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HVP contribution to the muon g-2 from ETMC

outline

- since several years ETMC has addressed the calculation of the hadronic leading-order quark-connected contributions to a_μ^{HVP} using the twisted-mass lattice setups with $N_f = 2$ (including also the physical pion point) [PRL '11, PRD '17] and $N_f = 2+1+1$ [JHEP '14] dynamical quarks (Jansen et al.)

$$a_\mu^{HVP}(ud;\text{conn.}) = 572(16)\cdot 10^{-10} \qquad [N_f = 2]$$

$$a_\mu^{HVP}(ud;\text{conn.}) = 567(11)\cdot 10^{-10} \qquad [N_f = 2+1+1] \qquad a_\mu^{HVP}(udsc;\text{conn.}) = 674(21)\cdot 10^{-10} \qquad [N_f = 2+1+1]$$

- recently ETMC has calculated both a_μ^{HVP} and the isospin-breaking (IB) corrections δa_μ^{HVP} for the strange and charm quarks [JHEP '17], adopting the RM123 method [JHEP '12, PRD '13] in which the path integral is expanded at leading order in both $(m_d - m_u) / \Lambda_{QCD}$ and α_{em} (RM123 people)

$$a_\mu^{HVP}(s;\text{conn.}) = 53.1(2.5)\cdot 10^{-10} \qquad a_\mu^{HVP}(c;\text{conn.}) = 14.75(0.56)\cdot 10^{-10}$$

$$\delta a_\mu^{HVP}(s;\text{conn., qQED}) = -0.018(11)\cdot 10^{-10} \qquad \delta a_\mu^{HVP}(c;\text{conn., qQED}) = -0.030(13)\cdot 10^{-10} \qquad [N_f = 2+1+1]$$

- the Rome branch of ETMC (special thanks to D. Giusti and F. Sanfilippo) has extended the calculations to the light u- and d-quark contributions for both the lowest order and the leading IB corrections. The new results will be presented in this talk and they include an explicit lattice evaluation of Finite Volume Effects (FVEs)

$$a_\mu^{HVP}(ud;\text{conn.}) = 622.8(12.8)\cdot 10^{-10} \qquad a_\mu^{HVP}(udsc;\text{conn.}) = 690.7(13.1)\cdot 10^{-10}$$

$$\delta a_\mu^{HVP}(ud;\text{conn., qQED}) = 6.9(1.9)\cdot 10^{-10} \qquad \delta a_\mu^{HVP}(udsc;\text{conn., qQED}) = 6.9(1.9)\cdot 10^{-10} \qquad [N_f = 2+1+1]$$

master formula

$$a_\mu^{HVP} = 4\alpha_{em}^2 \int_0^\infty dQ^2 \, \frac{1}{m_\mu^2} \, f\left(\frac{Q^2}{m_\mu^2}\right) \left[\Pi\left(Q^2\right) - \Pi\left(0\right)\right]$$

$Q =$ Euclidean 4-momentum

lepton kernel: $f\left(s\right) = \frac{1}{s}\sqrt{\frac{s}{4+s}}\left(\frac{\sqrt{4+s}-\sqrt{s}}{\sqrt{4+s}+\sqrt{s}}\right)^2$ peaked at $s = \frac{Q^2}{m_\mu^2} = \sqrt{5} - 2 \simeq 0.24$

$\Pi\left(Q^2\right) =$ HVP form factor appearing in the covariant decomposition of the HVP tensor:

$$\Pi_{\mu\nu}\left(Q\right) = \int d^4x \, e^{iQ\cdot x} \left\langle J_\mu\left(x\right) J_\nu\left(0\right)\right\rangle = \left[\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu\right] \Pi\left(Q^2\right)$$

$$J_\mu\left(x\right) = \sum_{f=u,d,s,c,...} q_f \, \bar{\psi}_f\left(x\right) \gamma_\mu \psi_f\left(x\right) \qquad \left(\text{quark e.m. current}\right)$$

- lattice data for $\Pi(Q^2)$ have been calculated with N_f=2 and N_f=2+1+1 ETMC ensembles, and then interpolated (and extrapolated) according to:

$$\Pi\left(Q^2\right) = \left[1 - \Theta\left(Q^2 - Q_{match}^2\right)\right] \Pi_{low}\left(Q^2\right) + \Theta\left(Q^2 - Q_{match}^2\right) \Pi_{high}\left(Q^2\right)$$

[PRL ’11, JHEP ’14, PRD ’17]

$$\Pi_{low}\left(Q^2\right) = \sum_{i=1}^{M} \frac{f_i^2}{m_i^2 + Q^2} + \sum_{j=0}^{N-1} a_j Q^{2j}$$

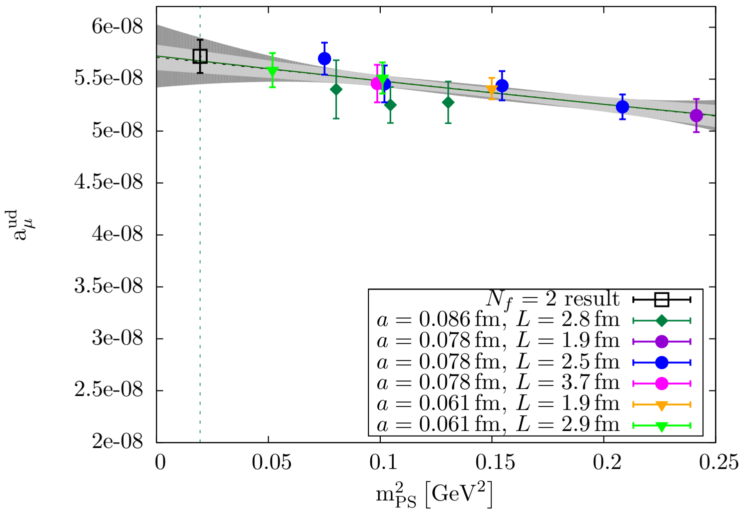
MNBC fit (Jansen et al.)

$$\Pi_{high}\left(Q^2\right) = \log\left(Q^2\right) \sum_{k=0}^{B-1} b_k Q^{2k} + \sum_{p=0}^{C-1} c_p Q^{2p}$$

- ETMC trick: rescale Q as $Q\frac{H}{H^{phys}}$ with $H = M_V, f_V, \ldots$
- simple chiral and continuum extrapolations: $a_\mu^{HVP}(ud) = A_0 + A_1 M_\pi^2 (+\ldots) + Da^2$

$$\boxed{a_\mu^{HVP}(ud;\,\text{conn.}) = 567(11)\cdot 10^{-10}}$$

$$\boxed{a_\mu^{HVP}(udsc;\,\text{conn.}) = 674(21)\cdot 10^{-10}}$$



$H = M_V$

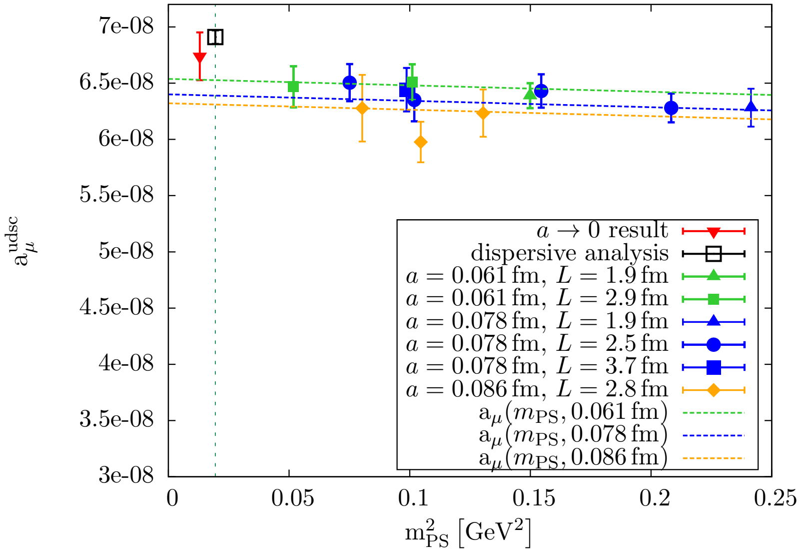


Figure 3. Light-quark contribution to $a_\mu^{\rm hvp}$ on $N_f = 2+1+1$ sea.

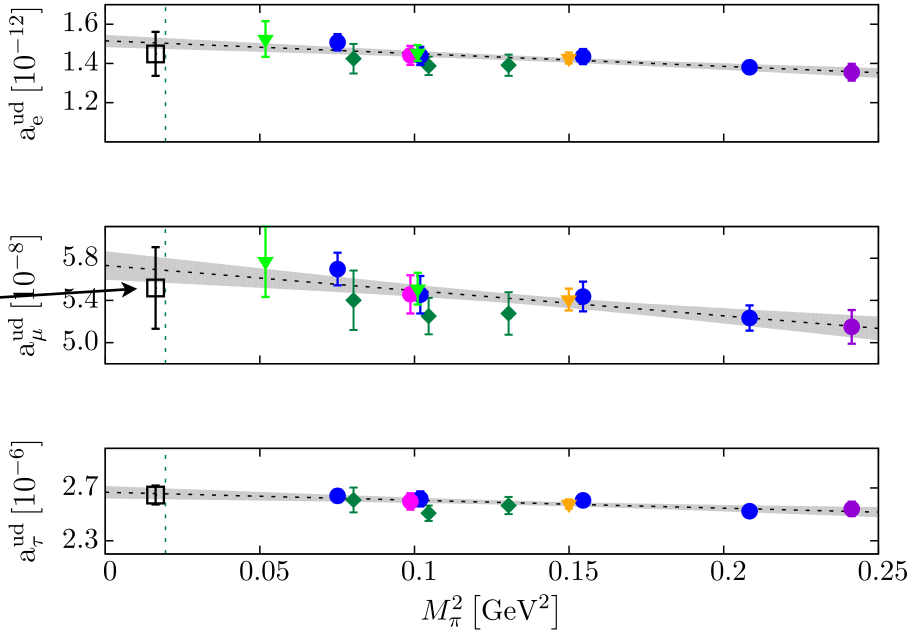
Figure 8. $N_f = 2+1+1$ result for $a_\mu^{\rm hvp}$.

$N_f = 2$ @ M_π^{phys}

$a_\mu^{ud} = 552(39)\cdot 10^{-10}$

$a \sim 0.091$ fm, $L \sim 4.4$ fm

$M_\pi L \sim 3.0$



$\ell = e$

$\ell = \mu$

$\ell = \tau$

time-momentum representation (TMR)

$$a_\mu^{HVP} = 4\alpha_{em}^2 \int_0^\infty dt \, \tilde{f}(t) V(t)$$

[Bernecker&Meyer ’11]

$$V(t) = \frac{1}{3} \sum_{i=1,2,3} \int d\vec{x} \left\langle \sum_f q_f \bar{\psi}_f(\vec{x},t) \gamma_i \psi_f(\vec{x},t) \sum_{f'} q_{f'} \bar{\psi}_{f'}(0) \gamma_i \psi_{f'}(0) \right\rangle$$

$$\tilde{f}(t) \equiv 2 \int_0^\infty dQ^2 \frac{1}{m_\mu^2} f\left(\frac{Q^2}{m_\mu^2}\right) \left[\frac{\cos(Qt) - 1}{Q^2} + \frac{1}{2} t^2 \right]$$

$$a_\mu^{HVP} = \sum_{f=u,d,s,c} 4\alpha_{em}^2 q_f^2 \left\{ \sum_{t=0}^{T_{data}} \tilde{f}(t) V^f(t) + \sum_{t=T_{data}+a}^{\infty} \tilde{f}(t) \frac{G_V^f}{2M_V^f} e^{-M_V^f t} \right\}$$

(quark connected terms only)

directly from lattice data

analytic representation
(ground-state dominance)

up to 10% of the sum
for light u- and d-quarks

* the sum ($t \le T_{data}$ + $t > T_{data}$) turns out to be almost independent on the specific choice of T_{data}

ETMC ensembles with $N_f = 2+1+1$

	ensemble	β	V/a^4	N_{cfg}	$a\mu_{sea} = a\mu_{ud}$	$a\mu_\sigma$	$a\mu_\delta$	$a\mu_s$	M_π(MeV)	M_K(MeV)	$M_\pi L$
→	A40.40	1.90	$40^3 \times 80$	100	0.0040	0.15	0.19	0.02363	317 (12)	576 (22)	5.7
	A30.32		$32^3 \times 64$	150	0.0030				275 (10)	568 (22)	3.9
→	A40.32			100	0.0040				316 (12)	578 (22)	4.5
	A50.32			150	0.0050				350 (13)	586 (22)	5.0
→	A40.24		$24^3 \times 48$	150	0.0040				322 (13)	582 (23)	3.5
	A60.24			150	0.0060				386 (15)	599 (23)	4.2
	A80.24			150	0.0080				442 (17)	618 (14)	4.8
	A100.24			150	0.0100				495 (19)	639 (24)	5.3
→	A40.20		$20^3 \times 48$	150	0.0040				330 (13)	586 (23)	3.0
	B25.32	1.95	$32^3 \times 64$	150	0.0025	0.135	0.170	0.02094	259 (9)	546 (19)	3.4
	B35.32			150	0.0035				302 (10)	555 (19)	4.0
	B55.32			150	0.0055				375 (13)	578 (20)	5.0
	B75.32			80	0.0075				436 (15)	599 (21)	5.8
	B85.24		$24^3 \times 48$	150	0.0085				468 (16)	613 (21)	4.6
	D15.48	2.10	$48^3 \times 96$	100	0.0015	0.1200	0.1385	0.01612	223 (6)	529 (14)	3.4
	D20.48			100	0.0020				256 (7)	535 (14)	3.9
	D30.48			100	0.0030				312 (8)	550 (14)	4.7

gluon action:
Iwasaki

fermion action:
Wilson twisted-mass

unitary in the light sector

OS in the valence strange
and charm sectors

a = {0.089, 0.082, 0.062} fm
at
β = {1.90, 1.95, 2.10}

pion masses in the range
210 - 450 MeV

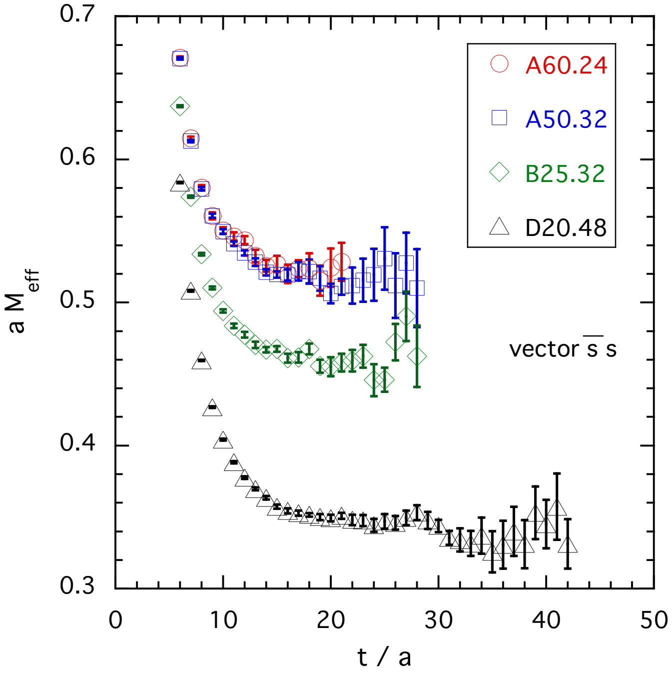
isosymmetric setup

$m_d = m_u = m_{ud}$

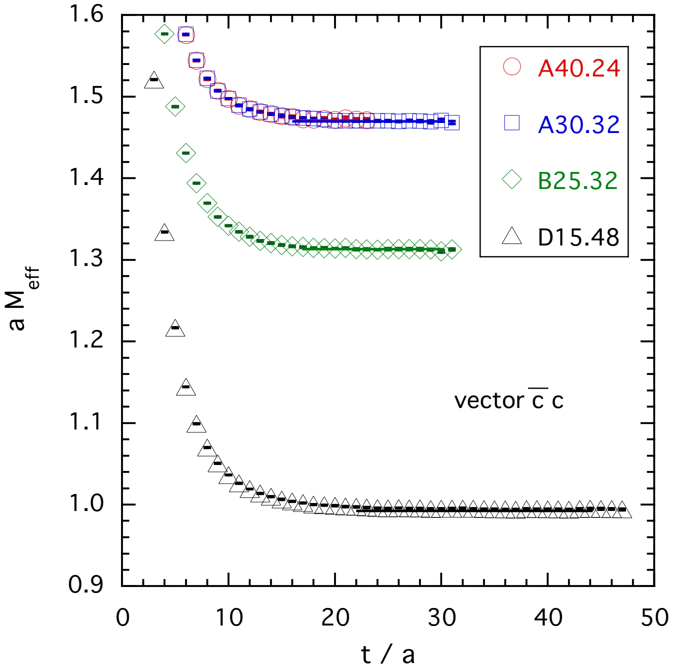
*** ensembles A40.XX: four volumes @ M_π ~ 320 MeV and a ~ 0.09 fm

ground-state identification

s-quark contribution



c-quark contribution

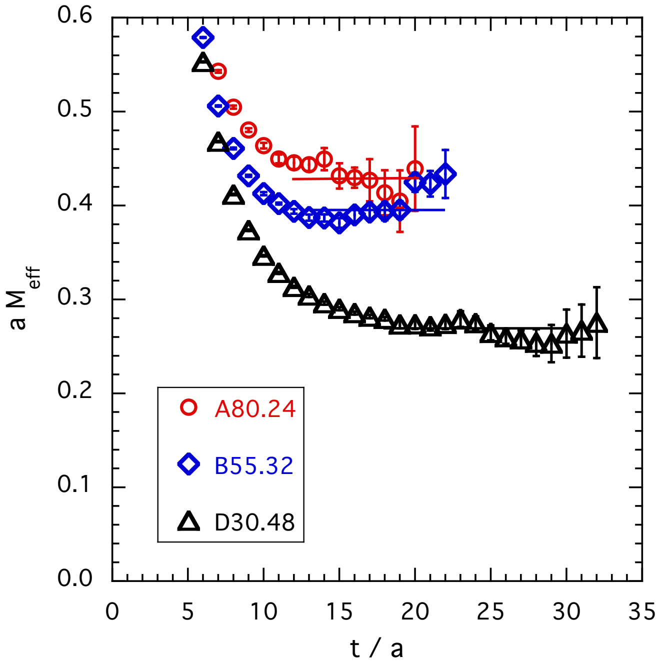


~ OK for the strange contribution

OK for the charm contribution

$$M_{eff}\left(t\right)=\log\frac{V\left(t\right)}{V\left(t-a\right)}$$

u- and d-quark contributions



though we have improved the number of stochastic sources, the quality is not OK for the light-quark contribution

↓

need of a more elaborated representation of the vector (ud) correlator strange contribution:

$$a_\mu^s(phys) = \left(53.1 \pm 1.6_{stat+fit} \pm 1.5_{input} \pm 1.3_{a^2} \pm 0.2_{FVE} \pm 0.1_{chiral}\right)\cdot 10^{-10}$$

$$= \left(53.1 \pm 2.5\right)\cdot 10^{-10}$$ [ETMC ’17]

$$a_\mu^s(phys) = \left(53.41 \pm 0.59\right)\cdot 10^{-10}$$ [HPQCD '14, $N_f = 2+1+1$]

$$= \left(53.1 \pm 0.9^{+0.1}_{-0.3}\right)\cdot 10^{-10}$$ [RBC/UKQCD '16, $N_f = 2+1$]

$$= \left(51.1 \pm 1.7 \pm 0.4\right)\cdot 10^{-10}$$ [CLS/Mainz '17, $N_f = 2$]

$$= \left(53.7 \pm 0.2 \pm 0.4\right)\cdot 10^{-10}$$ [BMW '17, $N_f = 2+1+1$]

$$= \left(53.2 \pm 0.4 \pm 0.3\right)\cdot 10^{-10}$$ [RBC/UKQCD '18, $N_f = 2+1$]

charm contribution:

$$a_\mu^c(phys) = \left(14.75 \pm 0.42_{stat+fit} \pm 0.36_{input} \pm 0.10_{a^2} \pm 0.03_{FVE} \pm 0.01_{chir}\right)\cdot 10^{-10}$$

$$= \left(14.75 \pm 0.56\right)\cdot 10^{-10}$$ [ETMC ’17]

$$a_\mu^c(phys) = \left(14.42 \pm 0.39\right)\cdot 10^{-10}$$ [HPQCD '14, $N_f = 2+1+1$]

$$= \left(14.3 \pm 0.2 \pm 0.1\right)\cdot 10^{-10}$$ [CLS/Mainz '17, $N_f = 2$]

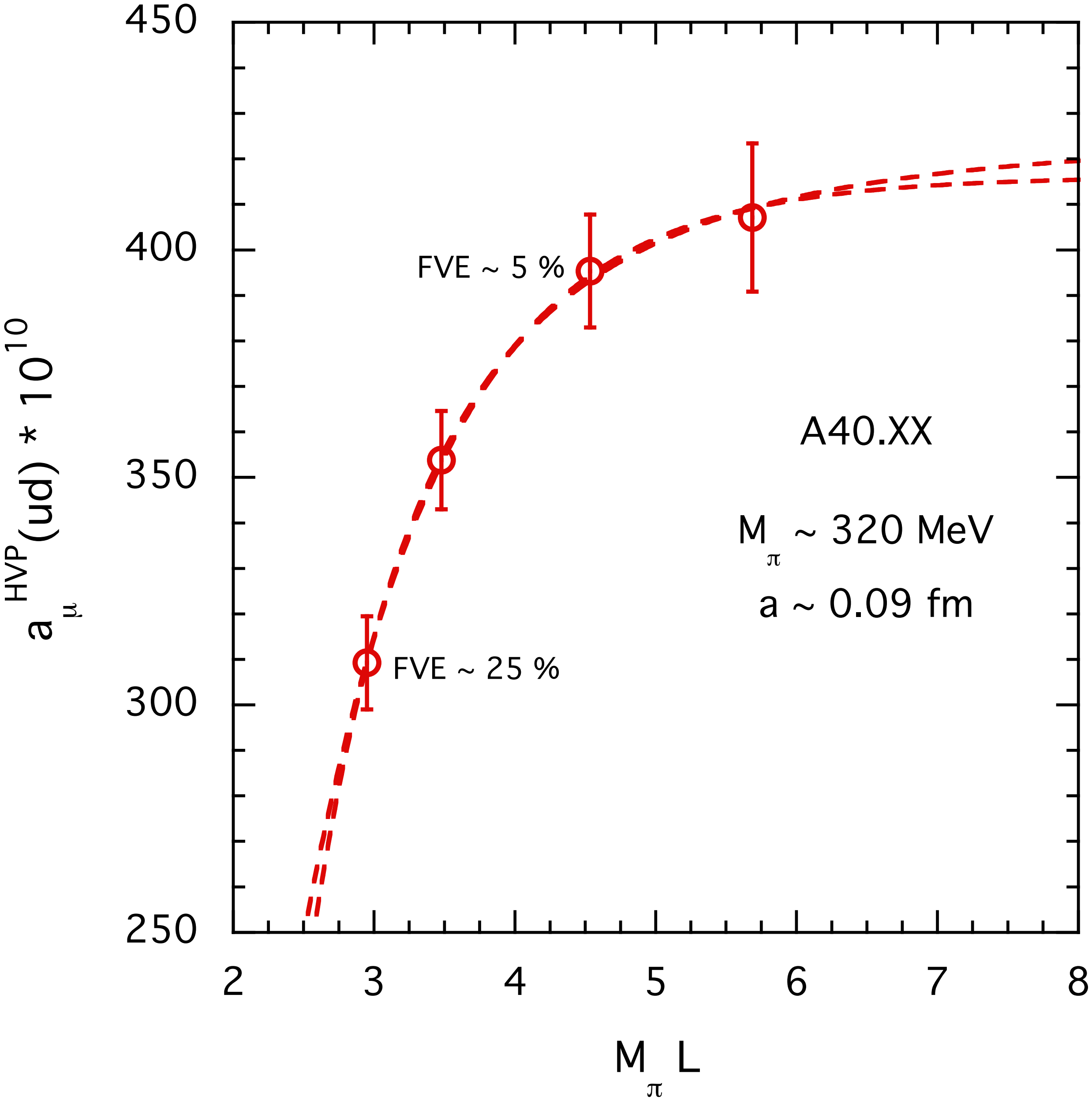
$$= \left(14.7 \pm 0.1 \pm 0.1\right)\cdot 10^{-10}$$ [BMW '17, $N_f = 2+1+1$]

$$= \left(14.3 \pm 0.7 \pm 0.1\right)\cdot 10^{-10}$$ [RBC/UKQCD '18, $N_f = 2+1$]

***** nice agreement *****

Finite Volume Effects on a_μ^{HVP}

u- and d-quark (connected) contributions



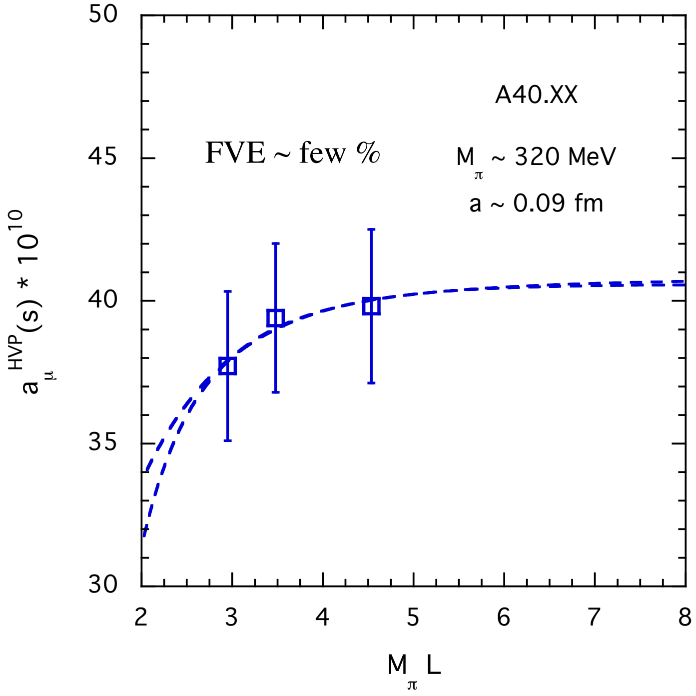
dashed lines are simple phenom. fits

$$a_\mu = a_\mu^{(\infty)}\left[1 + F e^{-M_\pi L}\right]$$

or

$$a_\mu = a_\mu^{(\infty)}\left[1 + \frac{F'}{\left(M_\pi L\right)^3}\right]$$

s-quark contribution



- ✻ our aim is to construct a representation of the vector correlator that allows to correct the FVEs directly on the correlator itself

- the starting point is the π-π contribution in a finite box of size L [Lüscher ’91]:

$$V_{\pi\pi}(t) = \sum_n \nu_n |A_n|^2 e^{-\omega_n t} \qquad \omega_n = 2\sqrt{M_\pi^2 + k_n^2}, \quad n = 1,2,\ldots$$

$\nu_n =$ number of vectors $\vec{z} \in \mathbb{Z}^3$ with $|\vec{z}|^2 = n$

$$k_n: \quad \delta_{11}(k_n) + \phi\left(\frac{k_n L}{2\pi}\right) = n\pi$$

δ_{11} = scattering phase shift (p-wave, I=1)

ϕ = known kinematical function $\qquad \tan\phi(z) = -\dfrac{2\pi^2 z}{\sum_{\vec{m}\in\mathbb{Z}^3}\left(|\vec{m}^2| - z^2\right)^{-1}}$

$$|A_n|^2: \qquad |F_\pi(\omega_n)|^2 = \left\{ k_n \frac{\partial\delta_{11}(k_n)}{\partial k_n} + \frac{k_n L}{2\pi}\phi'\left(\frac{k_n L}{2\pi}\right)\right\} \frac{3\pi\omega_n^2}{2k_n^5}\nu_n |A_n|^2$$

[Meyer ’11, Francis et al. ’13]

$\nwarrow$ time-like pion form factor

- need of a **realistic** model for the time-like pion form factor $F_\pi(\omega) = |F_\pi(\omega)| e^{i\delta_{11}}$ (Watson theorem)

- Gounaris-Sakurai (GS) parameterization [GS '68]

$$F_\pi\left(\omega\right)=\frac{\omega}{k^3}\frac{F_0}{\operatorname{cotg}\delta_{11}-i}$$

$$\frac{k^3}{\omega}\operatorname{cotg}\delta_{11}=k^2h\left(\omega\right)-k_\rho h\left(M_\rho\right)+b_\rho\left(k^2-k_\rho^2\right)$$

$$b_\rho=-\frac{24\pi}{g_{\rho\pi\pi}^2}-h\left(M_\rho\right)-2\frac{k_\rho^2}{M_\rho}h'\left(M_\rho\right)$$

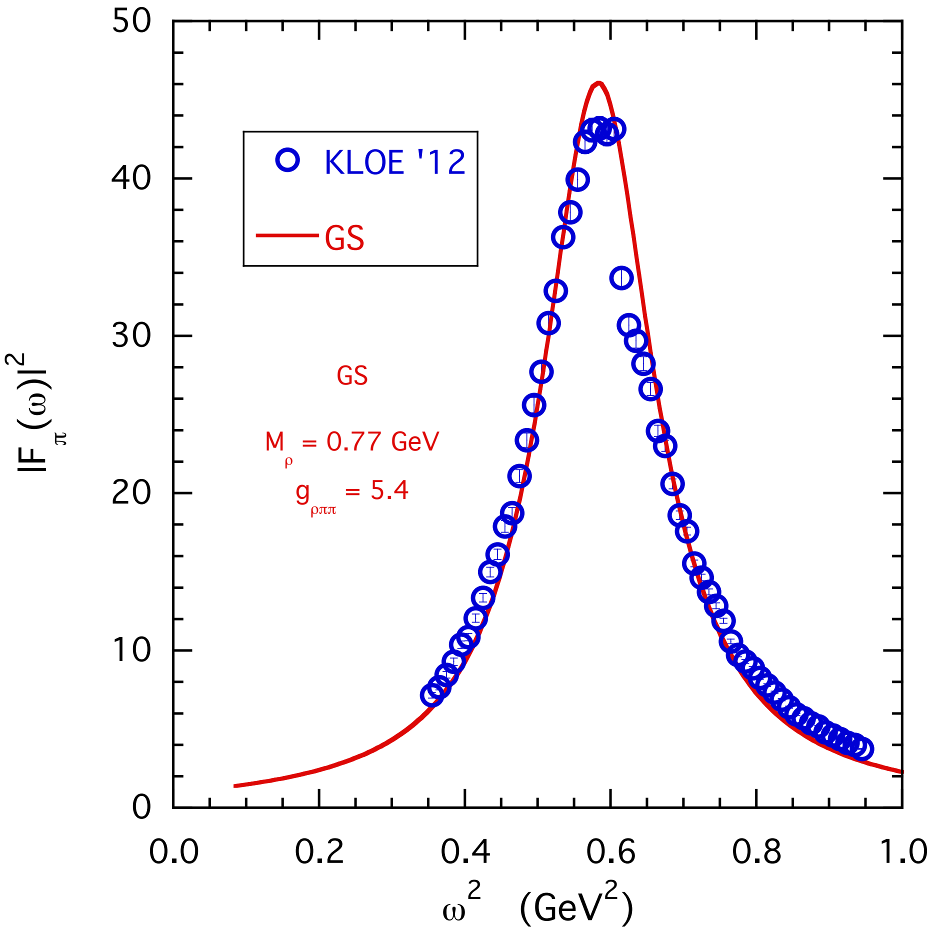
$$h\left(\omega\right)=\frac{2}{\pi}\frac{k}{\omega}\log\frac{\omega+2k}{2M_\pi}$$

$$h'\left(\omega\right)=\frac{1}{\pi\omega}\left[1+\frac{2M_\pi^2}{k\omega}\log\frac{\omega+2k}{2M_\pi}\right]$$

$$F_0=-\frac{M_\pi^2}{\pi}-k_\rho^2h\left(M_\rho\right)-b_\rho\frac{M_\rho^2}{4}$$

- the GS parameterization depends on two variables:

$$M_\rho \quad \text{and} \quad g_{\rho\pi\pi} \longrightarrow \Gamma_{\rho\pi\pi}=\frac{g_{\rho\pi\pi}^2}{6\pi}\frac{k_\rho^3}{M_\rho^2}$$



- reasonable description of experimental data from e^+e^-

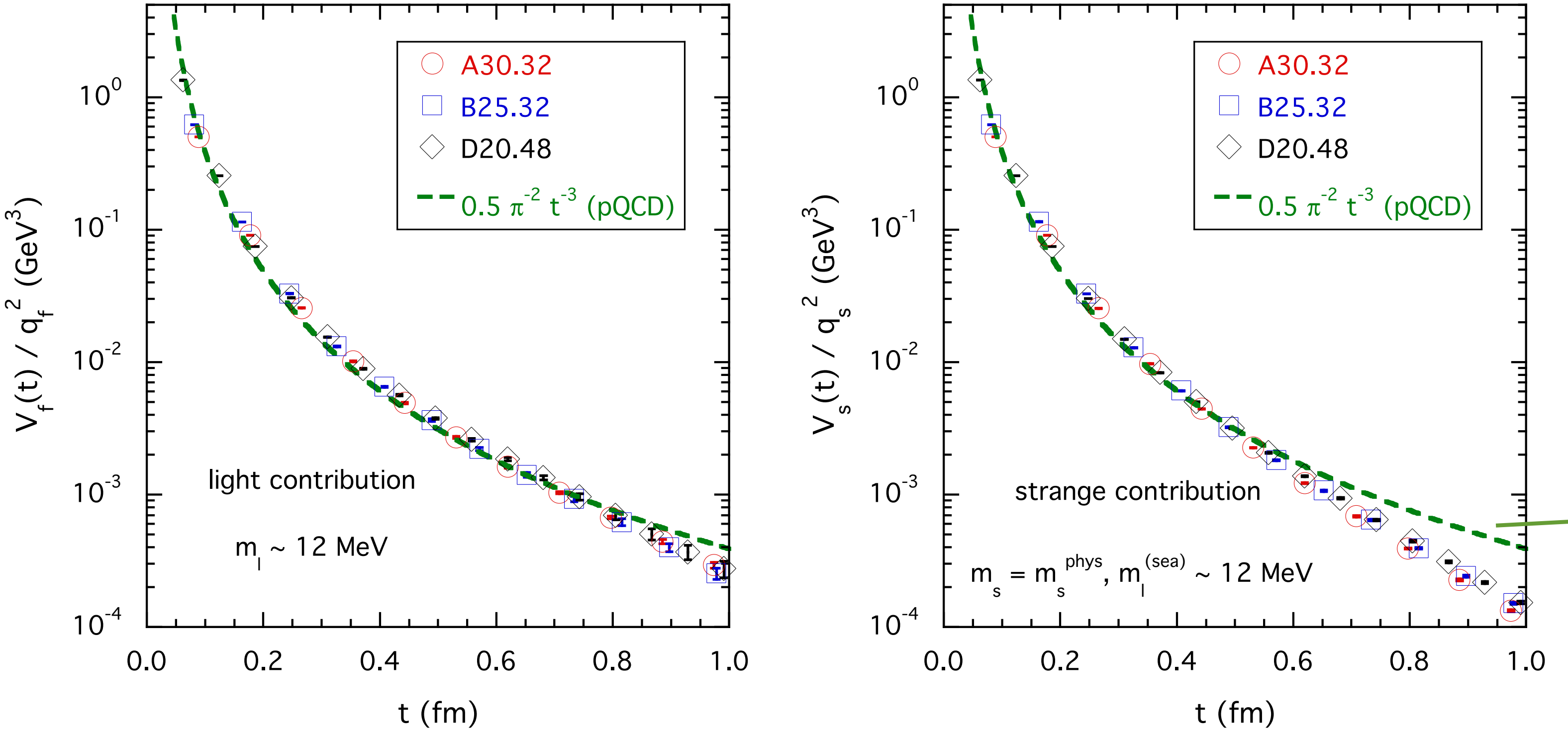
- it does not contain ϱ-ω mixing

OK for an isosymmetric lattice setup

- (isovector) π-π contribution is known to be OK for low-lying states (around the ϱ-resonance) for large time distances (t > 1 fm)

- we want a representation of the correlator also **at low and intermediate time distances**

* in JHEP ’17 we observed the onset of **quark-hadron duality** [SVZ ’79]



massless limit at LO

* the matching with pQCD (including quark mass effects) is present up to t ~ 1 fm

***** the sum of the contributions of intermediate and highly excited states is dual to pQCD *****

our representation: $V^{(ud)}(t) = V_{dual}(t) + V_{\pi\pi}(t)$

$$V_{dual}(t) \to \frac{1}{24\pi^2} \int_{s_{dual}}^{\infty} ds\, \sqrt{s}\, e^{-\sqrt{s}t} R^{pQCD}(s) = \frac{5}{9}\frac{1}{8\pi^2} \int_{s_{dual}}^{\infty} ds\, \sqrt{s}\, e^{-\sqrt{s}t} \sqrt{1-\frac{4m_{ud}^2}{s}} \left(1+\frac{2m_{ud}^2}{s}\right) + O(\alpha_s)$$

$$= \frac{5}{9}\frac{s_{dual}^{3/2}}{2\pi^2} \left\{ \frac{1}{\left(\sqrt{s_{dual}}t\right)^3} e^{-\sqrt{s_{dual}}t} \left(1+\sqrt{s_{dual}}t+\frac{1}{2}s_{dual}t^2\right) + O\left(\frac{m_{ud}^4}{s_{dual}^2}\right) \right\}$$

s_{dual} = effective threshold = $(M_\rho + E_{dual})^2$ with $E_{dual} \sim \Lambda_{QCD}$

introduce a multiplicative parameter: $R_{dual} = 1 + O(\alpha_s) + O(a^2)$ (see later on)

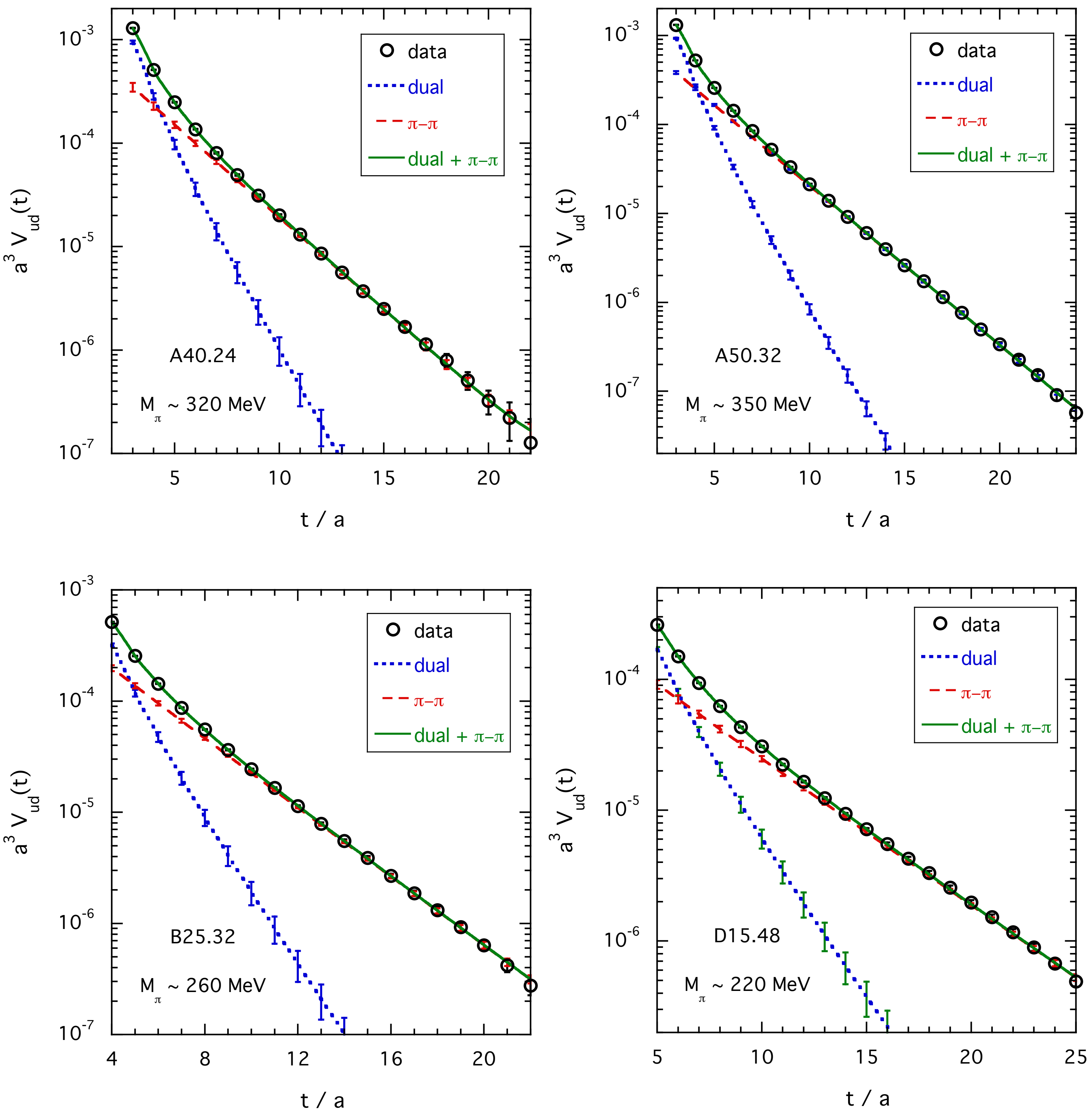
$$V_{dual}(t) = \frac{5}{18\pi^2} \frac{R_{dual}}{t^3} e^{-(M_\rho + E_{dual})t} \left[1 + (M_\rho + E_{dual})t + \frac{1}{2}(M_\rho + E_{dual})^2 t^2 \right]$$

* a total of four parameters: M_ρ and $g_{\rho\pi\pi}$ in the π-π term

R_{dual} and E_{dual} in the dual term

more precisely

M_ρ / M_π and E_{dual} / M_π



accurate reproduction of the vector (ud) correlators for all the ETMC ensembles for t ≥ 0.2 fm

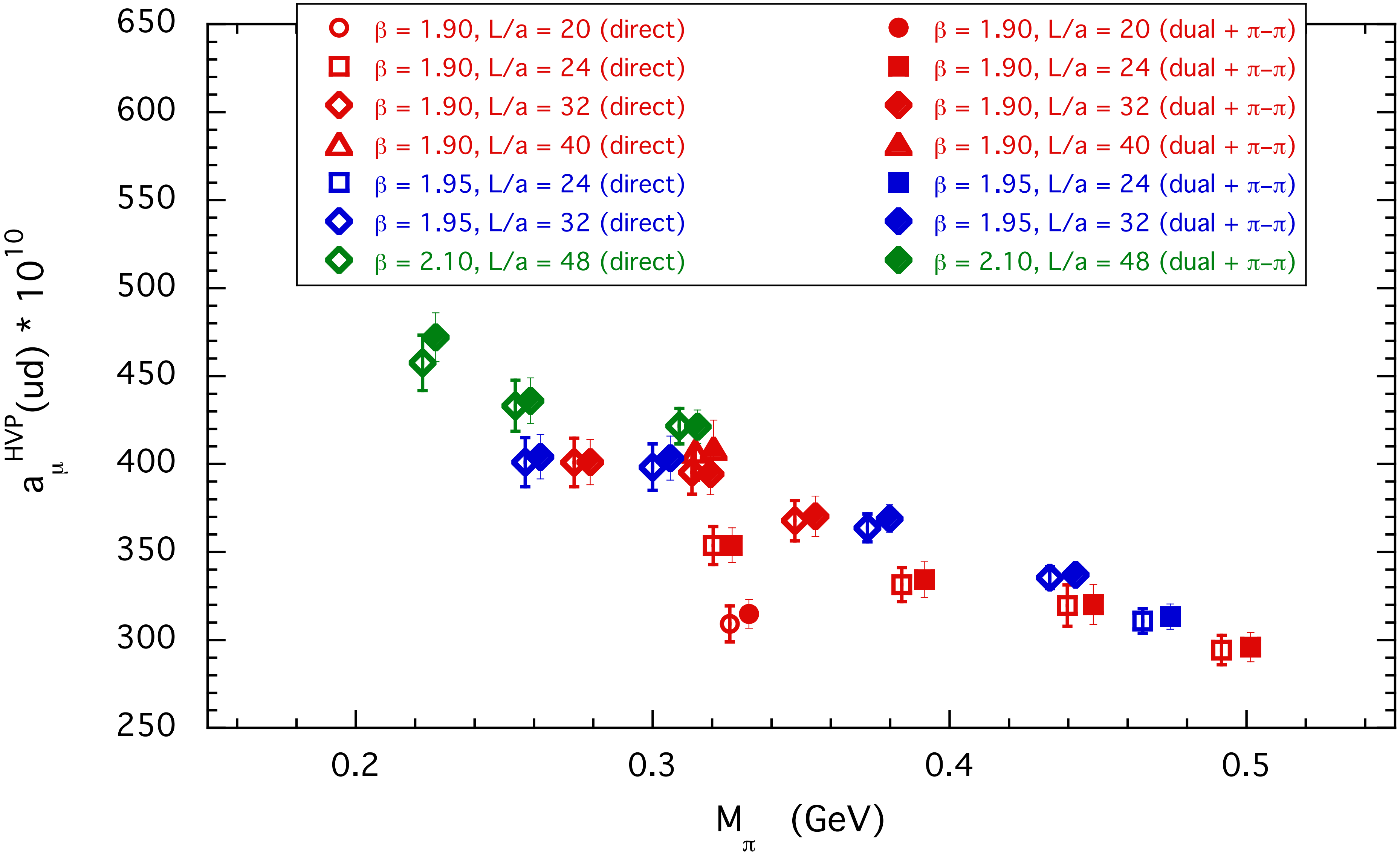
(χ^2/d.o.f. < 1)

fitting procedure entirely in lattice units

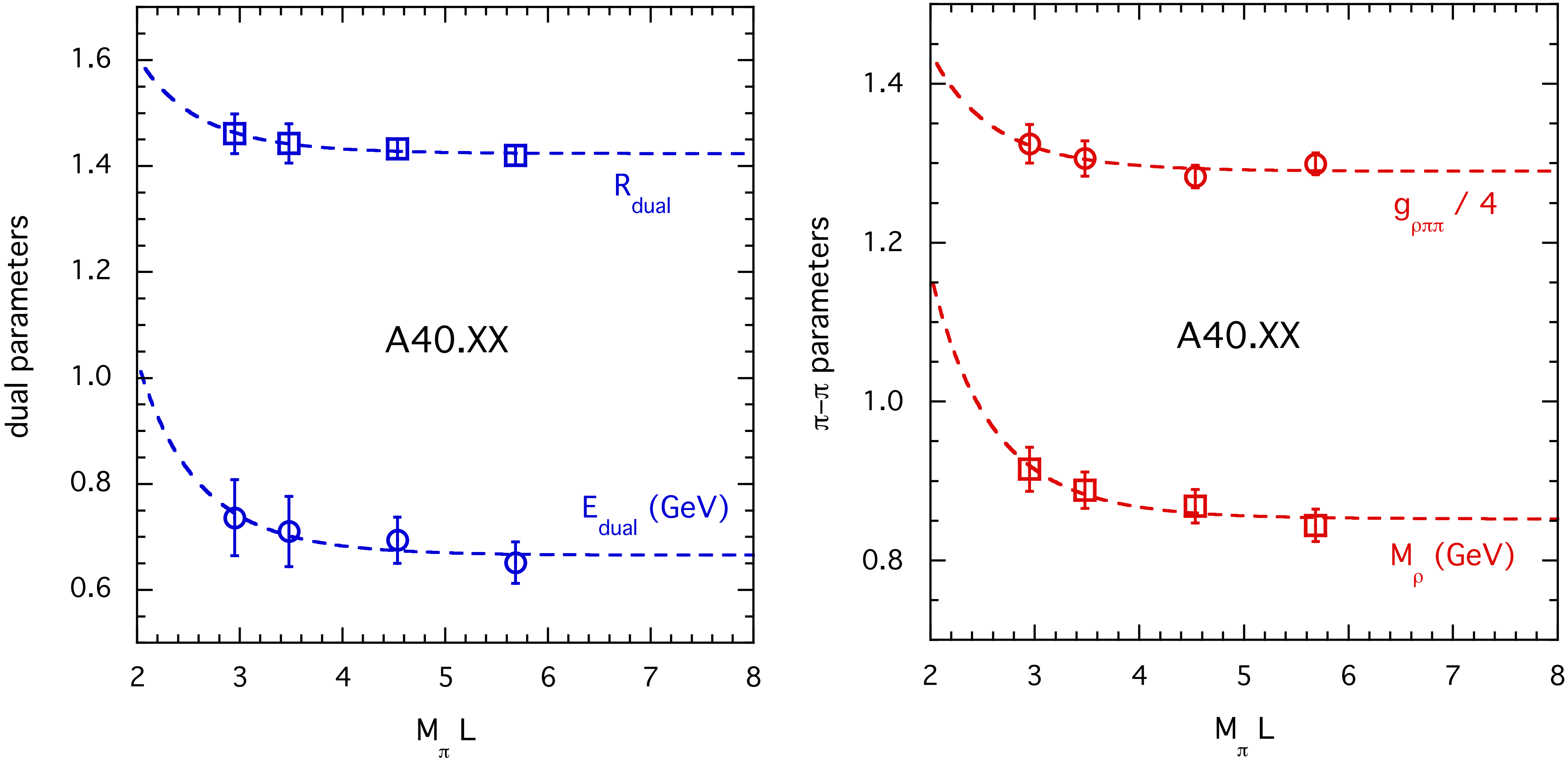
knowledge of the lattice spacing not required

4 energy levels for the π-π contribution are sufficient for all the ETMC ensembles

- nice agreement (within 1σ) for $a_\mu^{HVP}(ud)$ calculated using either the lattice points of the vector correlator (direct) or its dual + π-π representation



ETMC ensembles A40.XX: $M_\pi \sim 320$ MeV and a ~ 0.09 fm



dashed lines: $P_i = P_i^{(L=\infty)} \left[1 + F_{P_i} \frac{M_\pi^2}{16\pi^2 f_\pi^2} \frac{e^{-M_\pi L}}{(M_\pi L)^{3/2}} \right]$ $\quad P_i = R_{dual}, E_{dual}, M_\rho, g_{\rho\pi\pi}$

leading exponential correction in ChPT for M_π^2

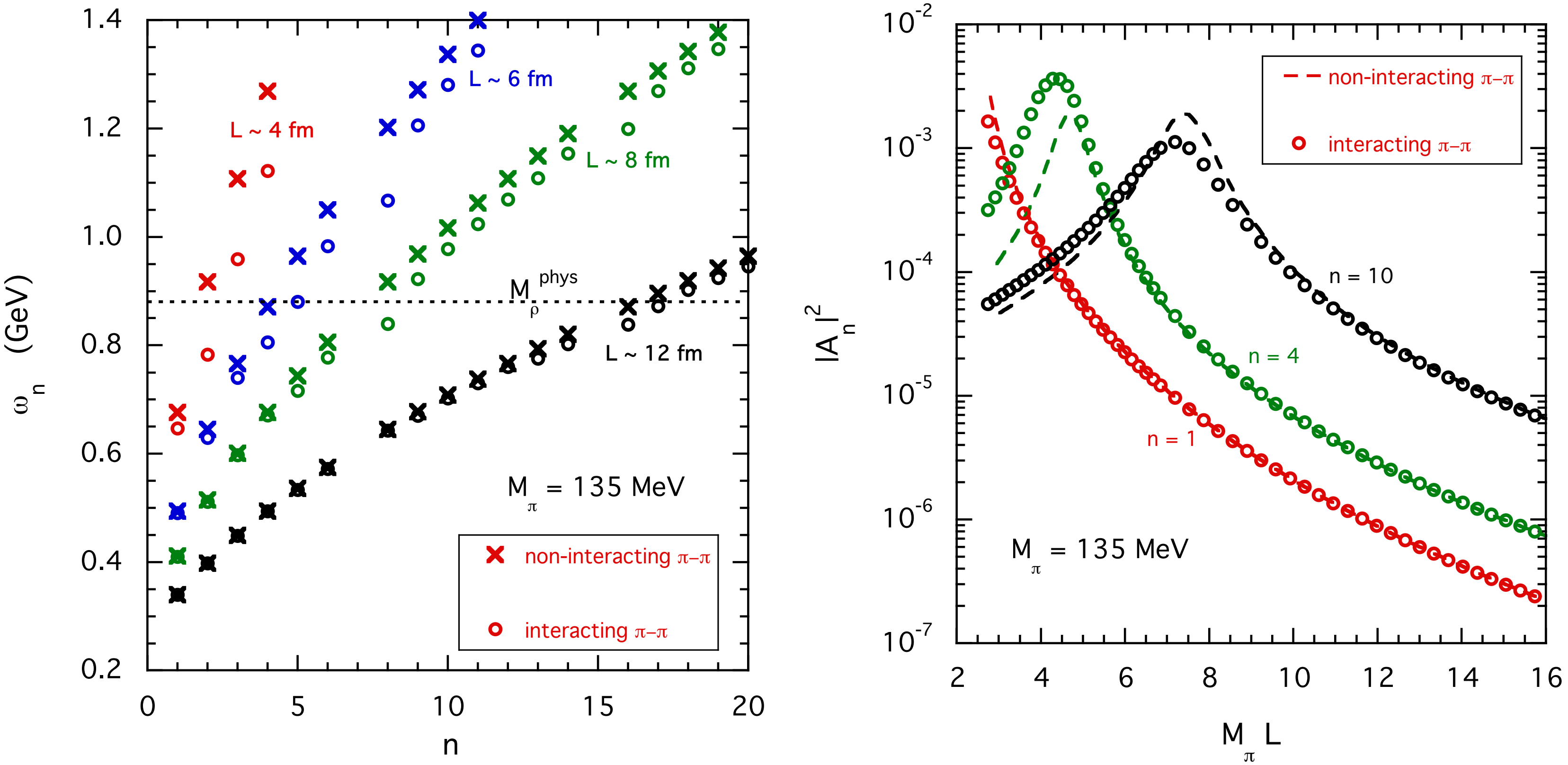
*** infinite volume limit:**

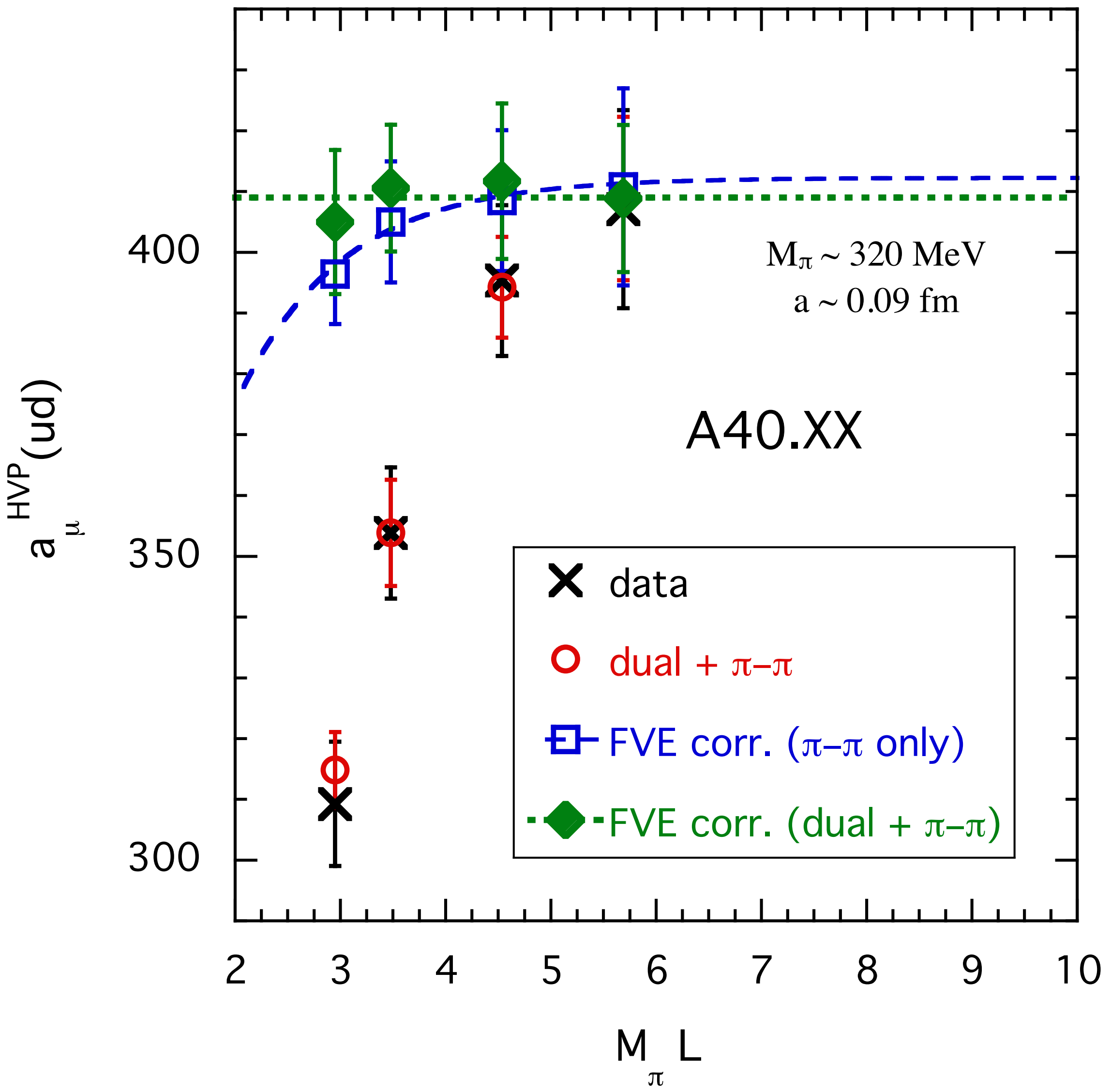
$$V_{dual}(t) \xrightarrow[L\to\infty]{} \frac{5}{18\pi^2}\frac{R_{dual}^{(L=\infty)}}{t^3} e^{-\left[M_\rho^{(L=\infty)}+E_{dual}^{(L=\infty)}\right]t} \left\{1+\left[M_\rho^{(L=\infty)}+E_{dual}^{(L=\infty)}\right]t+\frac{1}{2}\left[M_\rho^{(L=\infty)}+E_{dual}^{(L=\infty)}\right]^2 t^2\right\}$$

$$V_{\pi\pi}(t;L)=\sum_n \nu_n |A_n|^2 e^{-\omega_n t} \xrightarrow[L\to\infty]{} \frac{1}{48\pi^2}\int_{2M_\pi}^{\infty} d\omega\,\omega^2\left(1-\frac{4M_\pi^2}{\omega^2}\right)\left|F_\pi^{(L=\infty)}(\omega)\right|^2 e^{-\omega t}$$

[Meyer ’11]

evaluated with $M_\pi^{(L=\infty)}$, $M_\rho^{(L=\infty)}$ and $g_{\rho\pi\pi}^{(L=\infty)}$





FVE correction

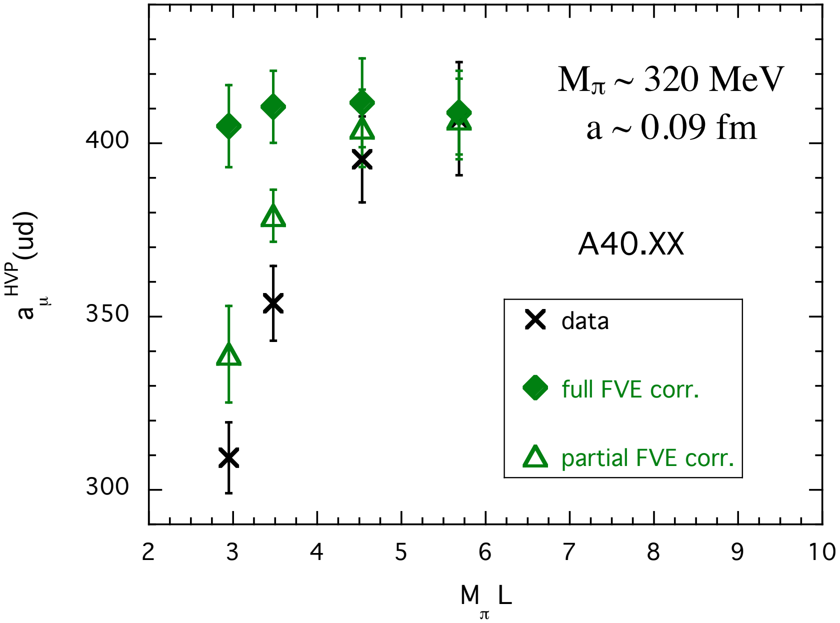
$$a_\mu^{HVP}(\infty) = a_\mu^{HVP}(L) + \Delta_{FVE} a_\mu^{HVP}$$

$\Delta_{FVE} a_\mu^{HVP}$ from π-π contribution only

$\Delta_{FVE} a_\mu^{HVP}$ from dual + π-π

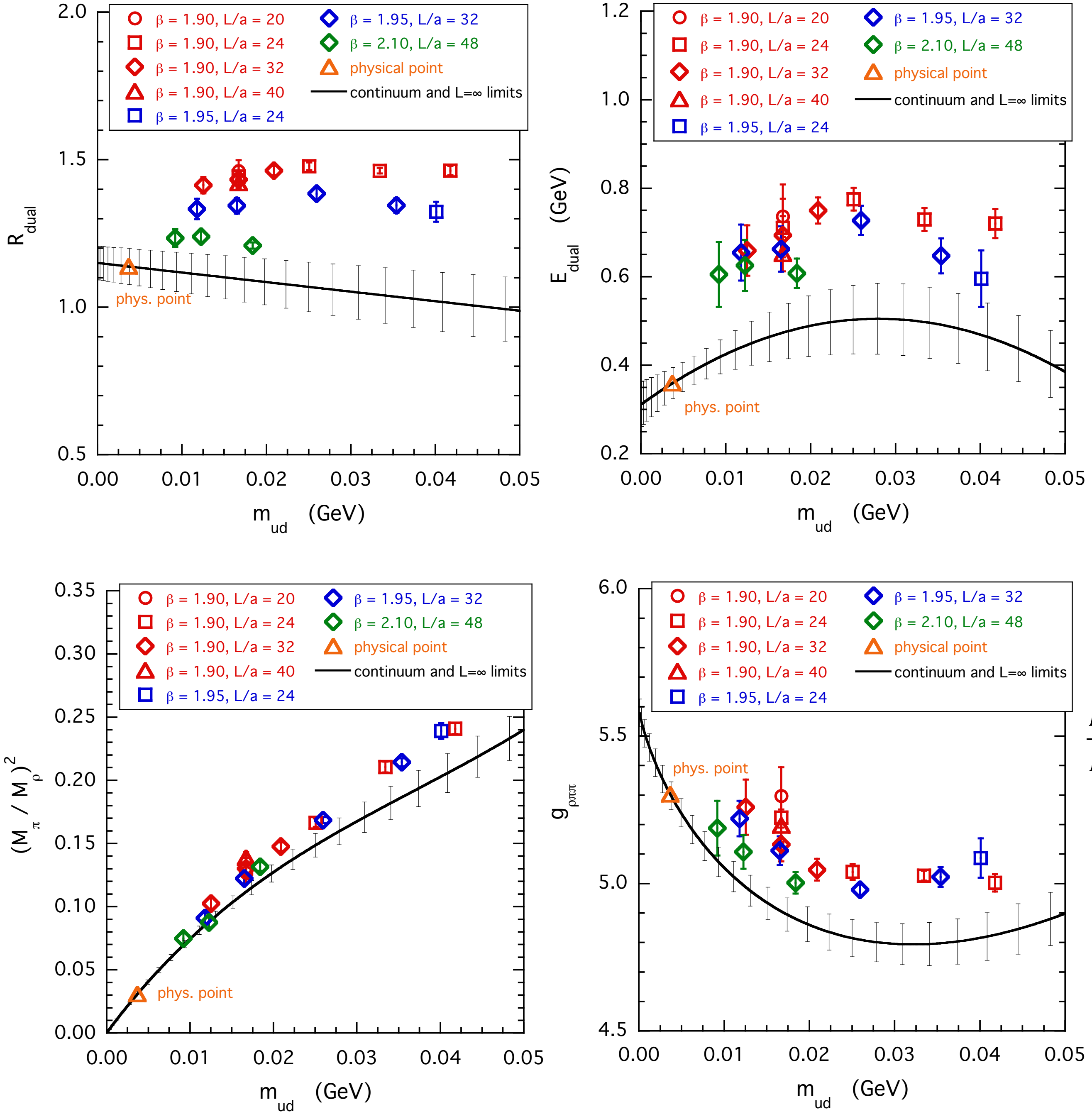
- large corrections from π-π contribution, but some residual FVE is still present

- no residual FVE using the dual + π-π representation



in order to correct properly for FVEs it is important to use in the (L=∞) formula the values of the pion mass and of the dual and π-π parameters estimated in the infinite volume

FVEs on M_π have been analyzed accurately in NPB ’14



combined fits

$$R_{dual} = R_0\left[1+R_1 m_{ud}+R_a a^2+R_{am} a^2 m_{ud}\right] \cdot\left[1+R_{FVE} M_\pi^2 \frac{e^{-M_\pi L}}{\left(M_\pi L\right)^{3/2}}\right]$$

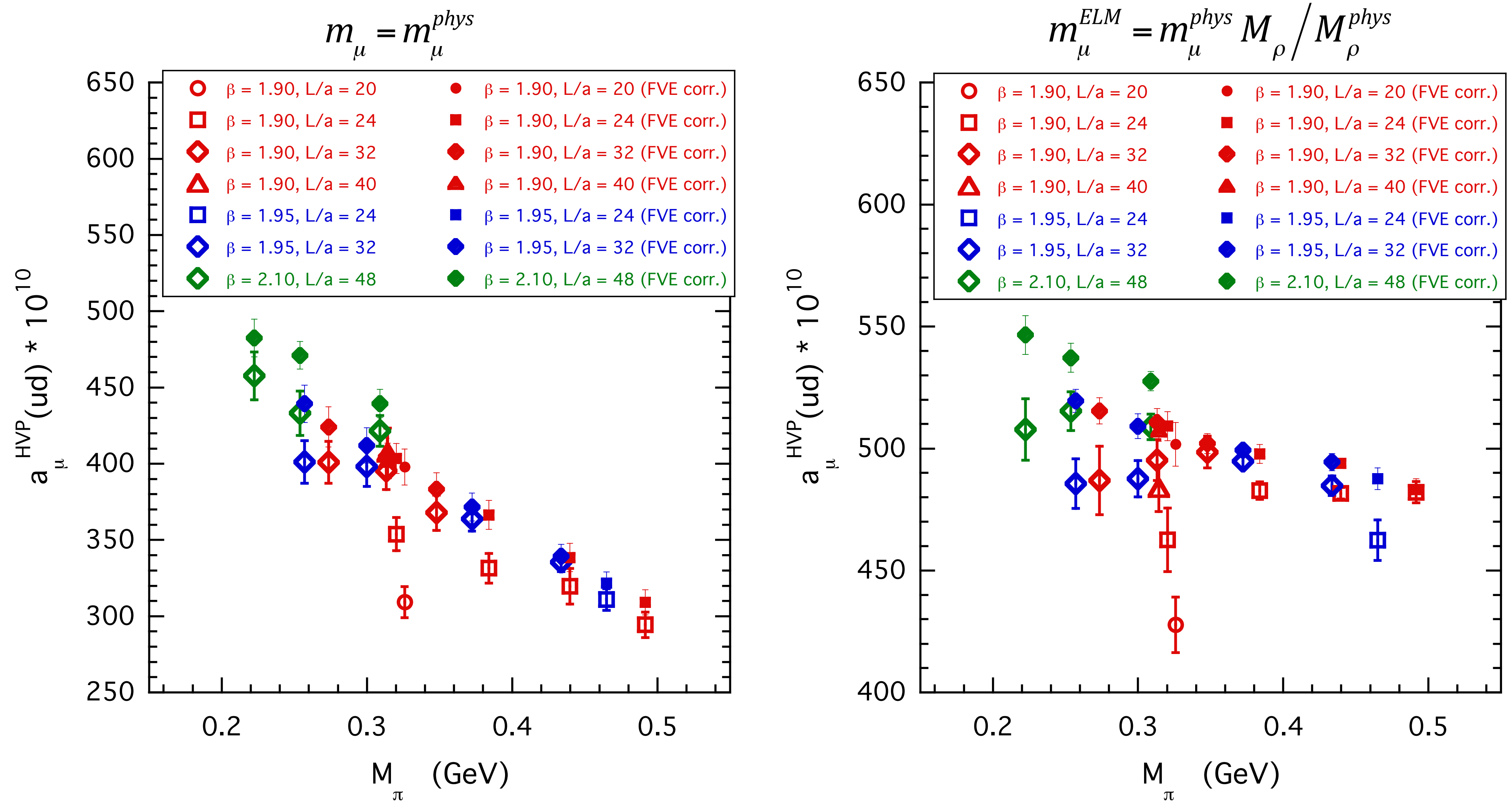
$$E_{dual} = E_0\left[1+E_1 m_{ud}+E_2 m_{ud}^2+E_a a^2\right] \cdot\left[1+E_{FVE} M_\pi^2 \frac{e^{-M_\pi L}}{\left(M_\pi L\right)^{3/2}}\right]$$

$$\frac{M_\pi^2}{M_\rho^2} = V_0 m_{ud}\left[1+V_1 m_{ud}+\frac{2B_0 m_{ud}}{16\pi^2 f_0^2}\log\frac{2B_0 m_{ud}}{16\pi^2 f_0^2}+V_2 m_{ud}^2+V_a a^2\right]$$

$$g_{\rho\pi\pi} = g_0\left[1+g_1 m_{ud}+2\frac{2B_0 m_{ud}}{16\pi^2 f_0^2}\log\frac{2B_0 m_{ud}}{16\pi^2 f_0^2}+g_a a^2\right] \cdot\left[1+g_{FVE} M_\pi^2 \frac{e^{-M_\pi L}}{\left(M_\pi L\right)^{3/2}}\right]$$

- for each ETMC ensemble the values of the four parameters in the infinite volume limit can be obtained and used to evaluate the FVEs on $a_\mu^{HVP}(ud)$

FVE correction: $a_\mu^{HVP}(\infty) = a_\mu^{HVP}(L) + \left[a_\mu^{HVP}(\infty) - a_\mu^{HVP}(L)\right]_{dual+\pi-\pi}$

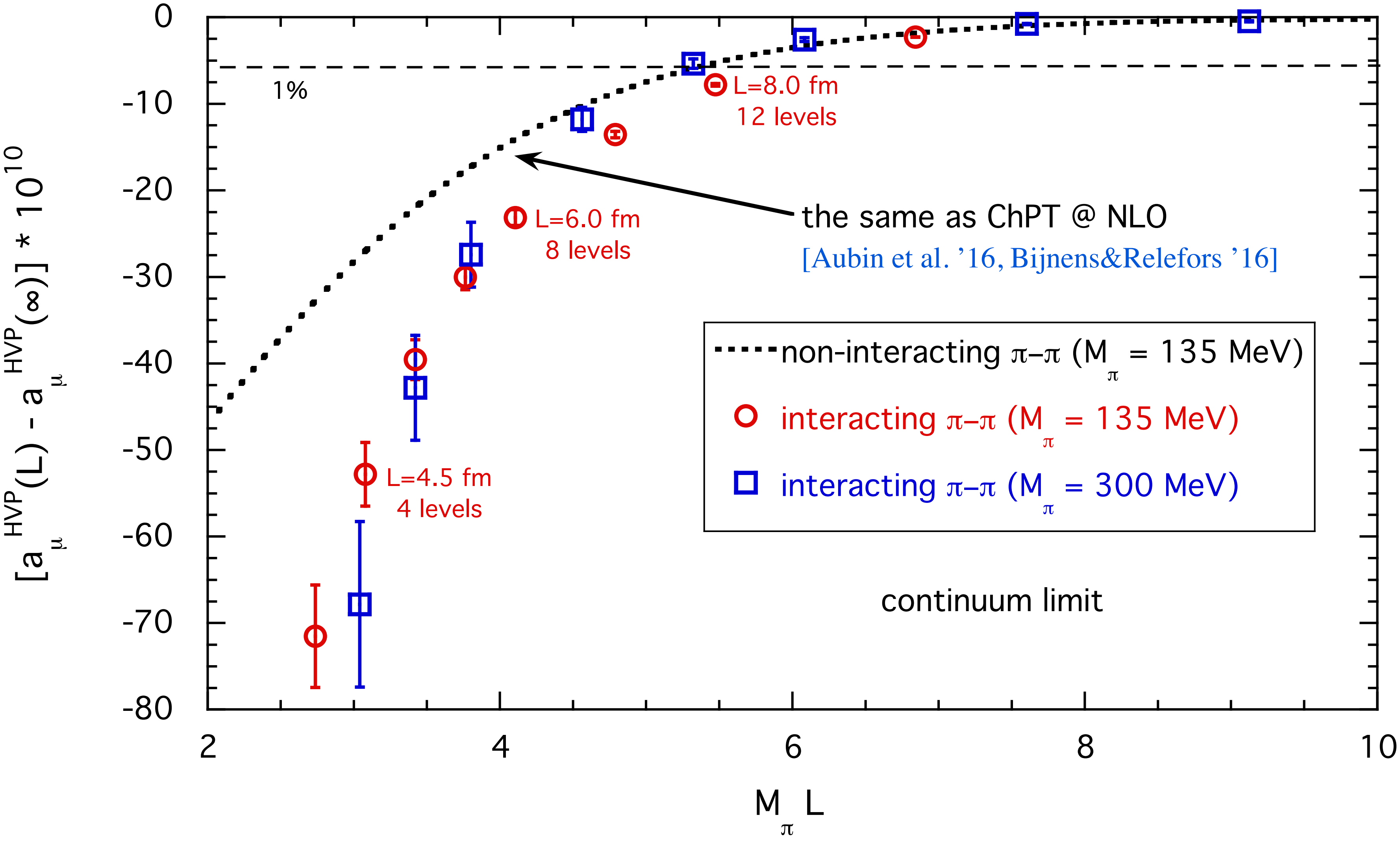


$m_\mu = m_\mu^{phys}$

$m_\mu^{ELM} = m_\mu^{phys} M_\rho / M_\rho^{phys}$

- after applying the FVE correction the m_{ud} dependence of $a_\mu^{HVP}(ud)$ is more pronounced (both with and without the ELM procedure for the muon mass)

FVE correction @ $a^2 \to 0$



[Francis et al. '13]

non-interacting π-π: $$V_{\pi\pi}^{(L)}(t) - V_{\pi\pi}^{(\infty)}(t) = \frac{M_\pi^4}{3\pi^2} t \sum_{\vec{n} \neq 0} \left\{ \frac{K_2\left[M_\pi \sqrt{L^2\vec{n}^2 + 4t^2}\right]}{M_\pi^2\left(L^2\vec{n}^2 + 4t^2\right)} - \frac{1}{M_\pi L|\vec{n}|} \int_1^\infty dy\, K_0\left[M_\pi y \sqrt{L^2\vec{n}^2 + 4t^2}\right] \sinh\left[M_\pi L|\vec{n}|(y-1)\right] \right\}$$

interacting π-π: dual + π-π representation [note that $\Delta a_\mu^{HVP}(L)$ depends approximately on $M_\pi L$ only]

chiral and continuum limit extrapolations

*** in the chiral limit $m_{ud} \rightarrow 0$ the polarization function $\Pi_R(Q^2)$ is not analytic at $Q^2=0$ and $a_\mu^{HVP}(ud)$ is logarithmically divergent ***

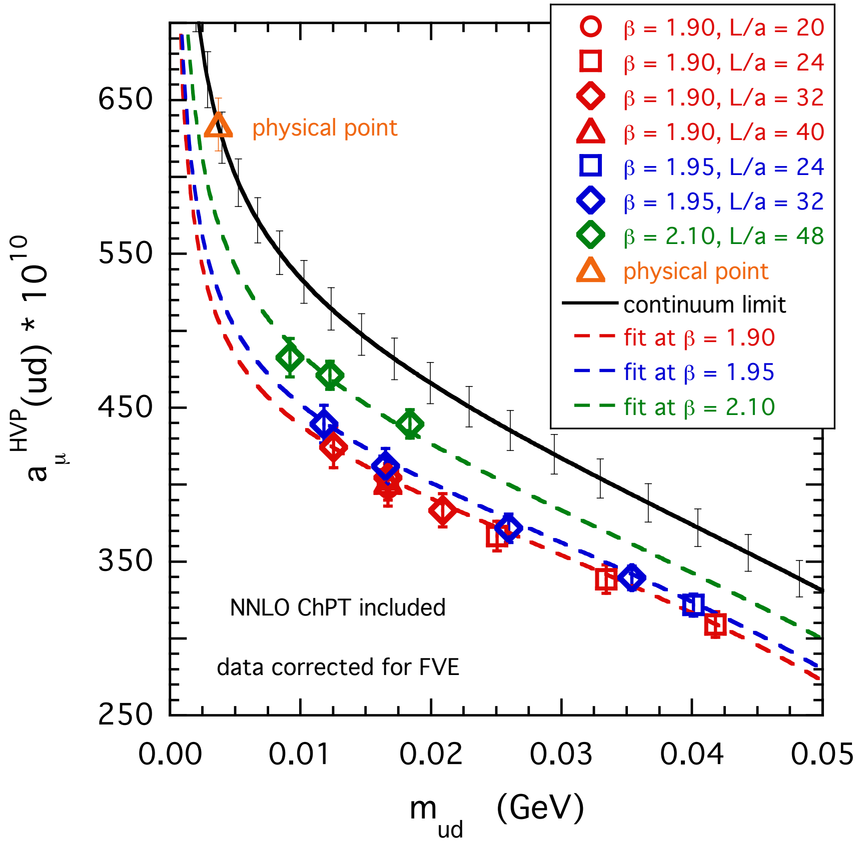
$$a_\mu^{HVP}(ud) = \left\{ \left[a_\mu^{HVP} \right]^{NLO} + \left[a_\mu^{HVP} \right]^{NNLO}_{L_9, C_{93}} + A_0 + A_1 m_{ud} \right\} \left(1 + D_0 a^2 + D_1 a^2 m_{ud} \right)$$

log(m_{ud}) independent on LECs

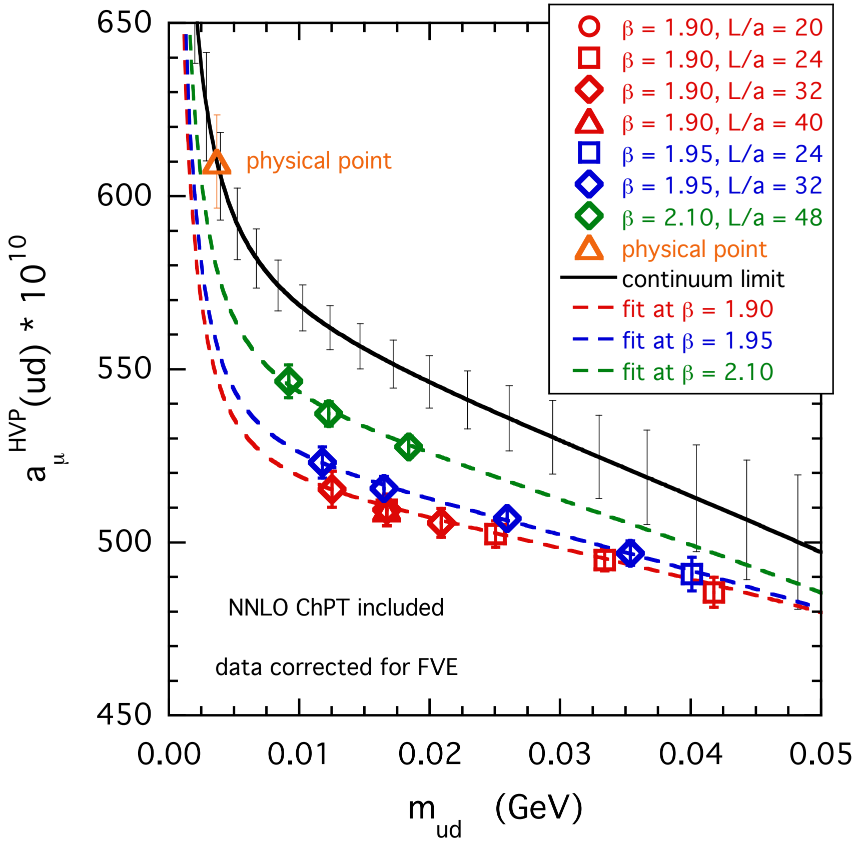
dependence on two LECs

Golowitch&Kambor '95, Amoros et al '00, ...
Bijnens&Relefors '16, Golterman at al. '17

$m_\mu = m_\mu^{phys}$



$m_\mu^{ELM} = m_\mu^{phys} M_\rho / M_\rho^{phys}$



our results:

$$L_9(0.77\,GeV) = 0.00273(143)$$
$$C_{93}(0.77\,GeV) = -0.0136(20)\,GeV^{-2}$$

Goltermann et al. '17:

$$L_9(0.77\,GeV) = 0.00593(43)$$
$$C_{93}(0.77\,GeV) = -0.0154(4)\,GeV^{-2}$$

* we have tried also a fit with free logs: $a_\mu^{HVP}(ud) = \left(A_0 + A_0^{\log} \log m_{ud}\right)\left(1 + A_1 m_{ud} + A_1^{\log} m_{ud} \log m_{ud}\right)\left(1 + D_0 a^2 + D_1 a^2 m_{ud}\right)$

results for $a_\mu^{HVP}(ud)$ $\left(\text{in units of } 10^{-10}\right)$

	including NLO ChPT	including NNLO ChPT	free logs
$m_\mu = m_\mu^{phys}$	625.3 (9.1)	635.8 (13.1)	613.1 (13.2)
$m_\mu^{ELM} = m_\mu^{phys} \dfrac{M_\rho}{M_\rho^{phys}}$	615.6 (7.2)	610.0 (13.4)	616.4 (17.5)

"ETMC" average

$$\langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 + \frac{1}{N} \sum_{i=1}^{N} \left(x_i - \langle x \rangle\right)^2$$

"ETMC" average: $a_\mu^{HVP}(ud) = 619.4\,(12.7)_{stat+fit}\,(8.7)_{syst} \cdot 10^{-10}$

$= 619.4\,(15.4) \cdot 10^{-10}$

$\searrow$ $(6.8)_{chir.}\,(5.4)_{disc.}\,(...)_{FVE}$

to be estimated, but
expected to be small

ETMC $N_f = 2$ @ M_π^{phys}

$a_\mu^{HVP}(ud) = 552(39) \cdot 10^{-10}$

$a \sim 0.091$ fm, L ~ 4.4 fm

$M_\pi L \sim 3.0$

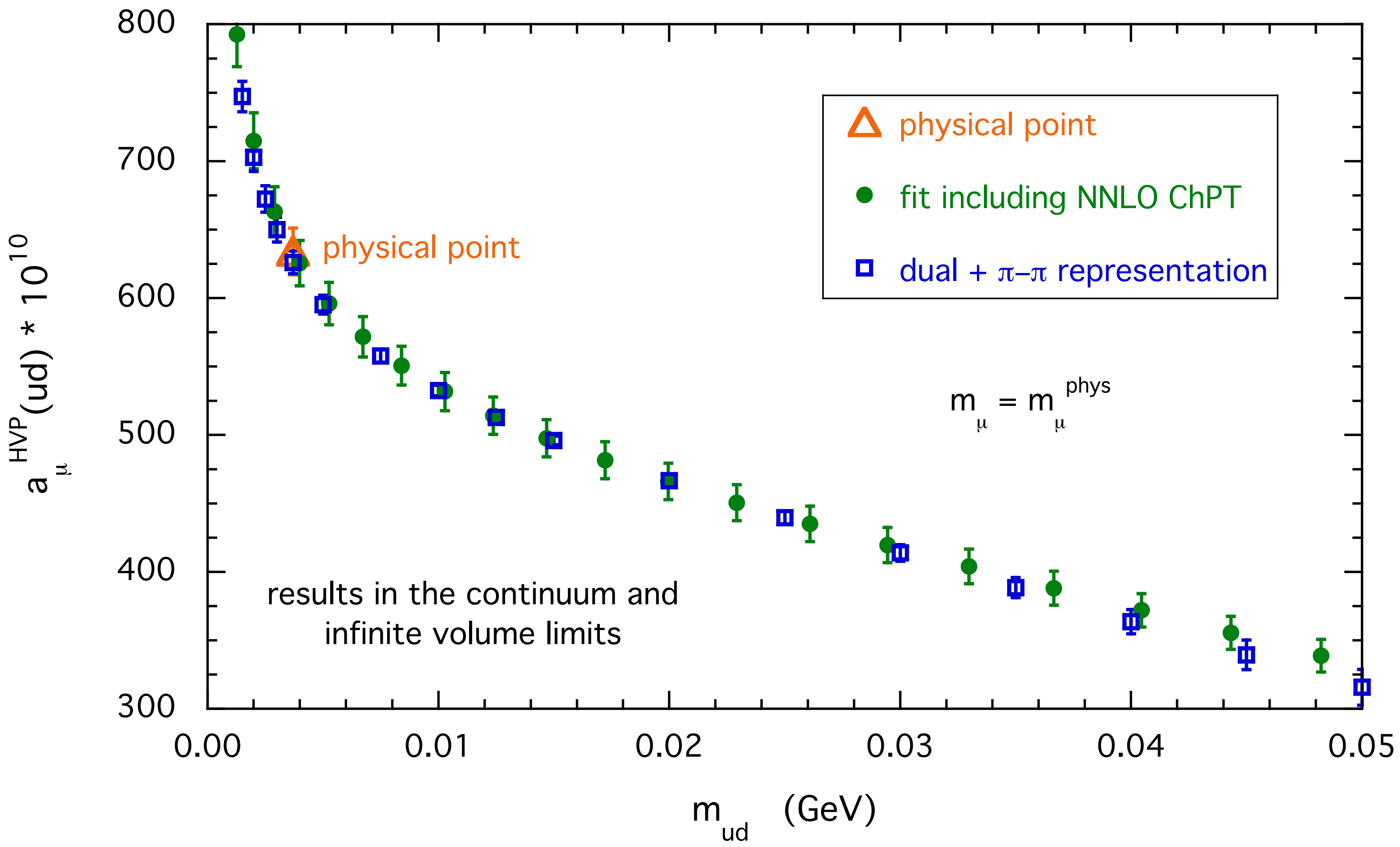
$$\xrightarrow[(\sim 11\%)]{\text{FVE correction}}$$

$a_\mu^{HVP}(ud) \sim 610(40) \cdot 10^{-10}$

an alternative route is based on our (dual + π-π) representation of the vector correlator

take the infinite volume formula and use the (dual + π-π) parameters evaluated in the limits L → ∞ and $a^2 \to 0$ as a function of the light-quark mass

- removal of lattice artifacts directly on the vector correlator
- light-quark mass dependence of the vector correlator



$$a_\mu^{HVP}(ud) = 635.8(16.1) \cdot 10^{-10}$$

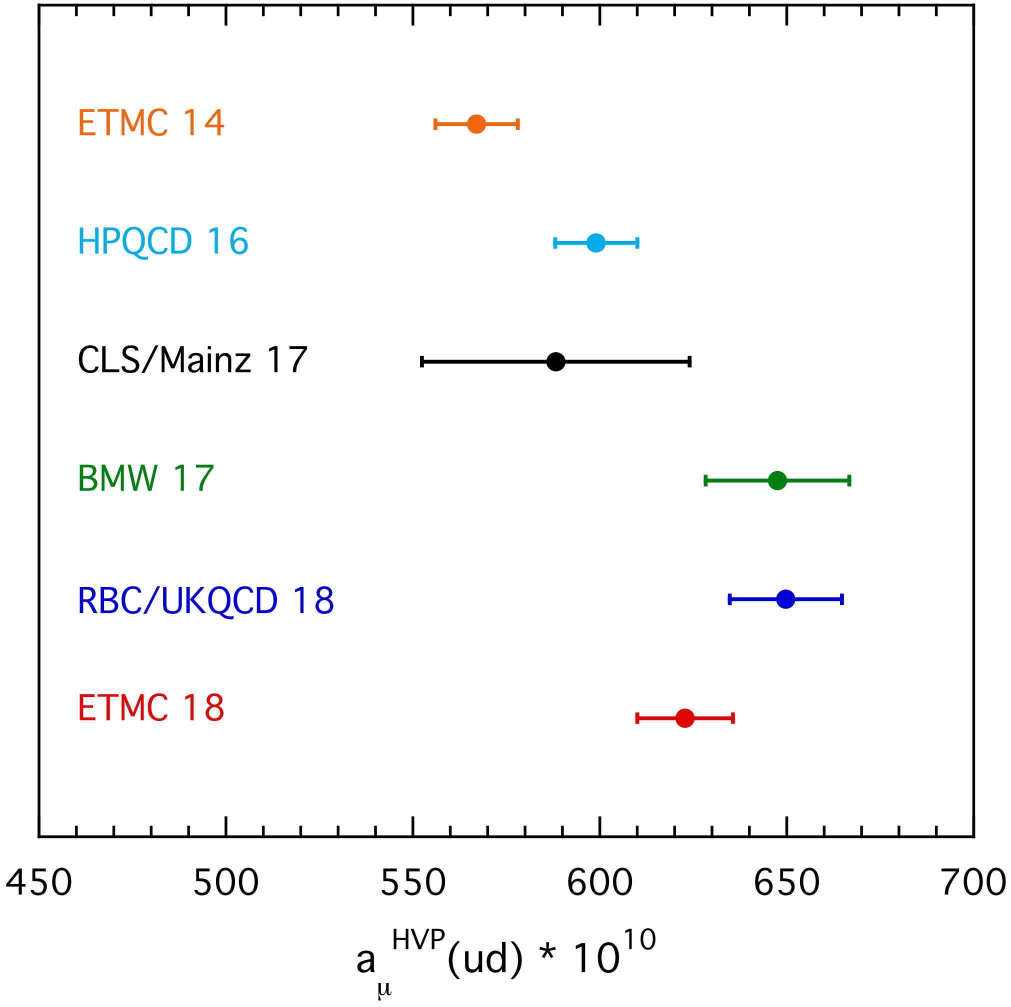
$$a_\mu^{HVP}(ud) = 626.2(8.2) \cdot 10^{-10}$$

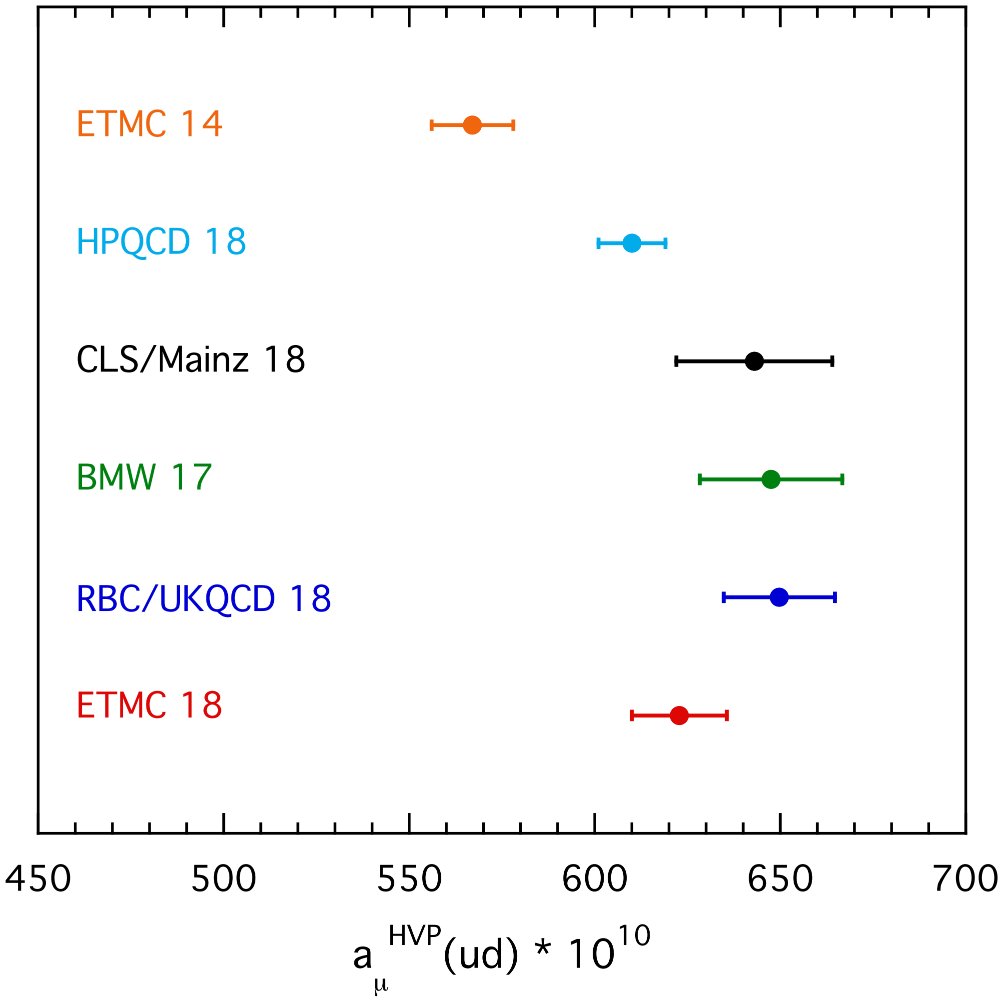
the blue points do not know anything about ChPT

note the sensitivity to m_{ud} around the physical point

final ETMC estimate: $a_\mu^{HVP}(ud) = 622.8(12.8) \cdot 10^{-10}$

u- and d-quark connected terms only


up to two days ago ...


now ...

ETMC 18 differs by 1.4σ from HPQCD 16 and RBC/UKQCD 18, and by 1.1σ from BMW 17

leading-order IB corrections δa_μ^{HVP}

[ETMC '17]

* RM123 approach: the path integral is expanded at leading order in both $(m_d - m_u) / \Lambda_{QCD}$ and α_{em}

$$\delta a_\mu^{HVP} = \delta a_\mu^{HVP}(QCD) + \delta a_\mu^{HVP}(QED)$$

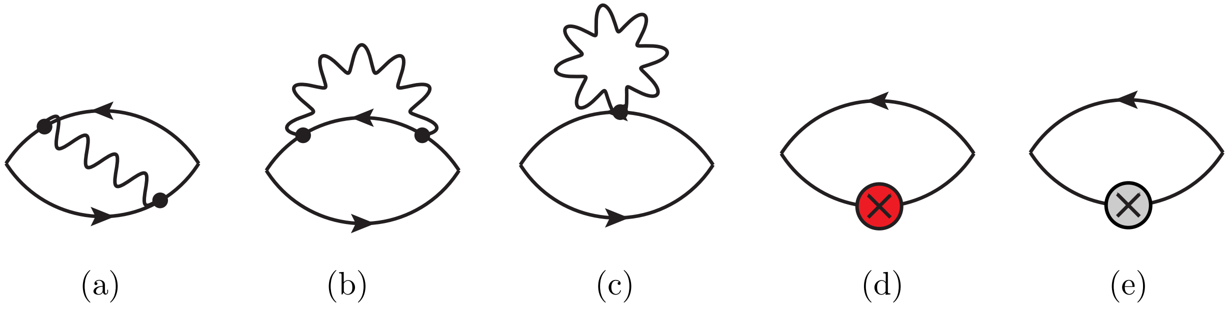
$$\delta a_\mu^{HVP}(QCD) = 4\alpha_{em}^2 \sum_{t=0}^{\infty} \tilde{f}(t)\, \delta V^{QCD}(t) \qquad \text{(u- and d-quark only)}$$

$$\delta a_\mu^{HVP}(QED) = 4\alpha_{em}^2 \sum_{t=0}^{\infty} \tilde{f}(t) \sum_{f=u,d,s,c} q_f^4 \delta V_f^{QED}(t) \qquad \text{(quark connected terms only)}$$

Strong IB: $\delta V^{QCD}(t) \rightarrow (m_d - m_u)\frac{Z_P}{3}\sum_{i=1,2,3}\sum_{\vec{x},y}\langle 0|T\left\{J_i^\dagger(\vec{x},t)\left[q_d^2\bar{\psi}_d\psi_d - q_u^2\bar{\psi}_u\psi_u\right]J_i(0)\right\}|0\rangle$ (scalar density insertions)

QED: $\delta V^{QED}(t) \rightarrow \delta V^{exch}(t) + \delta V^{self}(t) + \delta V^{tad}(t) + \delta V^{PS}(t) + \delta V^{S}(t) + \delta V^{ZA}(t)$ (for each quark flavor f)

e.m. effect on the RC of the local (twisted) vector current

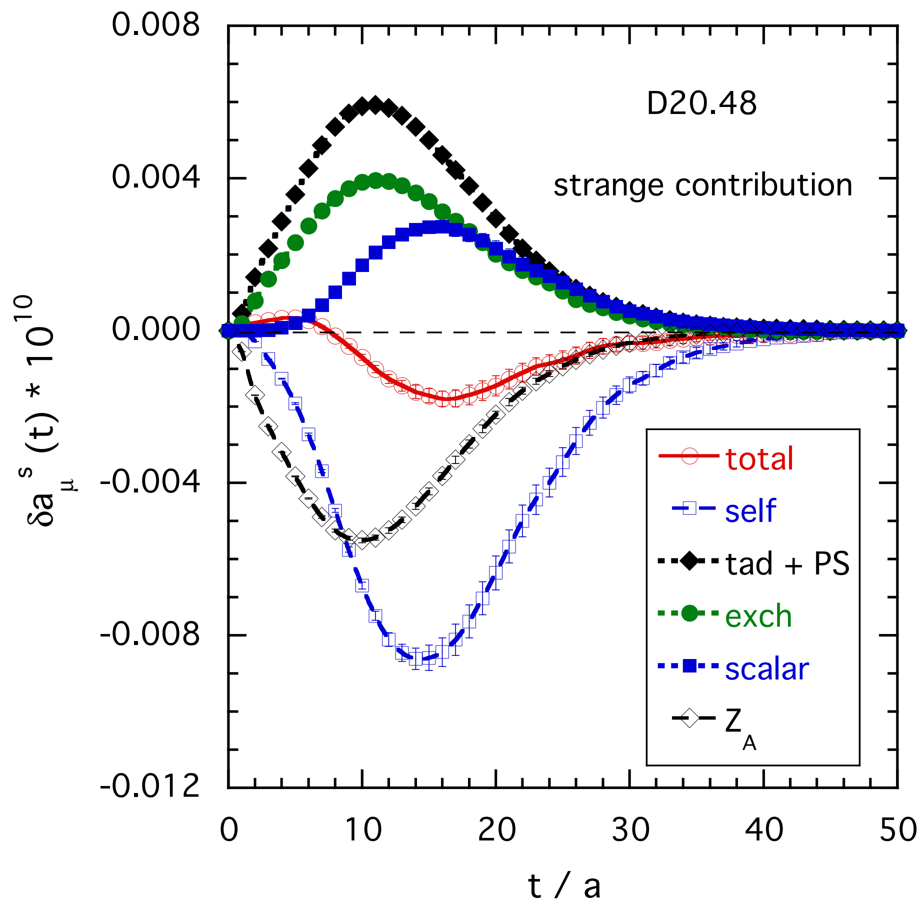


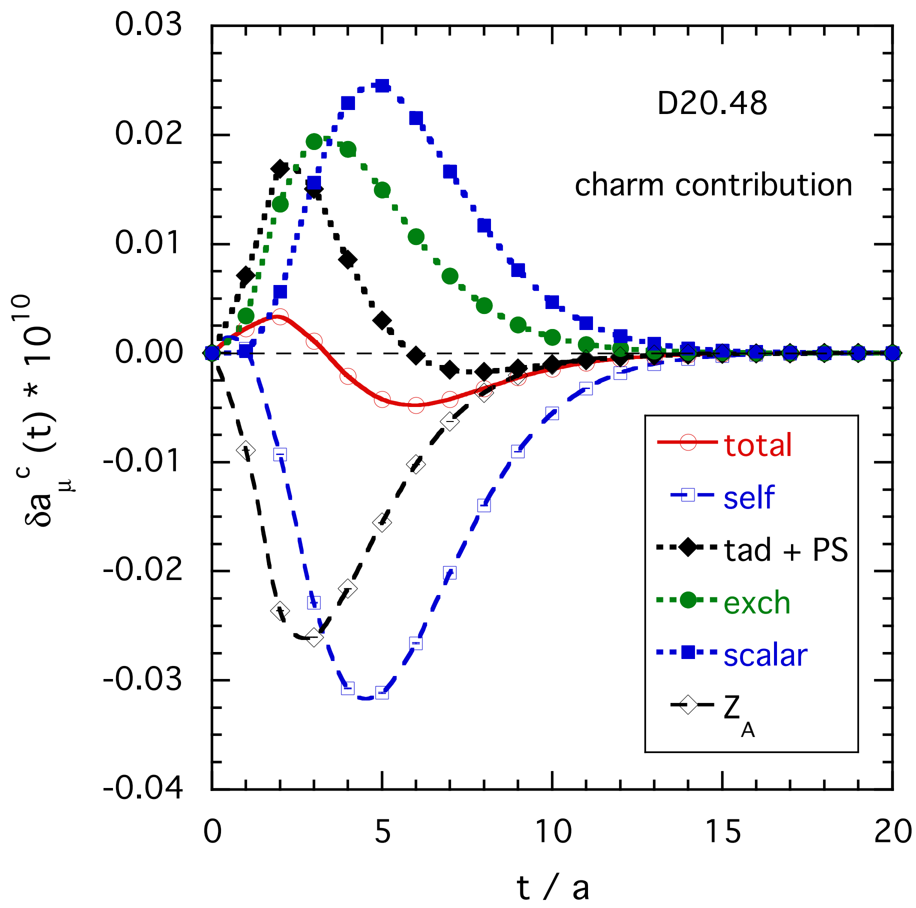
$$\boxed{q_f^{sea} = 0}$$

quenched QED

FIG. 8: *Fermionic connected diagrams contributing to the e.m. corrections to a_μ^{had}: exchange (a), self energy (b), tadpole (c), pseudoscalar (d) and scalar (e) insertions. Solid lines represent quark propagators.*

* strange and charm quark contributions: $\delta V^{QED}(t) \to \delta V^{exch}(t) + \delta V^{self}(t) + \delta V^{tad}(t) + \delta V^{PS}(t) + \delta V^{S}(t) + \delta V^{ZA}(t)$





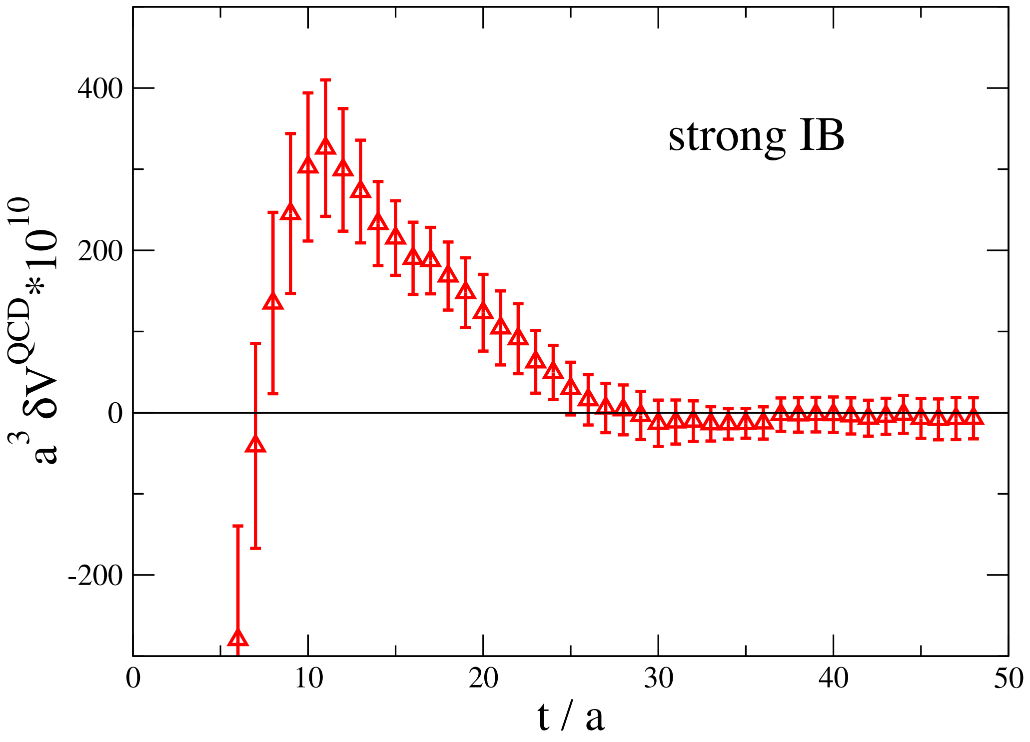
- contributions with different signs
- partial cancellations among the various terms
- total sum smaller than the separate terms

$@M_\pi^{phys}$

$$\delta a_\mu^{HVP}(s) = -0.018(11) \cdot 10^{-10}$$

$$\delta a_\mu^{HVP}(c) = -0.030(13) \cdot 10^{-10}$$

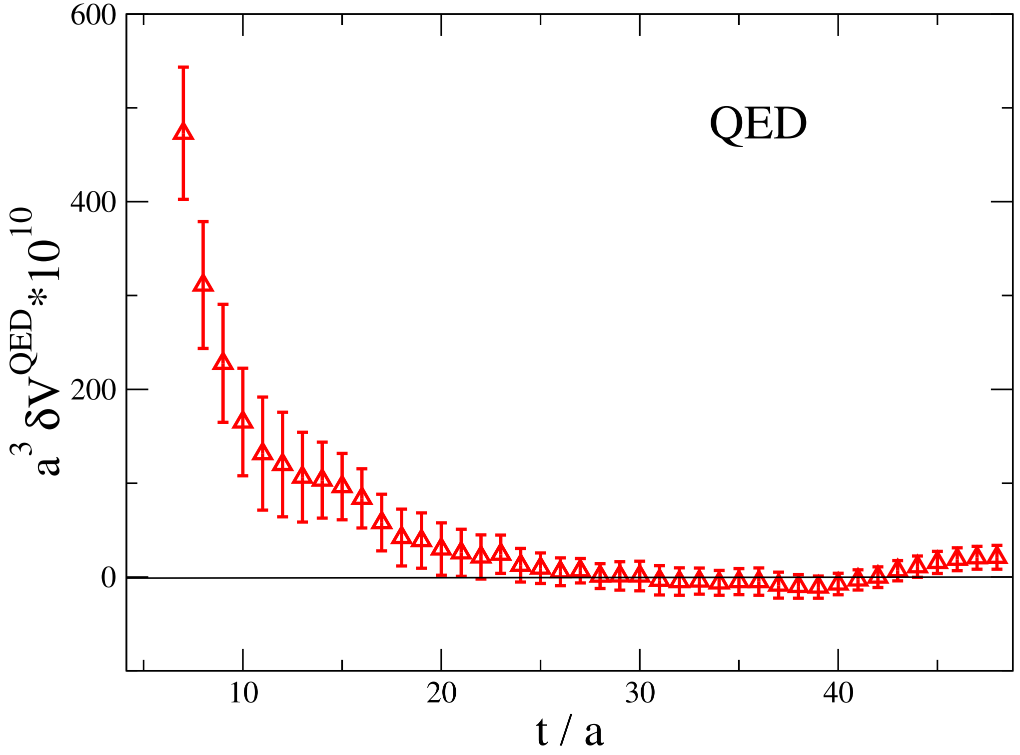
[ETMC ’17]



light-quark (connected) contributions

D20.48

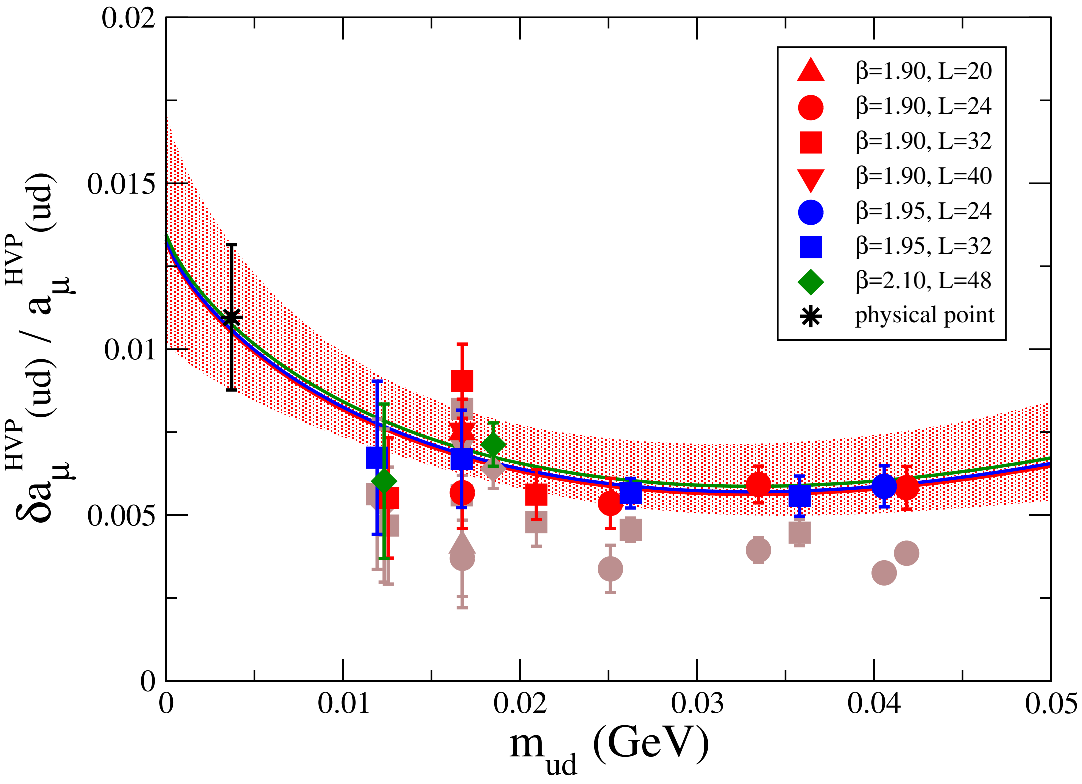
$M_\pi \sim 260$ MeV, a ~ 0.06 fm

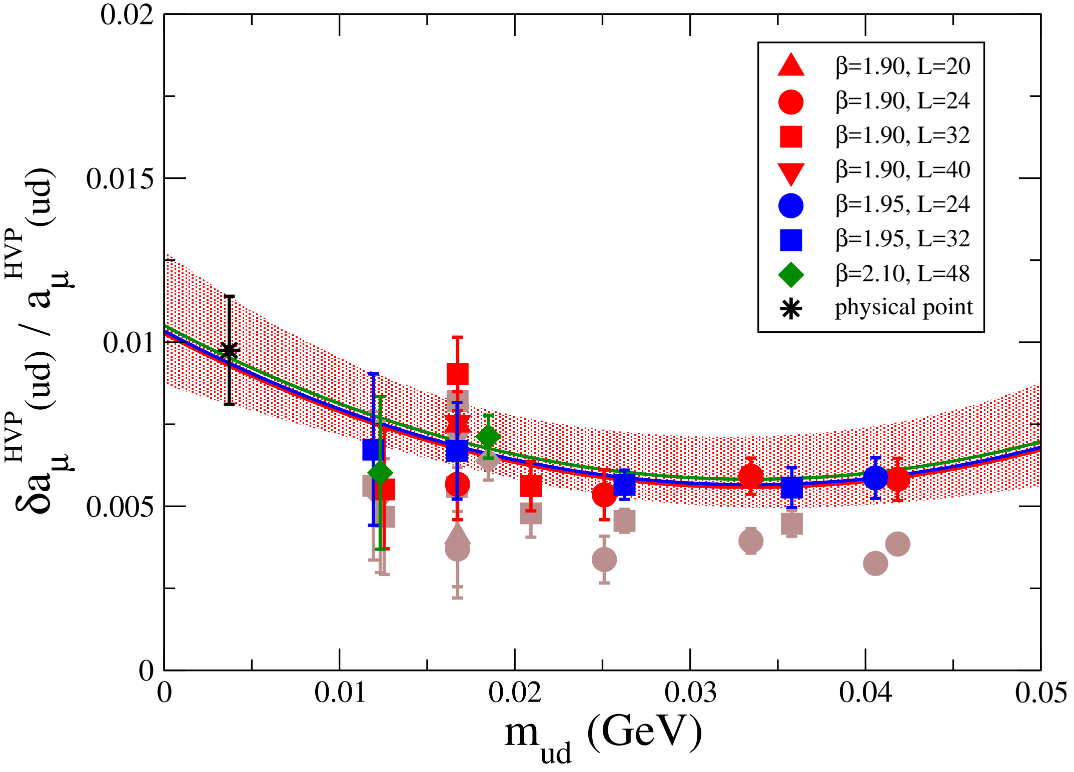


$$(m_d - m_u)(\overline{MS}, 2\,GeV) = 2.38(18)\,MeV$$

see RM123 '17 (arXiv:1704.0656)

strong IB is dominant





log fit

$$\frac{\delta a_\mu^{HVP}(ud)}{a_\mu^{HVP}(ud)} = \delta_0\left(1+\delta_1 m_{ud} + \delta_1^{\log} m_{ud}\log m_{ud} + Da^2 + FVE\right)$$

$$FVE = \frac{F}{L^3}$$ → expected in the case of neutral mesons with vanishing charge radius

[ETMC '17]

or

$$FVE = \tilde{F}e^{-M_\pi L}$$

quadratic fit

$$\frac{\delta a_\mu^{HVP}(ud)}{a_\mu^{HVP}(ud)} = \delta_0'\left(1+\delta_1' m_{ud} + \delta_2' m_{ud}^2 + D'a^2 + FVE'\right)$$

(quark connected only and qQED)

$$\boxed{\begin{array}{l} @M_\pi^{phys}: \quad \dfrac{\delta a_\mu^{HVP}(ud)}{a_\mu^{HVP}(ud)} = 0.011(3) \\ \qquad\qquad \delta a_\mu^{HVP}(ud) = 6.9(1.9)\cdot 10^{-10} \end{array}}$$

preliminary

~ 80% due to strong IB

$\delta a_\mu^{HVP}\left(ud;\text{ conn., qQED}\right) = 6.9\left(1.9\right)\cdot 10^{-10}$ (ETMC 18) → quark connected only and qQED

$\delta a_\mu^{HVP}\left(ud\right) = 7.8\left(5.1\right)\cdot 10^{-10}$ (BMW 17) → estimate from $\pi^0\gamma, \eta\gamma, \rho-\omega$ mixing, $M_{\pi^\pm}$

$\delta a_\mu^{HVP}\left(ud\right) = 9.5\left(10.2\right)\cdot 10^{-10}$ (RBC/UKQCD 18) → 1 disconnected QED diagram

$\delta a_\mu^{HVP}\left(ud;\text{ sIB}\right) = 9.0\left(4.5\right)\cdot 10^{-10}$ (FNAL/HPQCD/MILC 18) → strong IB only

ETMC 18

$$a_\mu^{HVP} = 685.6\left(13.8\right)\cdot 10^{-10}$$

$$\left(ud\right) = 622.8\left(12.8\right)\cdot 10^{-10}$$

$$\left(s\right) = 53.1\left(2.5\right)\cdot 10^{-10}$$

$$\left(c\right) = 14.75\left(0.56\right)\cdot 10^{-10}$$

$$\left(IB\right) = 6.9\left(1.9\right)\cdot 10^{-10}$$

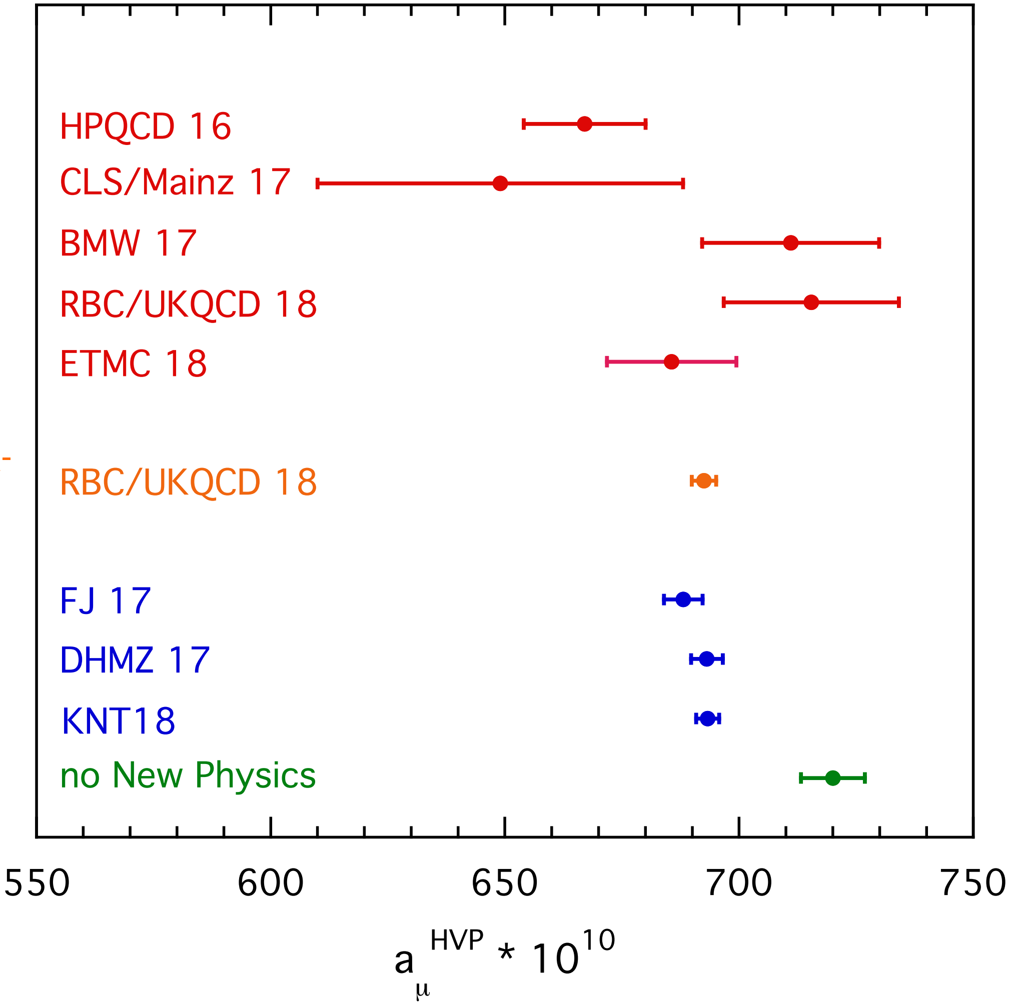
$$\left(disc\right) = -12\left(4\right)\cdot 10^{-10}$$

average of BMW 17 and RBC/UKQCD 18 estimates

lattice data
100%

lattice + e^+e^-
~ 30% + 70%

e^+e^- data
100%



CONCLUSIONS

* updated ETMC result for the hadronic leading-order (quark connected) contribution to a_μ^{HVP} obtained using a **representation of the vector correlator**, which allows to evaluate the FVEs on the lattice and to extrapolate to the physical pion point

$$a_\mu^{HVP}\left(udsc;\,\mathrm{conn.}\right)=690.7\left(13.1\right)\cdot 10^{-10}$$

* new ETMC result for the isospin-breaking correction δa_μ^{HVP} adopting the RM123 method, in which the path integral is expanded at leading order in both $(m_d - m_u) / \Lambda_{QCD}$ and α_{em}

$$\delta a_\mu^{HVP}\left(udsc;\,\mathrm{conn.,\,qQED}\right)=6.9\left(1.9\right)\cdot 10^{-10}$$

*** an accurate representation of the vector correlator allows to remove the lattice artifacts and to get its light-quark mass dependence

*** this represents a good strategy to achieve **a robust and trustworthy control** of the lattice prediction for a_μ^{HVP} below the percent level

TO DO ...

- use of the new ETMC lattice setup @ the physical pion point

- evaluation of the quark disconnected terms and removal of the qQED approximation

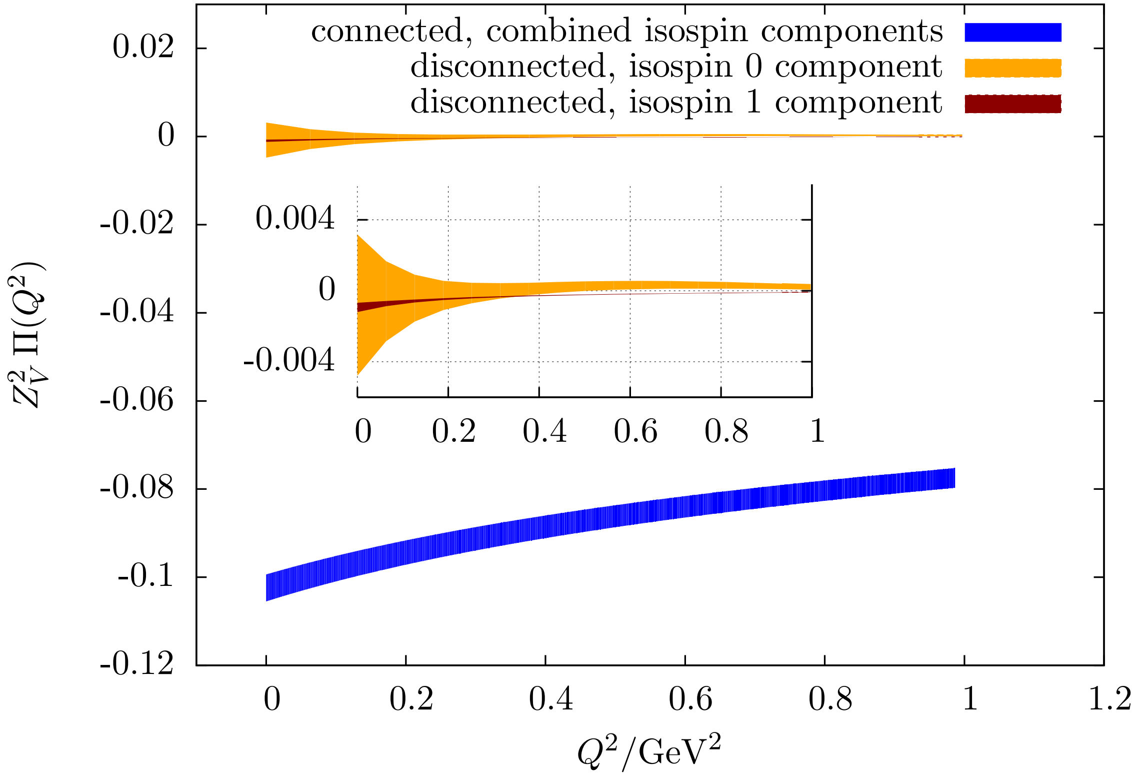
- study of FVEs in the strong and QED isospin breaking corrections

BACKUP SLIDES

on-going work

Quark-disconnected contribution

from Marcus' talk at the FNAL workshop (2017)



- isovector and isoscalar contribution

$$\Pi^3_{\mu\nu}(x,y) = \langle J^3_\mu(x)\, J^3_\nu(y)\rangle_{\rm disc} \quad \text{tm only, with one} - \text{end trick}$$
$$\Pi^0_{\mu\nu}(x,y) = \langle J^0_\mu(x)\, J^0_\nu(y)\rangle_{\rm disc}$$

- $a = 0.078\,\mathrm{fm}$, $m_\pi = 393\,\mathrm{MeV}$, $L = 2.5\,\mathrm{fm}$, $m_\pi L = 5.0$, up-down contribution
- 1548 × 24 + 4996 × 48 gauge configurations × stochastic volume sources

Effective Lepton Mass (ELM) procedure

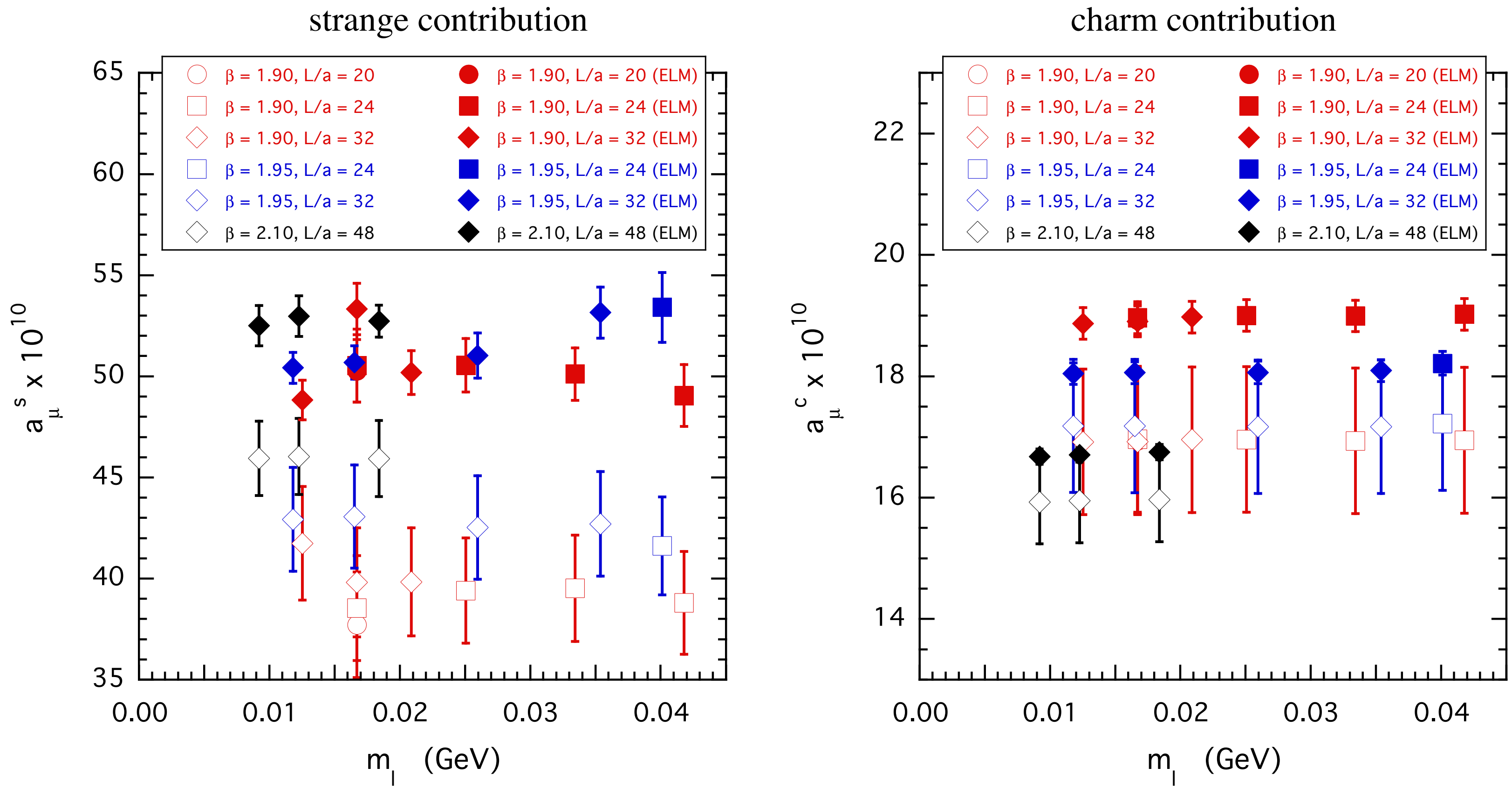
instead of am_{μ}^{phys} :

$$am_{\mu}^{ELM} = aM_{V} \frac{m_{\mu}^{phys}}{M_{V}^{phys}}$$

[ETMC '14]

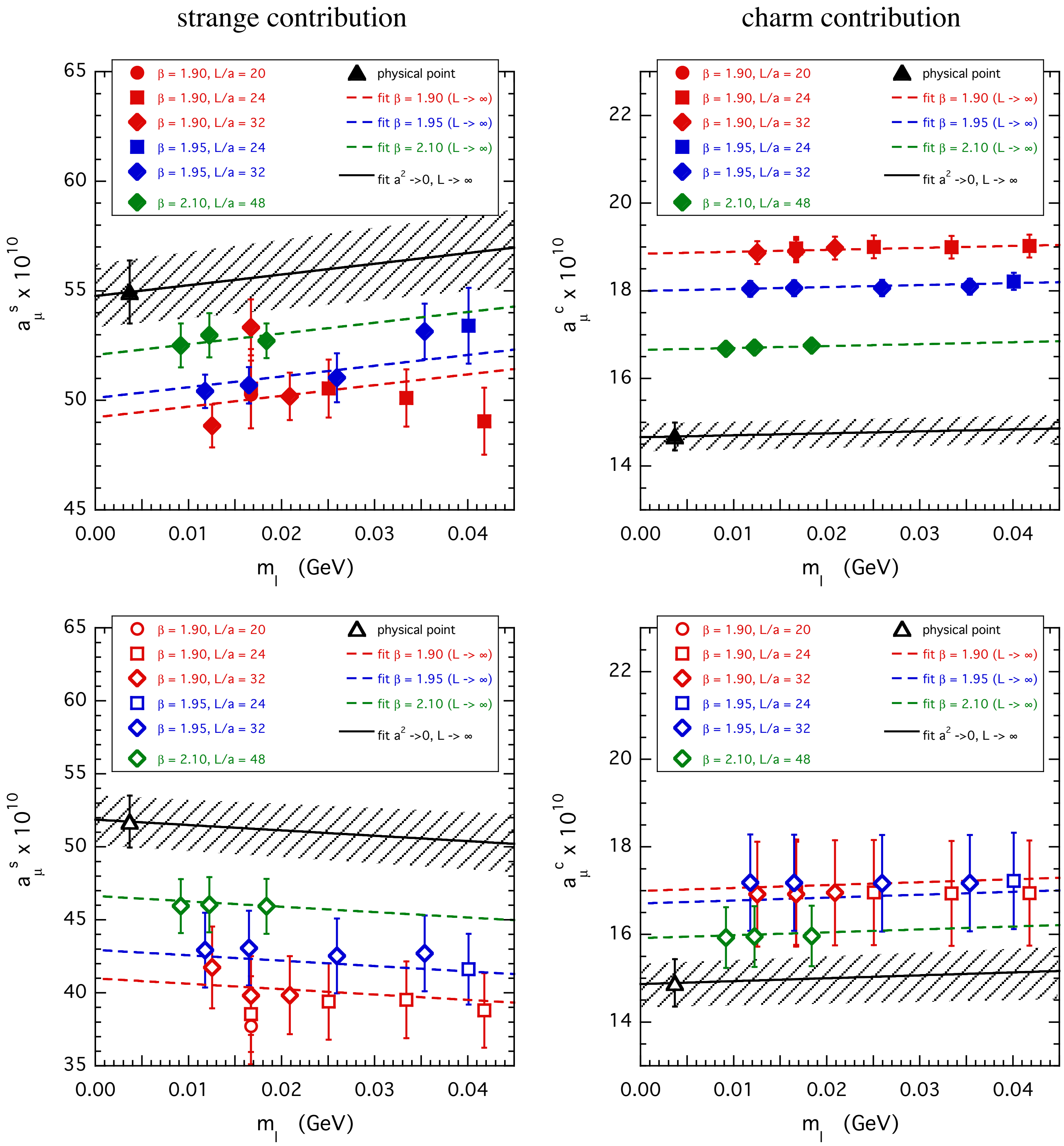
- no need of the value of the lattice spacing (no sensitivity to the lattice scale setting)

- sensitivity to the precision of the vector meson mass aM_V



[ETMC '17]

much better precision with the ELM procedure



[ETMC ’17]

with ELM

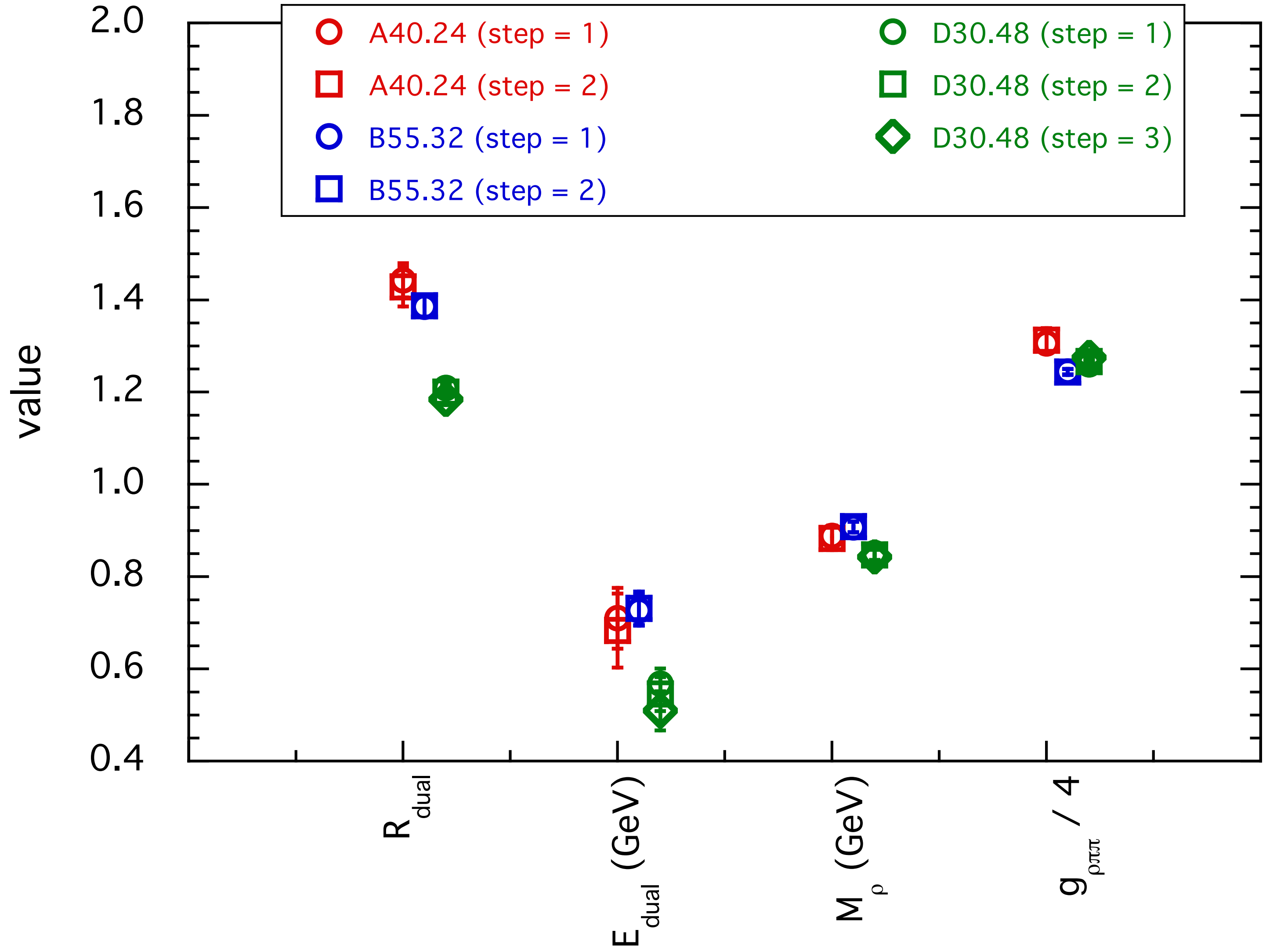
fitting functions:

$$a_\mu^{s,c} = A_0^{s,c}\left[1 + A_1^{s,c}\xi + D^{s,c}\,a^2 + F^{s,c}\,\xi\,\frac{e^{-M_\pi L}}{M_\pi L}\right]$$

without ELM

check temporal autocorrelations

step = 1 : include all points in the fitting procedure
step = 2 : include one point every two
step = 3 : include one point every three



ChPT contribution @ NLO and NNLO

$$a_\mu^{HVP}\left(ud\right)=4\alpha_{em}^2\int_0^\infty dQ^2\,\frac{1}{m_\mu^2}\,f\left(\frac{Q^2}{m_\mu^2}\right)\Pi_R^{(ud)}\left(Q^2\right)$$

$$\Pi_R^{(ud)}\left(Q^2\right)=\frac{5}{9}\left[\Pi_R^{NLO}\left(Q^2\right)+\Pi_R^{NNLO}\left(Q^2\right)\right]^{I=1}$$

Golowitch&Kambor ’95, Amoros et al ’00, ...
Bijnens&Relefors ’16, Golterman at al. ’17

$$\Pi_R^{NLO}\left(Q^2\right)=\frac{1}{24\pi^2}\left[2\hat{B}\left(Q^2,M_\pi^2\right)+\hat{B}\left(Q^2,M_K^2\right)\right]$$

$$\Pi_R^{NNLO}\left(Q^2\right)=\frac{1}{72\pi^2}\frac{Q^2}{\left(4\pi f_\pi\right)^2}\left[2B\left(Q^2,M_\pi^2\right)+B\left(Q^2,M_K^2\right)\right]^2-\frac{16}{3}L_9^r\frac{Q^2}{\left(4\pi f_\pi\right)^2}\left[2B\left(Q^2,M_\pi^2\right)+B\left(Q^2,M_K^2\right)\right]-8C_{93}^rQ^2$$

two LECs

$$B\left(Q^2,M^2\right)=B\left(0,M^2\right)+\hat{B}\left(Q^2,M^2\right)$$

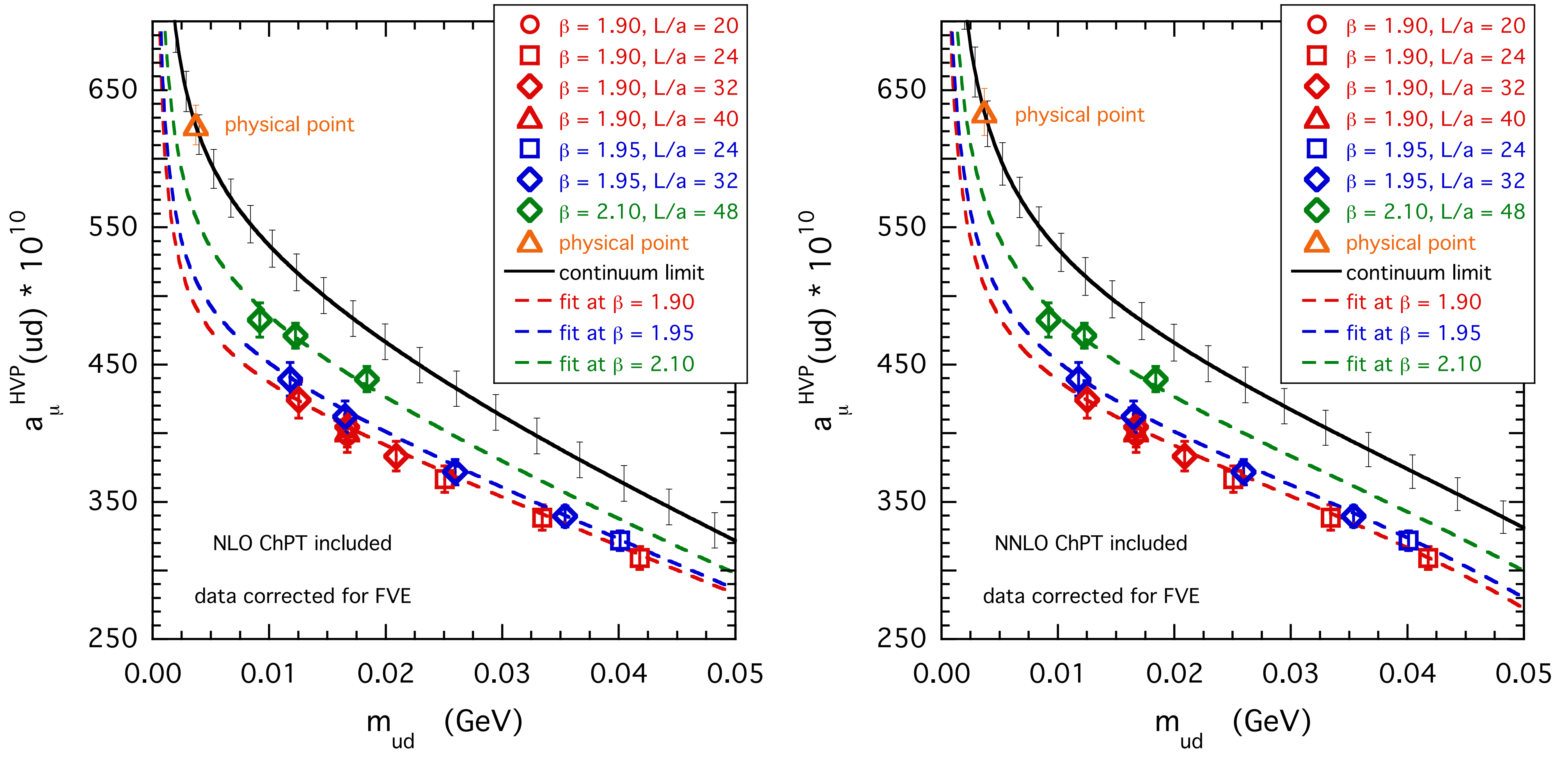
$$B\left(0,M^2\right)=\frac{1}{2}\left(1+\log\frac{M^2}{\mu^2}\right)$$

$$\hat{B}\left(Q^2,M^2\right)\equiv\hat{B}\left(x\right)=\left(1+x\right)^{3/2}\log\frac{1+\sqrt{1+x}}{\sqrt{x}}-x-\frac{4}{3}\qquad x\equiv\frac{4M^2}{Q^2}$$

fits including NLO and NNLO ChPT

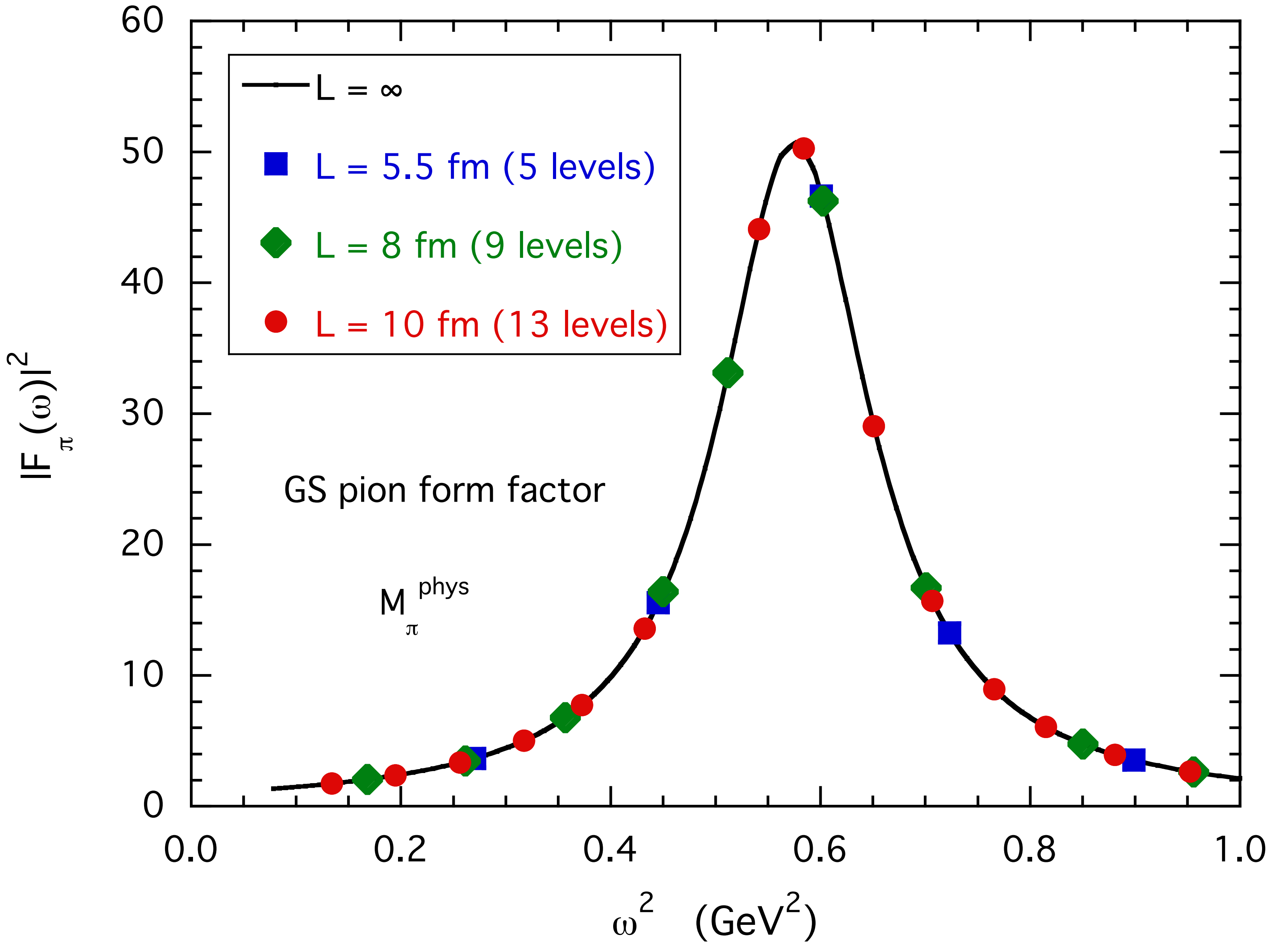
$$a_\mu^{HVP}(ud)=\left\{\left[a_\mu^{HVP}\right]^{NLO}+A_0+A_1m_{ud}+A_2m_{ud}^2\right\}\left(1+D_0a^2+D_1a^2m_{ud}\right)$$

$$a_\mu^{HVP}(ud)=\left\{\left[a_\mu^{HVP}\right]^{NLO}+\left[a_\mu^{HVP}\right]^{NNLO}_{L_9,C_{93}}+A_0+A_1m_{ud}\right\}\left(1+D_0a^2+D_1a^2m_{ud}\right)$$

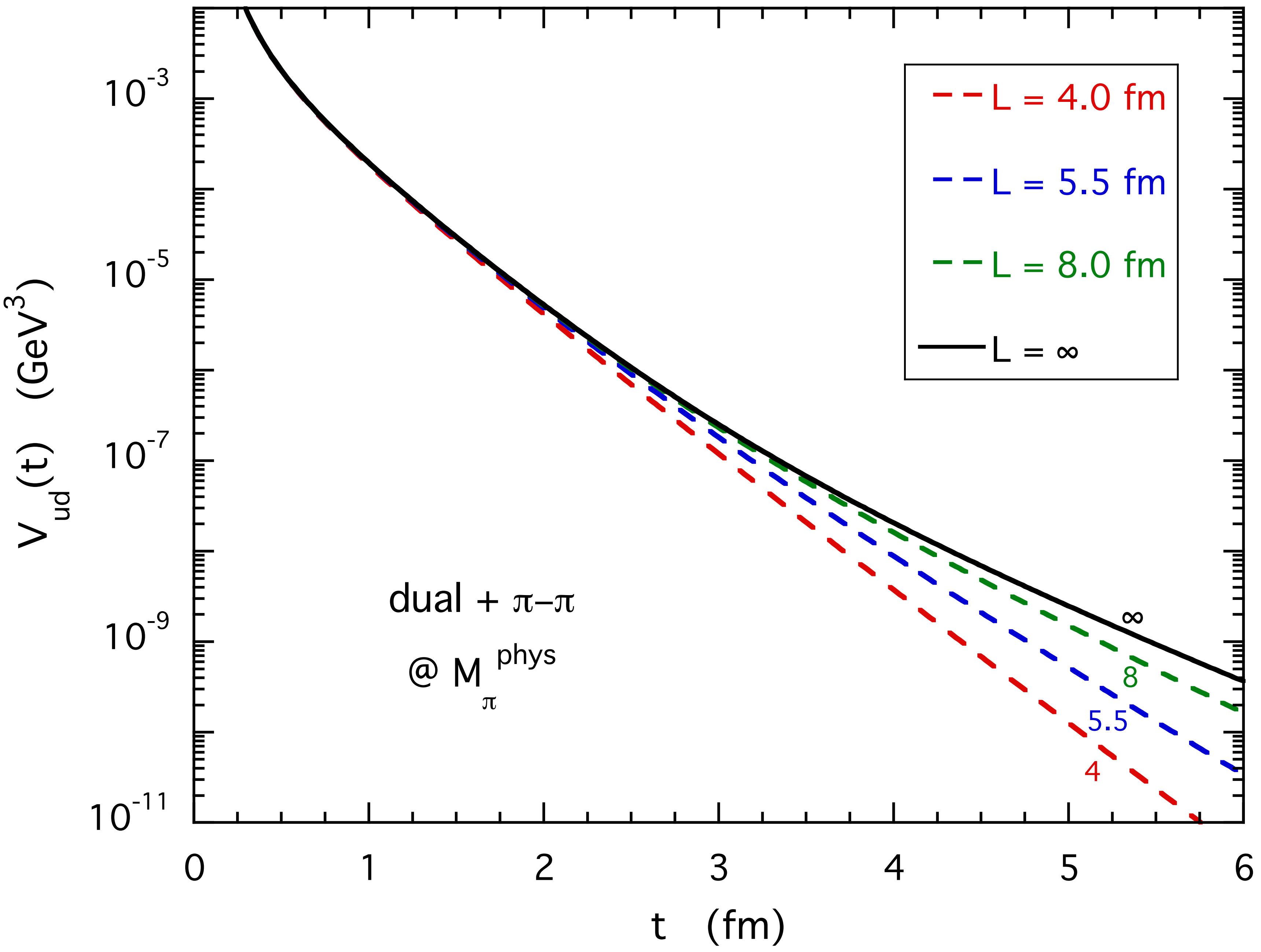


$$m_\mu = m_\mu^{phys}$$

no. of energy levels for the π-π contribution @ M_π^{phys}



u- and d-quark (connected) vector correlator @ M_π^{phys}



leading-order IB corrections

[ETMC’17]

For each quark flavor f one has :

$$\delta V(t) = \delta V^{self}(t) + \delta V^{exch}(t) + \delta V^{tad}(t) + \delta V^{PS}(t) + \delta V^{S}(t)$$

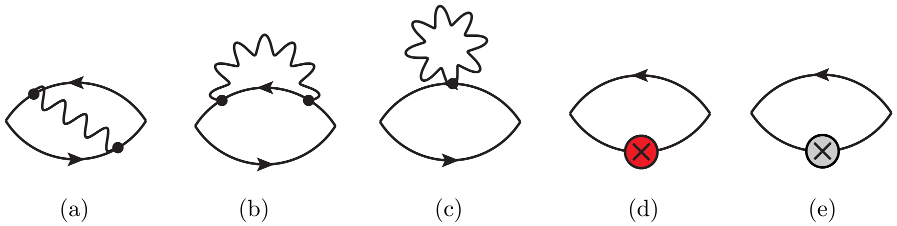


FIG. 8: *Fermionic connected diagrams contributing to the e.m. corrections to a_μ^{had}: exchange (a), self energy (b), tadpole (c), pseudoscalar (d) and scalar (e) insertions. Solid lines represent quark propagators.*

$$\delta V^{self}(t) + \delta V^{exch}(t) = \frac{4\pi\alpha_{em}}{3} \sum_{i=1,2,3} \sum_{\vec{x},y_1,y_2} \langle 0|T\left\{ J_i^\dagger(\vec{x},t) \sum_\mu J_\mu^C(y_1) J_\mu^C(y_2)\ J_i(0) \right\}|0\rangle \ ,$$

$$\delta V^{tad}(t) = \frac{4\pi\alpha_{em}}{3} \sum_{i=1,2,3} \sum_{\vec{x},y} \langle 0|T\left\{ J_i^\dagger(\vec{x},t) \sum_\nu T_\nu(y)\ J_i(0) \right\}|0\rangle \ ,$$

$$\delta V^{PS}(t) = \frac{2\delta m_f^{crit}}{3} \sum_{i=1,2,3} \sum_{\vec{x},y} \langle 0|T\left\{ J_i^\dagger(\vec{x},t)\ i\overline{\psi}_f(y)\gamma_5\psi_f(y)\ J_i(0) \right\}|0\rangle \ ,$$

$$\delta V^{S}(t) = -\frac{2m_f}{3Z_m\mathcal{Z}_f} \sum_{i=1,2,3} \sum_{\vec{x},y} \langle 0|T\left\{ J_i^\dagger(\vec{x},t)\ \overline{\psi}_f(y)\psi_f(y)\ J_i(0) \right\}|0\rangle \ ,$$

$$\boxed{q_f^{sea} = 0}$$

quenched QED

lattice conserved current

$$J_\mu^C(x) = q_f\,\frac{1}{2}\left[\bar{\psi}_f(x)(\gamma_\mu - i\tau^3\gamma_5)U_\mu(x)\psi_f(x+a\hat{\mu}) + \bar{\psi}_f(x+a\hat{\mu})(\gamma_\mu + i\tau^3\gamma_5)U_\mu^\dagger(x)\psi_f(x)\right] .$$

tadpole operator

$$T_\nu(y) = q_f^2\,\frac{1}{2}\left[\bar{\psi}_f(y)(\gamma_\nu - i\tau^3\gamma_5)U_\nu(y)\psi_f(y+a\hat{\nu}) - \bar{\psi}_f(y+a\hat{\nu})(\gamma_\nu + i\tau^3\gamma_5)U_\nu^\dagger(y)\psi_f(y)\right] .$$

$\delta m_f^{crit} \propto \alpha_{em} q_f^2$ = e.m. shift of the critical mass (breaking of chiral symmetry)

$\frac{1}{Z_m} = Z_P(\overline{MS},\mu) \rightarrow$ mass RC (maximally twisted LQCD)

$\frac{1}{Z_f}(\overline{MS},\mu) = \frac{\alpha_{em} q_f^2}{4\pi}\left[6\log(a\mu) - 22.596\right] \rightarrow$ mass RC (LO in QED)

* separation of QCD and QED effects is prescription dependent [see Gasser et al. '03]

- mass anomalous dimensions in QCD and QCD+QED are different
- matching possible only at a given renormalization scale μ* [our choice: μ* = 2 GeV]

$$\hat{m}_f\left(\overline{MS},\mu^*\right)=m_f\left(\overline{MS},\mu^*\right)$$

renormalized mass in QCD+QED → $\hat{m}_f$

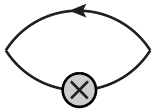
renormalized mass in QCD only → m_f

- bare mass difference: $\hat{\mu}_f-\mu_f=\frac{\hat{m}_f}{\hat{Z}_{m,f}}-\frac{m_f}{Z_m}$ Z_m = mass RC in QCD

$$\frac{1}{\hat{Z}_{m,f}}=\frac{1}{Z_m}\left(1+\frac{1}{Z_f}\right)\Rightarrow\hat{\mu}_f-\mu_f\simeq\frac{1}{Z_m}\left(\hat{m}_f-m_f\right)+\frac{1}{Z_mZ_f}m_f$$

$$\longrightarrow\quad\frac{1}{Z_f}=\frac{1}{Z_f^{em}Z_m^{fact}}=1+O\left(\alpha_{em},\alpha_{em}\alpha_s\right)$$

$$\left(\hat{\mu}_f-\mu_f\right)\times\left(\text{insertion of scalar density}\right)\xrightarrow[f=s,c]{\mu=\mu^*}\frac{m_f}{Z_mZ_f^{em}Z_m^{fact}}$$



for details see PRD95 ('17) 114504 (arXiv:1704.0656)

$\frac{1}{Z_f^{em}}$ is LO in QED and O$\left(\alpha_s^0\right)$ in QCD: $\frac{1}{Z_f^{em}}\left(\overline{MS},\mu^*\right)=\frac{\alpha_{em}q_f^2}{4\pi}\left[6\log\left(a\mu^*\right)-22.596\right]$ **[Martinelli&Zhang '82, Aoki et al. '98]**

Z_m^{fact} is $1+O\left(\alpha_{em}\alpha_s\right)$ $Z_m^{fact}=1$ ⟶ "factorization approximation" between QED and QCD vertex corrections

* e.m. corrections to the renormalization of the (local) e.m. current:

we have adopted a maximally twisted-mass setup with quarks and anti-quarks regularized with opposite values of the Wilson r-parameter: the vector current renormalizes multiplicatively with Z_A

$$Z_A = Z_A^{(0)} + \alpha_{em} Z_A^{(1)} + O\left(\alpha_{em}^2\right) = Z_A^{(0)}\left(1 - 2.51406\,\alpha_{em} q_f^2 Z_A^{fact}\right) + O\left(\alpha_{em}^2\right)$$

→ perturbative estimate at LO in α_{em}

[Martinelli&Zhang '82]

$Z_A^{fact} = 0.9 \pm 0.1$ → correction to the "factorization approximation" between QED and QCD vertex corrections based on WI

doublet of mass- and charge-degenerate TM quarks $\Rightarrow \partial_\mu A_\mu^{1p-split}(x) = 2mP_5(x)$

$$\begin{cases} Z_V\, \partial_\mu A_\mu^{TM,local}(x) = 2mP_5^{TM}(x) + O\left(a^2\right) \\ Z_V \langle 0 | A_0^{TM,local} | PS \rangle = Z_A \langle 0 | A_0^{OS,local} | PS \rangle + O\left(a^2\right) \end{cases}$$

[ETMC '17]

β	$Z_V^{(fact)}$	$Z_A^{(fact)}$
1.90	1.027 (5)	0.85 (5)
1.95	1.033 (4)	0.93 (5)
2.10	1.034 (3)	0.87 (6)

→ $Z_V^{(fact)} = 1.03 \pm 0.01\,,\quad Z_A^{(fact)} = 0.9 \pm 0.1$

relevant uncertainty

* addition of a further contribution: $\delta V(t) = \delta V^{self}(t) + \delta V^{exch}(t) + \delta V^{tad}(t) + \delta V^{PS}(t) + \delta V^{S}(t) + \delta V^{Z_A}(t)$

$$\delta V^{Z_A}(t) = -2.51406\,\alpha_{em} q_f^2\, Z_A^{fact}\, V(t)$$