the formalism

Unitarity relation for the pion form factor

- **Unitarity** for **pion vector form factor**

$$\operatorname{Im} F_\pi^V(s) = \theta\big(s - 4M_\pi^2\big)\, F_\pi^V(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



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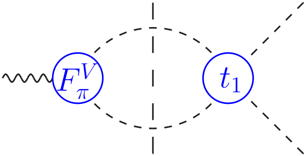

$\hookrightarrow$ **final-state theorem**: phase of F_π^V equals $\pi\pi$ P-wave phase δ_1 Watson 1954

- Solution in terms of **Omnès function** Omnès 1958

$$F_\pi^V(s) = P(s)\Omega_1(s) \qquad \Omega_1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1(s')}{s'(s'-s)}\right\}$$

- Asymptotics + normalization $\Rightarrow P(s) = 1$
- In practice: inelastic corrections $F_\pi^V(s) = G_3(s) G_4(s) \Omega_1(s)$

Intermediate states beyond $\pi\pi$

- **3π states**: forbidden for $m_u = m_d$, but otherwise correction factor

$$G_3(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^{\infty} \mathrm{d}s' \frac{\operatorname{Im} G_3(s')}{s'(s'-s)} \qquad \operatorname{Im} G_3(s) \sim (s - 9M_\pi^2)^4$$

- In practice: completely dominated by ω pole

$$G_3(s) = 1 + \epsilon_{\rho\omega}\frac{s}{s_\omega - s} \qquad s_\omega = \left(M_\omega - i\frac{\Gamma_\omega}{2}\right)^2$$

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- **4π states**: correction factor

$$G_4(s) = 1 + \frac{s}{\pi}\int_{16M_\pi^2}^{\infty} ds' \frac{\text{Im}\, G_4(s')}{s'(s'-s)} \qquad \text{Im}\, G_4(s) \sim (s - 16M_\pi^2)^{9/2}$$

- In practice: negligible below $\pi\omega$ threshold Eidelman, Łukaszuk 2003

$$G_4(s) = 1 + \sum_{i=1}^{p} c_i\big(z(s)^i - z(0)^i\big) \qquad z(s) = \frac{\sqrt{s_{\pi\omega} - s_1} - \sqrt{s_{\pi\omega} - s}}{\sqrt{s_{\pi\omega} - s_1} + \sqrt{s_{\pi\omega} - s}} \qquad s_{\pi\omega} = (M_\pi + M_\omega)^2$$

- **Inelastic phase** above $s_{\pi\omega}$ constrained by P-wave behavior and Eidelman–Łukaszuk bound

Parameterization of the $\pi\pi$ phase shift

- Isospin $I = 1$ P-wave t_1 related to other $\pi\pi$ channels by **Roy equations**

  $\hookrightarrow$ manifestation of analyticity, unitarity, and crossing symmetry

- Mathematical properties well understood Gasser, Wanders 1999

  $\hookrightarrow$ **uniqueness properties** depend on the phase shift

- Solving δ_1 below $\sqrt{s_\text{m}} = 1.15\,\text{GeV}$, there are **two free parameters**

  $\hookrightarrow$ take $\delta_1(s_\text{m})$ and $\delta_1(s_\text{A})$, $\sqrt{s_\text{A}} = 0.8\,\text{GeV}$

- Family of solutions from Caprini, Colangelo, Leutwyler 2011

  $\hookrightarrow$ effective parameterization in terms of $\delta_1(s_\text{m})$ and $\delta_1(s_\text{A})$

- In total: **$3 + p$ fit parameters** for F_π^V

Fit to $\pi\pi$ data sets: strategy

- For now: **one fixed representation** for $F_\pi^V(s)$, e.g. 1 free parameter in conformal polynomial
- For now: fix ω parameters to PDG values

  $\hookrightarrow$ 4 fit parameters in total
- Full statistical and systematic covariance matrices

  $\hookrightarrow$ **iterative fit** to avoid d'Agostini bias
- VP excluded by definition Tsukuba talk
- In practice, take **bare cross section**, remove FSR
- In calculation of HVP, add FSR in the end via

$$|F_\pi^V(s)|^2 \to |F_\pi^V(s)|^2\Big(1+\frac{\alpha}{\pi}\eta(s)\Big)$$

Fit to $\pi\pi$ data sets: fixed ω parameters

	$\delta(s_A)$ [°]	$\delta(s_m)$ [°]	$10^3\epsilon_{\rho\omega}$	c_1	χ^2/dof	p	$10^{10} a_\mu^{\pi\pi}\|_{[0.6,0.9]}$ DR	1711.03085
SND	110.4	165.5	1.95	0.24	5.30	$7 \cdot 10^{-26}$	374.1(3.6)	371.7(5.0)
CMD2	109.8	165.5	1.80	0.20	3.37	$2 \cdot 10^{-8}$	368.3(3.0)	372.4(3.0)
BaBar	110.6	166.0	2.08	0.22	1.53	$7 \cdot 10^{-8}$	377.3(2.0)	376.7(2.7)
KLOE	110.5	165.8	1.87	0.15	1.67	$2 \cdot 10^{-8}$	367.1(1.1)	366.9(2.1)

- Some observations:
  - Caprini, Colangelo, Leutwyler 2011: $\delta(s_A) = 108.9(2.0)^\circ$, $\delta(s_m) = 166.5(2.0)^\circ$
    $\hookrightarrow$ $\pi\pi$ phases remarkably consistent among all fits
  - Differences mainly in $\epsilon_{\rho\omega}$ and c_1
  - Reduced χ^2 and p-values terrible, why?

Fit to $\pi\pi$ data sets: fitting the ω mass

	M_ω [MeV]	χ^2/dof	p-value	$10^{10} a_\mu^{\pi\pi}\|_{[0.6,0.9]}$ DR	1711.03085
SND	781.54(8)	1.37 [5.30]	5.8% [$7 \cdot 10^{-26}$]	373.9(3.6) [374.1(3.6)]	371.7(5.0)
CMD2	782.09(7)	1.38 [3.37]	10.1% [$2 \cdot 10^{-8}$]	370.7(3.0) [368.3(3.0)]	372.4(3.0)
BaBar	781.91(7)	1.13 [1.53]	7.3% [$7 \cdot 10^{-8}$]	375.6(2.1) [377.3(2.0)]	376.7(2.7)
KLOE	782.12(14)	1.60 [1.67]	$3 \cdot 10^{-7}$ [$2 \cdot 10^{-8}$]	366.6(1.1) [367.1(1.1)]	366.9(2.1)

- Further observations:
  - In general vast improvement, most fits acceptable now
  - PDG: $M_\omega = 782.65(12)$ MeV (dominated by $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$ SND, CMD2)
    $\hookrightarrow$ shifts much larger than $\Delta M_\omega = \bar{M}_\omega - M_\omega = 0.13$ MeV from radiative corrections
  - Fitting Γ_ω does not yield further improvements
  - For KLOE only modest improvement, why?

Fit to $\pi\pi$ data sets: energy rescaling

	ξ	χ^2/dof	p-value	$10^{10} a_\mu^{\pi\pi}\|_{[0.6,0.9]}$ DR	1711.03085
SND	1.00142(11)	1.37 [1.37]	5.9% [5.8%]	373.8(3.6) [373.9(3.6)]	371.7(5.0)
CMD2	1.00071(10)	1.38 [1.38]	10.1% [10.1%]	370.6(3.0) [370.7(3.0)]	372.4(3.0)
BaBar	1.00095(9)	1.13 [1.13]	7.4% [7.3%]	375.5(2.1) [375.6(2.1)]	376.7(2.7)
KLOE	1.00069(18)	1.59 [1.60]	$3\cdot 10^{-7}$ [$3\cdot 10^{-7}$]	366.5(1.1) [366.6(1.1)]	366.9(2.1)
KLOE ($3\xi_i$)	1.00125(20)	1.36	$8\cdot 10^{-4}$	365.3(1.1)	366.9(2.1)
	1.00023(16)				
	1.00041(28)				
KLOE ($2\xi_i$)	1.00122(19)	1.35	$9\cdot 10^{-4}$	365.2(1.1)	366.9(2.1)
	1.00025(16)				

- Further observations:
  - Energy rescaling $\sqrt{s} \to \xi\sqrt{s}$ equivalent to fit of ω mass
  - KLOE fit improves significantly by allowing for different rescalings in KLOE08 and KLOE10/KLOE12

Fit to $\pi\pi$ data sets: systematics

	ξ	χ^2/dof	p-value	$10^{10} a_\mu^{\pi\pi}\|_{[0.6,0.9]}$ DR	1711.03085
SND	1.00142(11) [1.00142(11)]	1.43 [1.37]	4.2% [5.9%]	375.6(4.5) [373.8(3.6)]	371.7(5.0)
CMD2	1.00069(10) [1.00071(10)]	1.40 [1.38]	10.2% [10.1%]	372.9(3.4) [370.6(3.0)]	372.4(3.0)
BaBar	1.00096(9) [1.00095(9)]	1.13 [1.13]	7.2% [7.4%]	375.9(2.2) [375.5(2.1)]	376.7(2.7)
KLOE ($2\xi_i$)	1.00121(19) [1.00122(19)]	1.30 [1.35]	0.4% [$9 \cdot 10^{-4}$]	367.2(1.4) [365.2(1.1)]	366.9(2.1)
	1.00023(16) [1.00025(16)]				

- Systematic uncertainties:
  - Dominant effect: order of the **conformal polynomial** (here: $p = 3$)
    $\hookrightarrow$ some further improvement for KLOE
  - Others: asymptotics of phase (negligible), uncertainties in Roy phase ($\sim$ 0.5 units), s_1 ($\sim$ 0.5 units)

Fit to $\pi\pi$ data sets: a first look at combinations

	χ^2/dof	p-value	$10^{10} a_\mu^{\pi\pi}\|_{[0.6,0.9]}$ DR	KNT18
direct scan	1.40	1.7%	373.8(2.7)	370.8(2.6)
BaBar	1.13	7.2%	375.9(2.2)	376.7(2.7)
KLOE	1.30	0.4%	367.2(1.4)	366.9(2.1)
all	1.31	$3 \cdot 10^{-6}$	369.9(1.1)	369.4(1.3)

- Caveats:
  - Systematic errors missing
    $\hookrightarrow$ total errors likely larger than in direct integration (but not much)
  - Fits not perfect: PDG **scale factors**?
- Very stable prediction for low-energy region: $a_\mu^{\pi\pi}\big|_{\leq 0.63} = 133.0(3)(5) \cdot 10^{-10}$
  $\hookrightarrow$ compare to 131.1(1.0) KNT18, 133.3(7) Ananthanarayan et al. 2016

Conclusions

- Better understanding of $\pi\pi$ channel from **analyticity** and **unitarity**?
- Some preliminary fit results
  - Acceptable fit only for variable ω mass $\Rightarrow$ energy rescaling
  - KLOE08 and KLOE10/KLOE12 seem to favor different such rescalings
  - Systematic error dominated by order of conformal polynomial

    $\hookrightarrow$ **Eidelman–Łukaszuk bound**
  - For $[0.6, 0.9]$ GeV good agreement with direct integration within (comparable) errors
  - Parameterization becomes increasingly stringent for small energies
- Outlook
  - Combination strategy: PDG scale factors?
  - Space-like data
  - Can we help resolve the controversy in $\pi\pi$ channel?

How to define the pion form factor?

- In QCD: matrix element of the electromagnetic current $j^\mu_{\text{em}} = \bar{q}Q\gamma^\mu q$

$$\langle\pi^\pm(p')|j^\mu_{\text{em}}|\pi^\pm(p)\rangle = \pm(p+p')^\mu e F^V_\pi(s) \qquad s=(p'-p)^2$$

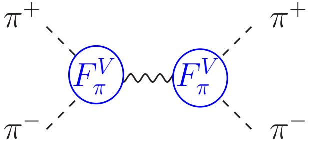
- Relation to cross section

$$\sigma(e^+e^-\to\pi^+\pi^-) = \frac{\pi\alpha(s)^2}{3s}\sigma^3_\pi(s)\big|F^V_\pi(s)\big|^2\frac{s+2m_e^2}{s\sigma_e(s)} \qquad \sigma_\pi(s)=\sqrt{1-\frac{4M_\pi^2}{s}}$$

- Two issues
  - **Vacuum polarization**: $\alpha(s)=\alpha(0)/(1-\Pi(s)) \qquad \alpha\equiv\alpha(0)$
  - **Final-state radiation**: $\sigma(e^+e^-\to\pi^+\pi^-(\gamma)) = \sigma(e^+e^-\to\pi^+\pi^-)\Big(1+\frac{\alpha}{\pi}\eta(s)\Big)$
- Usually
  - For HVP: bare cross section including FSR
    $\sigma_0(e^+e^-\to\pi^+\pi^-(\gamma)) = \frac{\pi\alpha^2}{3s}\sigma^3_\pi(s)\big|F^V_\pi(s)\big|^2\frac{s+2m_e^2}{s\sigma_e(s)}\Big(1+\frac{\alpha}{\pi}\eta(s)\Big)$
  - Absorb VP into form factor, i.e. $\sigma(e^+e^-\to\pi^+\pi^-) = \frac{\pi\alpha^2}{3s}\sigma^3_\pi(s)\big|\tilde{F}^V_\pi(s)\big|^2\frac{s+2m_e^2}{s\sigma_e(s)}$
- Here: keep the QCD $F^V_\pi(s)$!

Role of ρ–γ (and ρ–ω) mixing

- In the context of τ data, ρ^0–γ mixing critical Jegerlehner, Szafron 2011
- Reason: **isospin-breaking corrections** Cirigliano, Ecker, Neufeld 2001, 2002 expressed in terms of ρ^+ and ρ^0 Breit–Wigner parameters
  $\hookrightarrow$ need to identify a **physical ρ^0 state**

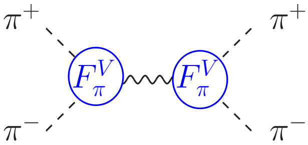


- Here: external states are e^+e^- and $\pi^+\pi^-$
- ρ^0–γ diagonalization related to $\pi^+\pi^- \to \gamma^* \to \pi^+\pi^-$ transition
- Consider coupled channel system of e^+e^- and $\pi^+\pi^-$ Hanhart 2012
- Similarly: 3π channel for ρ–ω mixing

Step 1: $\pi^+\pi^-$ scattering

- Partial-wave projected amplitude

$$t_\gamma(s) = -\frac{4\pi\alpha(s)}{s}\left(s - 4M_\pi^2\right)\left(F_\pi^V(s)\right)^2$$

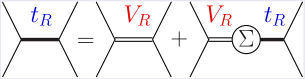


- VP from $\pi^+\pi^-$ states

$$\Pi_\pi(s) = -\frac{\alpha s}{12\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sigma_\pi^3(s')|F_\pi^V(s')|^2}{s'(s'-s-i\epsilon)}$$

Full amplitude

$$t(s) = \underbrace{\tilde{t}(s)}_{48\pi t_1(s)} + \underbrace{\xi_\pi(s)\Gamma_{\text{out}}(s)t_R(s)\Gamma_{\text{in}}^\dagger(s)\xi_\pi(s)}_{t_\gamma(s)}$$



$\hookrightarrow$ takes form of single-channel Bethe–Salpeter equation with

- "potential" $V_R(s) = -\frac{4\pi\alpha}{s}$
- centrifugal barrier factors $\xi_\pi(s) = \sqrt{s - 4M_\pi^2}$
- self energy $\Sigma_\pi(s) = \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\tilde{\sigma}_\pi(s')\xi_\pi^2(s')|\Gamma(s')|^2}{s'^2(s'-s-i\epsilon)}$
- $t_R(s) = \frac{V_R(s)}{1 - V_R(s)\Sigma_\pi(s)}$
- $\Gamma_{\text{out}}(s) = \Gamma_{\text{in}}^\dagger(s) = F_\pi^V(s)$

Step 2: $\pi^+\pi^-$ and e^+e^- (and $\mu^+\mu^-$) scattering

- **Lepton VP**

$$\Pi_\ell(s) = -\frac{4\pi\alpha}{s}\frac{s^2}{\pi}\int_{4m_\ell^2}^{\infty} ds' \frac{\tilde{\sigma}_\ell(s')4(s'+2m_\ell^2)}{s'^2(s'-s-i\epsilon)} \equiv V_R(s)\Sigma_\ell(s)$$

$\hookrightarrow$ same form as for $\pi^+\pi^-$ with $\xi_\ell(s) = 2\sqrt{s+2m_\ell^2}$ and $\Gamma = 1$

Full system

$$(t(s))_{ij} = \delta_{ij}\delta_{1i}\tilde{t}(s) + \xi_i(s)\big(\Gamma_{\text{out}}(s)\big)_i\big(t_R(s)\big)_{ij}\big(\Gamma_{\text{in}}^\dagger(s)\big)_j\xi_j(s)$$

with $t_R(s) = (\mathbb{1} - V_R(s)\Sigma(s))^{-1}V_R(s)$ $\quad V_R(s) = -\frac{4\pi\alpha}{s}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \Sigma(s) = \text{diag}\big(\Sigma_\pi(s), \Sigma_e(s)\big)$

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- From $(t(s))_{12}$ we find

$$\sigma(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3s}\frac{\sigma_\pi^3(s)|F_\pi^V(s)|^2}{|1-\Pi(s)|^2}\frac{s+2m_e^2}{s\sigma_e(s)} \qquad \Pi(s) = \Pi_\pi(s) + \Pi_e(s) + \Pi_\mu(s)$$

$\hookrightarrow$ no effect besides VP, in which $F_\pi^V(s)$ should be fit self-consistently!

- Note: no necessity to ever specify a ρ external state

- Can describe $e^+e^- \to 3\pi$ with dispersion relations MH, Kubis, Leupold, Niecknig, Schneider 2014
- Here: capture the dominant contribution from the ω, leading to the ansatz

$$V_R(s) = -\frac{4\pi\alpha}{s}\begin{pmatrix} 1 & g_3 s \\ g_3 s & (g_3 s)^2 \end{pmatrix} - \frac{1}{s - M_{\omega,0}^2}\begin{pmatrix} 0 & 0 \\ 0 & g_{\omega 3}^2 \end{pmatrix} \qquad \Sigma(s) = \text{diag}\big(\Sigma_e(s), \Sigma_{3\pi}\big)$$

- Bare parameters g_3, $g_{\omega 3}$, $\Sigma_{3\pi}$, $M_{\omega,0}$ via **matching to physical quantities**
- VP from 3π states

$$\Pi_\omega(s) = P_\omega(s) + \frac{e^2 s}{g_{\omega\gamma}^2}\frac{1}{s - M_\omega^2 + iM_\omega\Gamma_\omega}$$

$\hookrightarrow$ strictly speaking only valid near the resonance, set polynomial $P_\omega(s) = 0$

- $g_{\omega\gamma} = 16.7(2)$ determined from $\omega \to e^+e^-$ width

Step 3: ω parameters

- $(t(s))_{ij}$ all involve VP factor $\big(1-\Pi_e(s)-\Pi_\omega(s)\big)^{-1}$

  $\hookrightarrow$ ensures **universality of the ω pole**

- But: the pole parameters are shifted with respect to the ones from $\Pi_\omega(s)$

$$s-M_\omega^2+iM_\omega\Gamma_\omega-\frac{e^2 s}{g_{\omega\gamma}^2(1-\Pi_e(s))}\equiv\bigg(1-\frac{e^2}{g_{\omega\gamma}^2}\bigg)\Big(s-\bar{M}_\omega^2+i\bar{M}_\omega\bar{\Gamma}_\omega\Big)+\mathcal{O}(e^4)$$

with, up to $\mathcal{O}(e^4)$,

$$\bar{M}_\omega=\bigg(1+\frac{e^2}{2g_{\omega\gamma}^2}\bigg)M_\omega\qquad\bar{\Gamma}_\omega=\bigg(1+\frac{e^2}{2g_{\omega\gamma}^2}\bigg)\Gamma_\omega\qquad\bar{g}_{\omega\gamma}=\frac{g_{\omega\gamma}}{\sqrt{Z}}\qquad Z=1+\frac{e^2}{g_{\omega\gamma}^2}$$

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with, up to $\mathcal{O}(e^4)$,

$$\bar M_\omega=\bigg(1+\frac{e^2}{2g_{\omega\gamma}^2}\bigg)M_\omega\qquad\bar\Gamma_\omega=\bigg(1+\frac{e^2}{2g_{\omega\gamma}^2}\bigg)\Gamma_\omega\qquad\bar g_{\omega\gamma}=\frac{g_{\omega\gamma}}{\sqrt{Z}}\qquad Z=1+\frac{e^2}{g_{\omega\gamma}^2}$$

- Numerically

$$\Delta M_\omega=\bar M_\omega-M_\omega=0.13\,\text{MeV}\ [\text{PDG: }0.12\,\text{MeV}]\qquad\Delta g_{\omega\gamma}=\bar g_{\omega\gamma}-g_{\omega\gamma}=-3\times10^{-3}$$

$$\Delta\Gamma_\omega=\bar\Gamma_\omega-\Gamma_\omega=1.4\,\text{keV}\ [\text{PDG: }0.08\,\text{MeV}]$$

$\hookrightarrow$ potentially relevant for the mass

Step 4: full system

- Channels: $1 = \pi^+\pi^-$, $2 = e^+e^-$, $3 = \mu^+\mu^-$, $4 = 3\pi$

$$V_R(s) = -\frac{4\pi\alpha}{s}\begin{pmatrix} 1 & 1 & 1 & g_3 s \\ 1 & 1 & 1 & g_3 s \\ 1 & 1 & 1 & g_3 s \\ g_3 s & g_3 s & g_3 s & (g_3 s)^2 \end{pmatrix} - \frac{1}{s - M_{\omega,0}^2}\begin{pmatrix} g_{\omega 2}^2 & 0 & 0 & g_{\omega 2} g_{\omega 3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_{\omega 2} g_{\omega 3} & 0 & 0 & g_{\omega 3}^2 \end{pmatrix}$$

- $(1 - \Pi(s))^{-1}$ again factorizes in all amplitudes

$$\Pi(s) = \Pi_e(s) + \Pi_\mu(s) + \Pi_\pi(s)\left(1 + \frac{2s\epsilon_{\rho\omega}}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\right) + \Pi_\omega(s) + \mathcal{O}(g_{\omega 2}^2)$$

- Further renormalization of ω parameters

$$\Delta\Gamma_\omega \simeq -0.06\,\text{MeV}\ [\text{PDG: } 0.08\,\text{MeV}]$$

$\hookrightarrow$ enhanced by M_ρ/Γ_ρ, related to ρ–ω mixing ($g_{\omega 2} = \epsilon_{\rho\omega} g_{\omega\gamma}$)

Step 4: result for the form factor

Relation between $e^+e^- \to \pi^+\pi^-$ and the QCD pion form factor

$$\sigma(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3s}\frac{\sigma_\pi^3(s)\big|F_\pi^V(s)\big|^2}{|1-\Pi(s)|^2} \times \bigg|1+\frac{s\epsilon_{\rho\omega}}{M_\omega^2-s-iM_\omega\Gamma_\omega}\bigg|^2 \times \frac{s+2m_e^2}{s\sigma_e(s)}$$

$$\Pi(s) = \Pi_e(s)+\Pi_\mu(s)+\Pi_\pi(s)\bigg(1+\frac{2s\epsilon_{\rho\omega}}{M_\omega^2-s-iM_\omega\Gamma_\omega}\bigg)+\Pi_\omega(s)+\mathcal{O}(\epsilon_{\rho\omega}^2)$$

- Recognize $G_3(s)$

  $\hookrightarrow$ ρ–ω mixing reproduced

- No $G_4(s)$ without consideration of 4π channel

  $\hookrightarrow$ still parameterize by conformal polynomial

Step 4: result for the form factor

Relation between $e^+e^- \to \pi^+\pi^-$ and the QCD pion form factor

$$\sigma(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3s}\frac{\sigma_\pi^3(s)\big|F_\pi^V(s)\big|^2}{|1-\Pi(s)|^2}\big|G_3(s)\big|^2\frac{s+2m_e^2}{s\sigma_e(s)}$$

$$\Pi(s) = \Pi_e(s) + \Pi_\mu(s) + \Pi_\pi(s)\bigg(1 + \frac{2s\epsilon_{\rho\omega}}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\bigg) + \Pi_\omega(s) + \mathcal{O}(\epsilon_{\rho\omega}^2)$$

- Lessons for the fit of $F_\pi^V(s)$
  - Cleanest input should be pion form factor from experiment (no assumptions on VP), but: **unitarity/analyticity constraints apply to QCD form factor**
    $\hookrightarrow$ need to account for VP in the fit
  - Alternatively: use bare cross section, but need to remove FSR and rely on VP used by respective experiment
  - No further corrections from ρ–γ mixing (would only be relevant when using explicit ρ states, similarly to shifts in ω parameters)
  - ω parameters in $G_3(s)$ are not the physical pole parameters, potentially relevant shifts due to VP

Eidelman–Łukaszuk bound

- $\pi\pi$ amplitude

$$t_1 = \frac{\eta_1 e^{2i\delta_1} - 1}{2i\sigma_\pi}$$

- From unitarity relation Łukaszuk 1973

$$\left(\frac{1-\eta_1}{2}\right)^2 + \eta_1 \sin^2\delta_{\text{inel}} \leq \frac{1-\eta_1^2}{4} r \qquad\qquad r = \frac{\sigma^{I=1}_{\text{non-}2\pi}}{\sigma_{e^+e^-\to\pi^+\pi^-}}$$

- Implies bound Eidelman, Łukaszuk 2003

$$\sin^2\delta_{\text{inel}} \leq \frac{1}{2}\Big(1-\sqrt{1-r^2}\Big)$$

$\hookrightarrow$ shows that $\delta_{\text{inel}} \simeq 0$ below $s_{\pi\omega}$

- Better constraint on δ_{inel} when providing input for inelasticity η_1