The hadronic vacuum polarization contribution to $(g-2)_\mu$: status of the Mainz-CLS calculation

Harvey Meyer

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Cluster of Excellence

Outline

- Calculation in the time-momentum representation in $N_f = 2 + 1$ QCD
- Technical improvements over our $N_f = 2$ calculation [1705.01775 (JHEP)].
- Results for the strange and charm connected contributions.
- Status of the light-quark contribution.

CLS-Mainz HVP collaboration: A. Gérardin, T. Harris, G. von Hippel, B. Hörz, HM, D. Mohler, K. Ottnad, H. Wittig.

All numerical results in this talk are still preliminary!

HVP: definitions (Euclidean space)

- primary object on the lattice: $G_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0) \rangle$.
- polarization tensor:

$$\Pi_{\mu\nu}(Q) \equiv \int d^4x\, e^{iQ\cdot x} G_{\mu\nu}(x) = \big(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\big)\Pi(Q^2).$$

- $$a_\mu^{\rm hvp} = 4\alpha^2 \int_0^\infty dQ^2\; K(Q^2; m_\mu^2)\; [\Pi(Q^2) - \Pi(0)]$$

- Spectral representation: $\rho(s) = \frac{R(s)}{12\pi^2}$, $R(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{4\pi\alpha(s)^2/(3s)}$,

$$\Pi(Q^2) - \Pi(0) = Q^2 \int_{4m_\pi^2}^\infty ds \frac{\rho(s)}{s(s+Q^2)}.$$

Lautrup, Peterman & de Rafael Phys.Rept 3 (1972) 193; Blum hep-lat/0212018 (PRL)

The time-momentum representation (TMR)

- mixed-representation Euclidean correlator: (natural on the lattice)

$$G_{\rm TMR}(x_0) = -\frac{1}{3}\sum_{k=1}^{3}\int d^3\boldsymbol{x}\; G_{kk}(x),$$

- the spectral representation:

$$G_{\rm TMR}(x_0) = \int_0^\infty d\omega\; \omega^2 \rho(\omega^2)\; e^{-\omega|x_0|}, \qquad x_0 \neq 0.$$

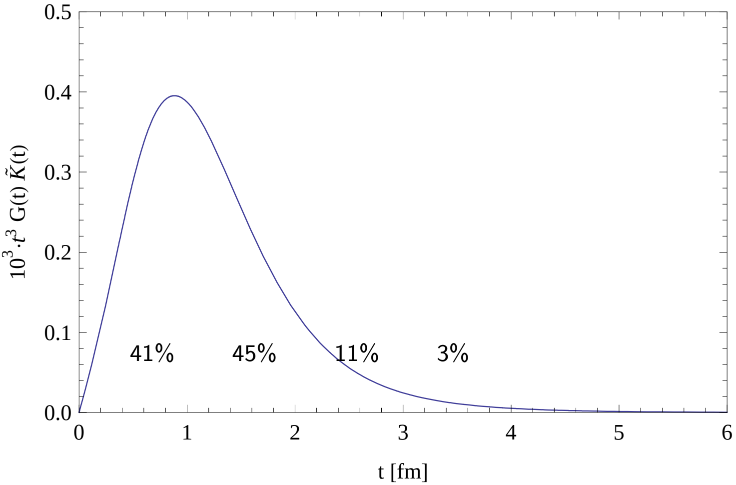
- Finally, the quantity $a_\mu^{\rm hvp}$ is given by

$$\begin{aligned} a_\mu^{\rm hvp} &= \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\; G(x_0)\; \tilde{f}(x_0), \\ \tilde{f}(x_0) &= \frac{2\pi^2}{m_\mu^2}\Big[-2 + 8\gamma_{\rm E} + \frac{4}{\hat{x}_0^2} + \hat{x}_0^2 - \frac{8}{\hat{x}_0} K_1(2\hat{x}_0) \\ &\quad + 8\log(\hat{x}_0) + G^{2,1}_{1,3}\left(\hat{x}_0^2 \Big|\; {\textstyle\frac{3}{2} \atop 0,\,1,\,\frac{1}{2}}\right)\Big] \end{aligned}$$

where $\hat{x}_0 = m_\mu x_0$, $\gamma_{\rm E} = 0.577216..$ and $G^{m,n}_{p,q}$ is Meijer's function.

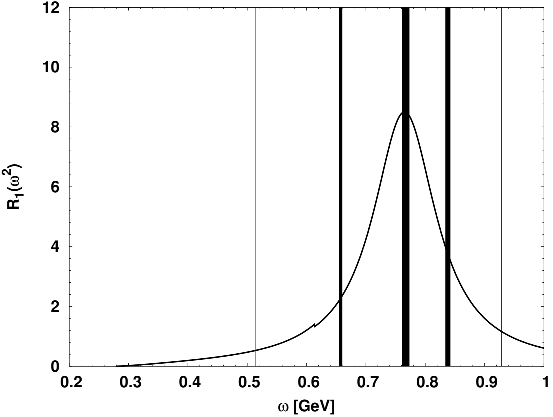
Bernecker & Meyer 1107.4388; Mainz-CLS 1705.01775.

Expected integrand for $a_\mu^{\rm hvp}$ (using pheno. R)



Bernecker & Meyer 1107.4388

Finite-size effects: discrete states on the torus

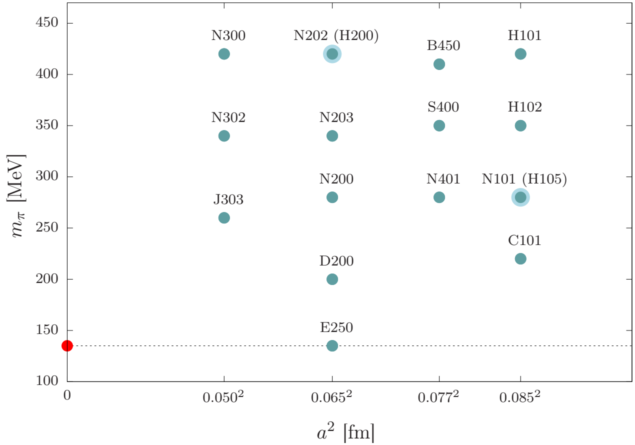


Isovector final states: $\rho(s) = \frac{1}{48\pi^2} \left(1 - 4m_\pi^2/s\right)^{3/2} |F_\pi(s)|^2 + \text{other channels}$

$$|F_\pi(s)|^2 = \left(q\phi'(q) + k\frac{\partial\delta_1(k)}{\partial k}\right) \frac{3\pi s}{2k^5 L^3} \left| {}_L\left\langle \pi\pi \left| \int \mathrm{d}\boldsymbol{x}\, j^z(\boldsymbol{x}) \right| 0 \right\rangle \right|^2 .$$

HM 1105.1892 (PRL); figure: model for $F_\pi(s)$.

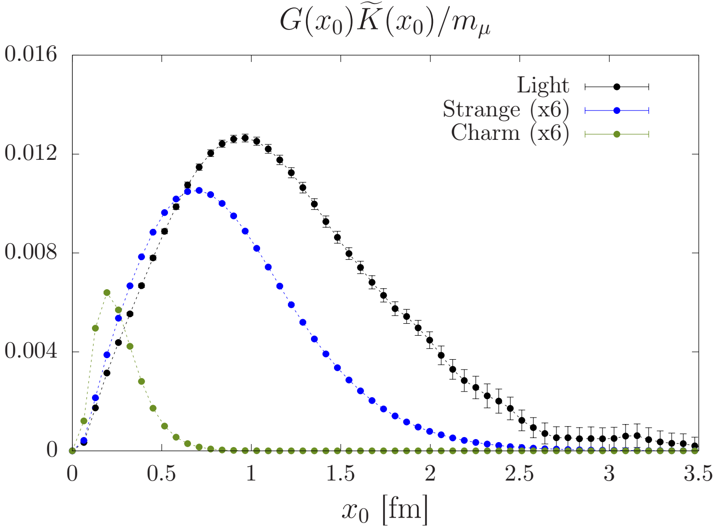
Landscape of CLS ensembles



$N_f = 2 + 1$ ensembles, $O(a)$ improved Wilson quarks, treelevel-improved Lüscher-Weisz gauge action. Algorithm uses twisted-mass reweighting. **Exact isospin symmetry** on the reweighted configurations.

I will often illustrate our calculations using ensemble D200.

A first look at the three connected integrands on ensemble D200



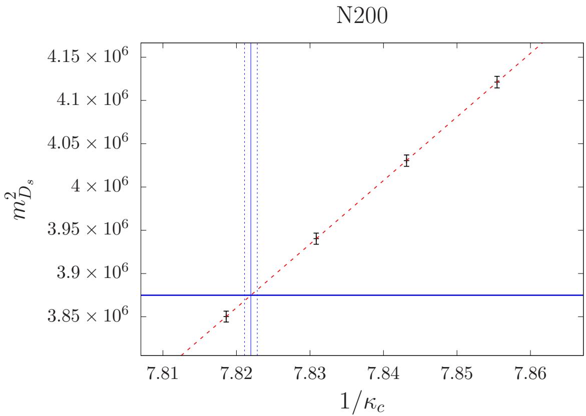
Technical aspects of the calculation

- scale setting: we use the lattice spacing values from [1608.08900 (PRD) Bruno, Korzec, Schaefer].
  For instance, $a[\mathrm{fm}] = 0.06440(65)(15)$ for D200: 1% precision.
  Dimensionful quantity used for scale-setting: $f_K + \frac{1}{2} f_\pi$.
- new: non-perturbative on-shell improvement of the vector current: calculation of c_V [Gérardin, Harris et al., in prep.].
  $\Rightarrow a_\mu^{\mathrm{hvp}}$ approaches its continuum value with O(a^2) corrections.
- local current $\bar\psi(x)\gamma_\mu\psi(x)$ as well as lattice Noether current
  $\Rightarrow$ use two discretizations of the current-current correlator (ll,lc).
  Perform constrained simultaneous continuum limit for a_μ^{hvp}.
- We benefit from a dedicated lattice calculation of the $I = \ell = 1$ scattering phase and of the pion form factor at timelike q^2: [Hörz et al. 1511.02351 and in prep.]. Used to control tail of isovector correlation function and for the finite-size correction.

Charm contribution

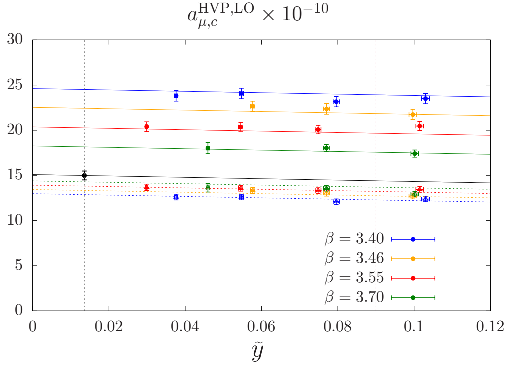
Tuning of κ_c, the bare charm mass parameter

Tuning of κ_c : by imposing $m_{c\bar{s}}^{\text{eff}} = m_{D_s}^{\text{exp}} = 1972$ MeV on each lattice ensemble.



- Interpolation : $m^2_{D_s}$ vs $1/\kappa_c \rightarrow$ linear behaviour
- $am_{D_s} = 0.86$ at $\beta = 3.40$: discretization effects are expected to be large.

Chiral & continuum extrapolation of a_μ^c $(\tilde{y} = m_\pi^2/(16\pi^2 f_\pi^2))$

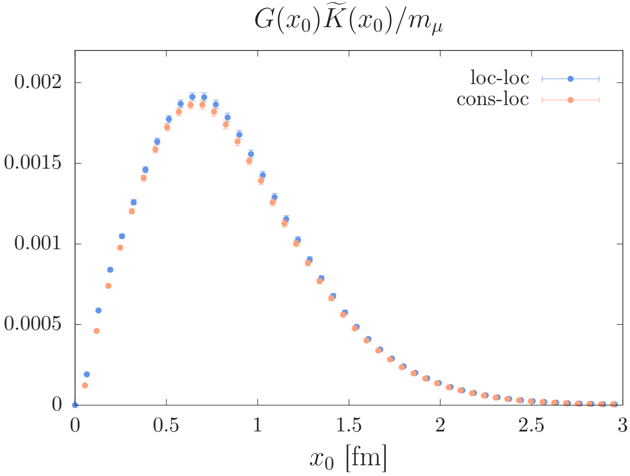


$$10^{10} \cdot a_\mu^c = 14.95(47)_{\mathrm{stat}}(11)_\chi$$

- The $\mathcal{O}(a)$−improvement reduces lattice artefacts significantly
- The simultaneous continuum extrapolation (linear in a^2) works well.

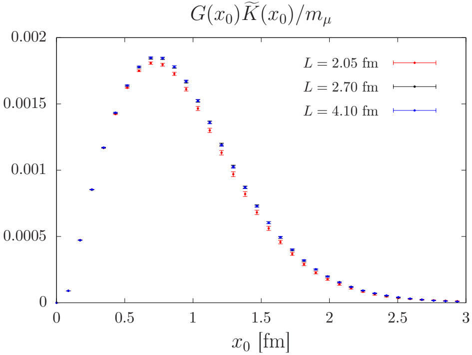
Strange contribution

Strange contribution: data at physical quark masses (E250)



- With full $\mathcal{O}(a)-$improvement of the vector currents
- Two discretizations almost coincide: remaining discretization errors small.

Finite size effects on strangeness correlator at $m_\pi = 280$ MeV

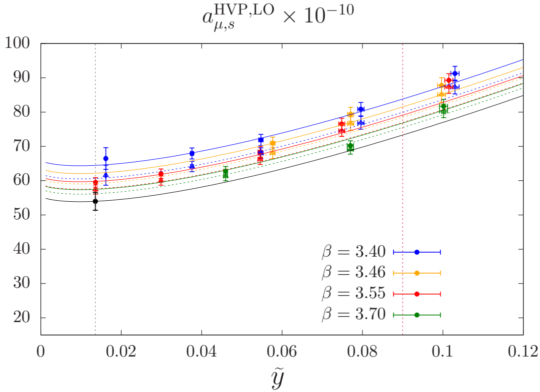


$\Rightarrow$ At the level of 0.5 % precision, volume effects are negligible for $L \geq 2.7$ fm.

CLS	$a_\mu^{\rm ll-imp}$	$a_\mu^{\rm lc-imp}$
U101 ($L = 2.05$ fm)	69.5(0.6)	65.8(0.6)
H105 ($L = 2.70$ fm)	71.8(0.4)	68.0(0.4)
N101 ($L = 4.10$ fm)	71.9(0.3)	68.0(0.3)

Chiral & continuum extrapolation ($\widetilde{y} = m_\pi^2/(16\pi^2 f_\pi^2)$)

- Fit : $a_\mu(a, \widetilde{y}, d) =$
  $a_\mu(0, \widetilde{y}_{\rm exp}) + \delta_d\, a^2 + \gamma_1\, (\widetilde{y} - \widetilde{y}_{\rm exp}) + \gamma_2\, (\widetilde{y} \log \widetilde{y} - \widetilde{y}_{\rm exp} \log \widetilde{y}_{\rm exp})$



$$a_{\mu,s}^{\rm HVP,LO} = 53.6(2.5)_{\rm stat}(0.8)_\chi$$

- The first error is the statistical error;
- second error : variation wrt setting the cut at $m_\pi = 360$ MeV;
- statistical error dominated by the scale setting error.

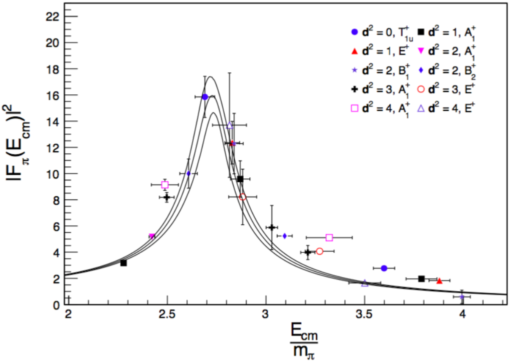
Light-quark contributions

Auxiliary calculation: time-like pion form factor

For all the ensembles with $m_\pi < 300$ MeV: dedicated study (except for E250)

- Overlap and energy levels to constrain the tail of the correlation function
- Time-like pion form factor to estimate finite-size effects.

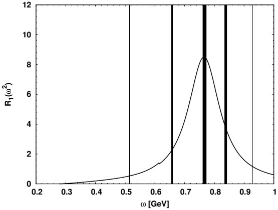
On **N200** ($m_\pi = 280\,$MeV): parametrizing the time-like pion form factor in the Gounaris-Sakurai form: $m_\rho^{\rm GS} = 776(4)\,$MeV $\quad \Gamma_\rho^{\rm GS} = 59(2)\,$MeV.
Other parametrizations are being investigated.

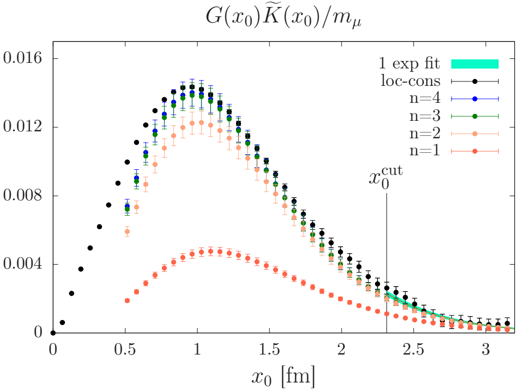


[Bulava, Hörz et al., 1511.02351 (LAT2015).]

Saturation of the correlator by the low-lying states (D200)

cf. slide 6:





$$G(x_0) = \sum_{n=1}^{\infty} A_n \, e^{-E_n x_0}$$

- Excellent cross-check that the tail is understood.

[Update from H. Wittig et al. 1710.10072 (LAT2017)]

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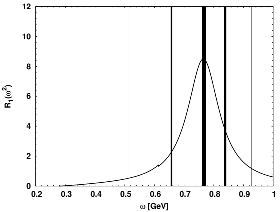

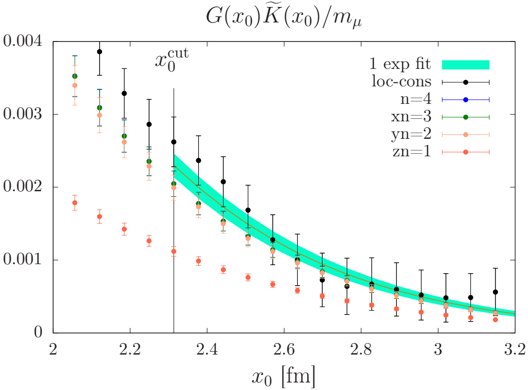

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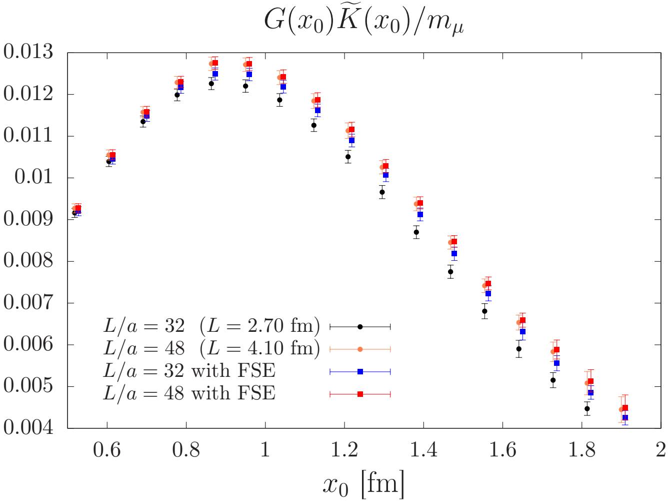
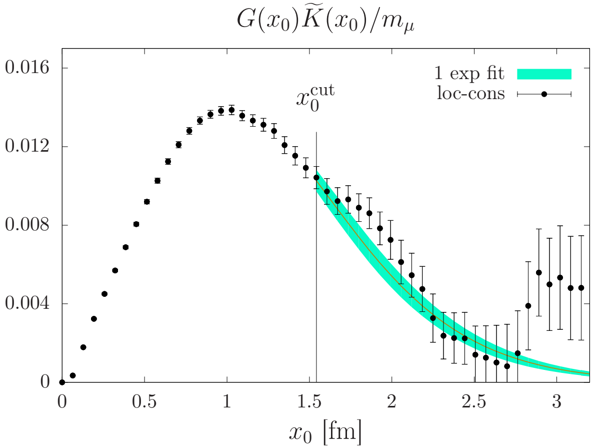
Finite size effects: check on the lattice ($m_\pi = 280\,$MeV H105 vs. N101)



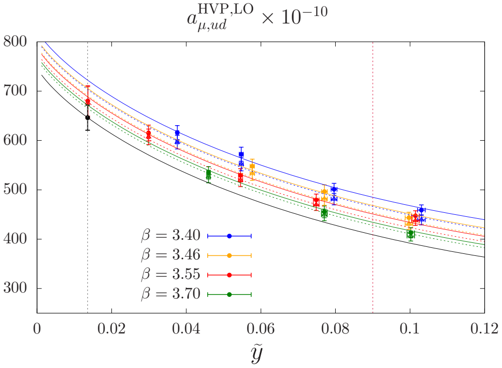
- FSE consistent with the estimate using the pion FF and Lüscher formalism (see [1705.01775 Mainz-CLS]).

Integrand at the physical pion mass (E250)



- statistics being increased
- FSE: for $m_\pi = 140\,\text{MeV}$ and $m_\pi L = 4$, our estimate: $10^{10} \cdot [a_\mu(\infty) - a_\mu(L)] = 20.4 \pm 4.1$ [1705.01775 Mainz-CLS];
- For $m_\pi L = 6$, the estimate goes down by a factor ≈ 10.

Chiral extrapolations (light contribution): $\widetilde{y} = m_\pi^2/(16\pi^2 f_\pi^2)$



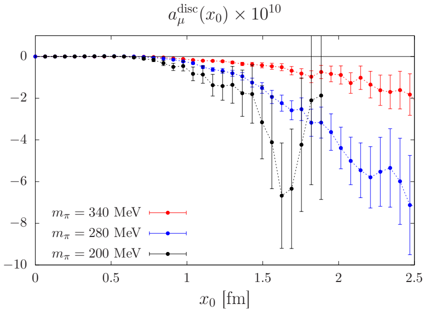
Fit ansatz:

$$a_\mu(a, \widetilde{y}, d) = a_\mu(0, \widetilde{y}_{\mathrm{exp}}) + \delta_d\, a^2 + \gamma_1\, (\widetilde{y} - \widetilde{y}_{\mathrm{exp}}) + \gamma_2\, (\widetilde{y}\, \log \widetilde{y} - \widetilde{y}_{\mathrm{exp}} \log \widetilde{y}_{\mathrm{exp}})$$

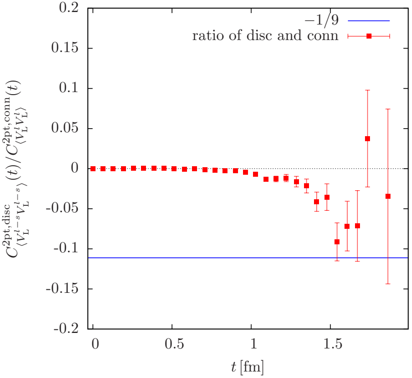
$$\rightsquigarrow \qquad 10^{10} \cdot a_{\mu,s}^{\mathrm{HVP,LO}} = 643(21)_{\mathrm{stat}}(\times\times)_{\mathrm{syst}}.$$

- chiral extrapolation in good agreement with direct calculation at the physical point (E250);
- other ensembles dictate the lattice-spacing dependence.

Disconnected contributions



Integrated contribution



N200 ($m_\pi = 280$ MeV)

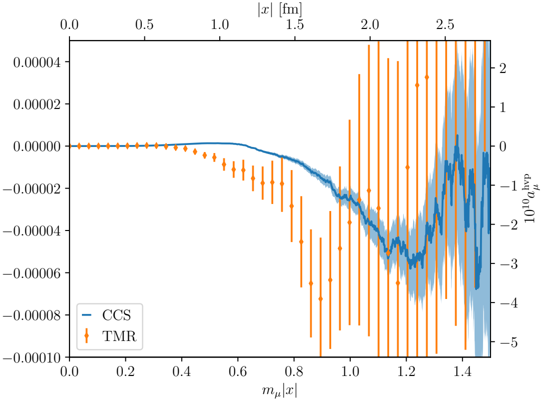
Since $2m_\ell + m_s =$ constant in these ensembles, we envisage an extrapolation in $(m_\ell - m_s)^2$.

At what distance do we reach the asymptotic behavior [1306.2532 Mainz-CLS]

$$\frac{G^{\rm disc}(t)}{G^{\rm isovector}(t)} \stackrel{t\to\infty}{\longrightarrow} -\frac{1}{9} \quad ?$$

Caveat: the relative finite-size effect on $a_\mu^{\rm disc}$ could be very large.

Disconnected contribution using a new Lorentz-covariant method



- at $|x|_{\rm CCS} = |x_0|_{\rm TMR}$, much smaller uncertainties in the covariant version.
- ensemble D200: 128×64^3, $m_\pi = 200\,$MeV, $a = 0.064\,$fm.

HM 1706.01139; M. Cè, K. Ottnad et al.

Conclusion

- Compared to our previous calculation [1705.01775]
  - O(a) improvement implemented + four lattice spacings $0.050 < a/\mathrm{fm} < 0.087 \Rightarrow$ exquisite control over cutoff effects
  - much higher statistics
  - dedicated calculation of the timelike pion form factor available.

- $\rightsquigarrow$ much better control of the tail of the correlators and of their finite-volume effects.

- Main sources of uncertainty in the calculation :

  $\rightarrow$ Statistics & scale setting

  $\rightarrow$ Disconnected diagrams

  $\rightarrow$ Chiral fits

  $\rightarrow$ Finite-size effects.

- gearing up for QED+isospin breaking terms.
  [A. Risch, H. Wittig 1710.06801].

Backup Slides

Finite-size effects on $M(\nu) = \int_0^\infty dt\, t^\nu\, G(t)$, $(\nu > 2)$

Non-interacting pions: $(\omega \equiv 2\sqrt{k^2 + m_\pi^2})$

$$M(\nu, L) - M(\nu, \infty) = \frac{4}{3}\,\Gamma(\nu+1)\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int \frac{d^3k}{(2\pi)^3}\right]\frac{\boldsymbol{k}^2}{\omega^{\nu+3}}.$$

Interacting case:

$$\begin{aligned} M(\nu, L) - M(\nu, \infty) &= \Gamma(\nu+1)\int_0^\infty \frac{4k\,dk}{\omega^\nu}\,[\rho(\omega, L) - \rho(\omega)], \\ \rho(\omega) = \frac{1}{12\pi^2}R(\omega^2) &= \frac{1}{48\pi^2}\left(1 - 4m_\pi^2/\omega^2\right)^{3/2}|F_\pi(\omega^2)|^2 + \dots, \\ \rho(\omega, L) &= \sum_{n=1}^{\infty}\frac{A_n}{E_n^2}\,\delta(\omega - E_n) \end{aligned}$$

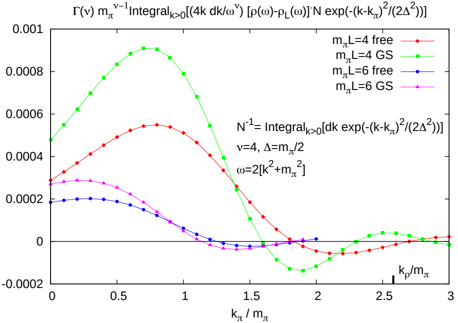
where (k_n, A_n) are related to the pion form factor $F(\omega^2) = |F(\omega^2)|e^{i\delta_{11}}$ by the Lüscher formalism.

Diagnostics of the estimated finite-size correction

Key question: for given L, what states dominate in the calculation of the finite-size effects? $\rightsquigarrow$ Compute

$$f_\Delta(L; k_\pi) = \Gamma(\nu+1) \int_0^\infty \frac{4k\,dk}{\omega^\nu}\,[\rho(\omega, L) - \rho(\omega)] \cdot \psi(k, k_\pi),$$

$$\psi(k, k_\pi) = N \cdot \exp\Big[- \frac{(k-k_\pi)^2}{2\Delta^2} \Big], \qquad \int_0^\infty dk\,\psi(k, k_\pi) = 1.$$



GS= Gounaris-Sakurai form of $F_\pi(\omega^2)$: yields a larger FSE than free pions

Larger volume $\Rightarrow$

1) reduced finite-size effect
2) dominated by softer pions,
hence better predicted by ChPT.

$m_\pi L = 4$: the 1-loop ChPT prediction is not quantitatively reliable yet.

A Lorentz-covariant coordinate-space approach

- primary object: $G_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0) \rangle$.
- $a_\mu^{\rm hvp} = \int d^4x\, G_{\mu\nu}(x)\, H_{\mu\nu}(x) = 2\pi^2 \int_0^\infty d|x|\, |x|^3\, [G_{\mu\nu}(x)\, H_{\mu\nu}(x)]$,

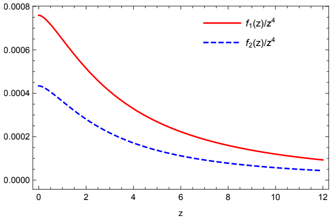
$$H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2}\,\mathcal{H}_2(|x|)$$

a transverse tensor with $\mathcal{H}_i(|x|) = \frac{8\alpha^2}{3m_\mu^2} f_i(m_\mu |x|)$ and

$$f_2(z) = \frac{G^{2,2}_{2,4}\left(z^2 \middle| \begin{matrix} \frac{7}{2}, 4 \\ 4, 5, 1, 1 \end{matrix}\right) - G^{2,2}_{2,4}\left(z^2 \middle| \begin{matrix} \frac{7}{2}, 4 \\ 4, 5, 0, 2 \end{matrix}\right)}{8\sqrt{\pi} z^4},$$

$$f_1(z) = f_2(z) - \frac{3}{16\sqrt{\pi}} \cdot \left[G^{2,3}_{3,5}\left(z^2 \middle| \begin{matrix} 1, \frac{3}{2}, 2 \\ 2, 3, -2, 0, 0 \end{matrix}\right) - G^{2,3}_{3,5}\left(z^2 \middle| \begin{matrix} 1, \frac{3}{2}, 2 \\ 2, 3, -1, -1, 0 \end{matrix}\right) \right] .v$$

$f_i(z)/z^4$



HM, 1706.01139.