Dispersive approach to hadronic light-by-light: partial-wave contributions

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and work in progress

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Second Plenary Workshop of the Muon $g-2$ Theory Initiative,
Helmholtz-Institut Mainz

Outline

1. Dispersive approach to HLbL
2. Helicity-partial-wave formalism
3. $\pi\pi$-rescattering: S-waves
4. $\pi\pi$-rescattering: D-waves and higher left-hand cuts
5. Outlook

Reminder: BTT Lorentz decomposition

Lorentz decomposition of the HLbL tensor:

→ Bardeen, Tung (1968) and Tarrach (1975)

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures manifestly gauge invariant
- scalar functions Π_i free of kinematic singularities
  ⇒ dispersion relation in the Mandelstam variables

Dispersive representation

- write down a double-spectral (Mandelstam) representation for the HLbL tensor
- split the HLbL tensor according to the sum over intermediate (on-shell) states in unitarity relations

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \dots$$

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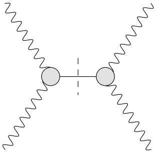
one-pion intermediate state

→ talk by B.-L. Hoid



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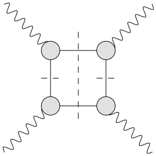
two-pion intermediate state in both channels

→ talk by G. Colangelo



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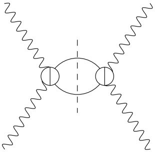
two-pion intermediate state in first channel

$\rightarrow$ this talk



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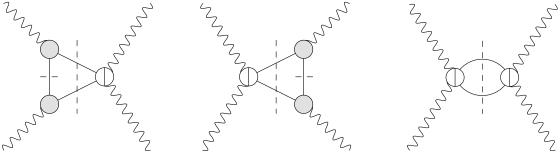
higher intermediate states

Resonance contributions to HLbL?

- unitarity: resonances unstable, not asymptotic states
  $\Rightarrow$ do not show up in unitarity relation
- analyticity: resonances are poles on unphysical Riemann sheets of partial-wave amplitudes
  $\Rightarrow$ describe in terms of multi-particle intermediate states that generate the branch cut
- here: resonant $\pi\pi$ contributions in S-wave (f_0) and D-wave (f_2)
- resonance model-independently encoded in $\pi\pi$-scattering phase shifts

Rescattering contribution



- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel:

$$\Pi_i^{\pi\pi} = \frac{1}{2}\left(\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \frac{\mathrm{Im}\Pi_i^{\pi\pi}(s,t',u')}{t'-t} + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} du' \frac{\mathrm{Im}\Pi_i^{\pi\pi}(s,t',u')}{u'-u}\right.$$
$$+\ \text{fixed-}t$$
$$\left.+\ \text{fixed-}u\right)$$

Helicity formalism and sum rules

Several challenges:

- ambiguities in the tensor decomposition: make sure that only physical helicity amplitudes contribute to the result (i.e. only ± 1 helicities of external photon)
- helicity amplitudes have kinematic singularities and a worse asymptotic behaviour than scalar functions Π_i
- find a good basis for the singly-on-shell case:
    - no subtractions necessary
    - no ambiguities due to tensor decomposition
    - longitudinal polarisations for external photon manifestly absent

Helicity formalism and sum rules

Crucial observation to solve these problems:

- uniform asymptotic behaviour of the full tensor together with BTT tensor decomposition leads to 9 HLbL sum rules
- sum rules derived for general $(g-2)_\mu$ kinematics
- can be expressed in terms of helicity amplitudes

Helicity formalism and sum rules

Singly-on-shell basis $\{\check{\Pi}_i\}$ for fixed-$s/t/u$ constructed:

- 27 elements – one-to-one correspondence to 27 physical helicity amplitudes

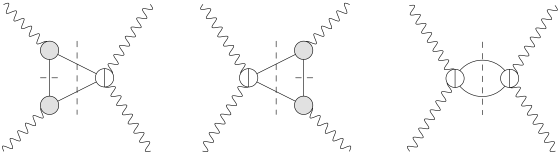
$$\check{\Pi}_i = \check{c}_{ij} H_j$$

  basis change (27×27 matrix $\check{c}_{ij}$) explicitly calculated

- unsubtracted dispersion relations for $\check{\Pi}_i$
- sum rules simple in terms of $\check{\Pi}_i$:

$$0 = \int ds' \mathrm{Im}\check{\Pi}_i(s') \Big|_{t=q_2^2, q_4^2=0} \quad \text{(for certain } i\text{)}$$

Rescattering contribution



- expansion into partial waves
- unitarity gives imaginary parts in terms of helicity amplitudes for $\gamma^*\gamma^{(*)} \to \pi\pi$:

$$\mathrm{Im}_{\pi\pi} h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s) \propto \sigma_\pi(s) h_{J,\lambda_1\lambda_2}(s) h^*_{J,\lambda_3\lambda_4}(s)$$

- framework valid for arbitrary partial waves
- resummation of PW expansion reproduces full result: checked for pion box

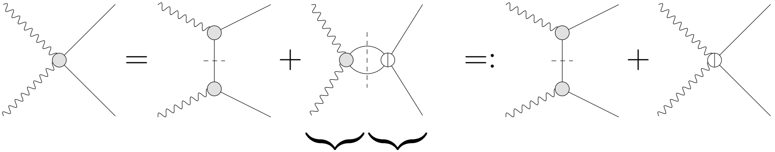
Convergence of partial-wave expansion

Relative deviation from full result: $1 - \frac{a_{\mu,J_{\text{max}}}^{\pi\text{-box, PW}}}{a_\mu^{\pi\text{-box}}}$

J_{max}	fixed-s	fixed-t	fixed-u	average
0	100.0%	-6.2%	-6.2%	29.2%
2	26.1%	-2.3%	7.3%	10.4%
4	10.8%	-1.5%	3.6%	4.3%
6	5.7%	-0.7%	2.1%	2.4%
8	3.5%	-0.4%	1.3%	1.5%
10	2.3%	-0.2%	0.9%	1.0%
12	1.7%	-0.1%	0.7%	0.7%
14	1.3%	-0.1%	0.5%	0.6%
16	1.0%	-0.0%	0.4%	0.4%

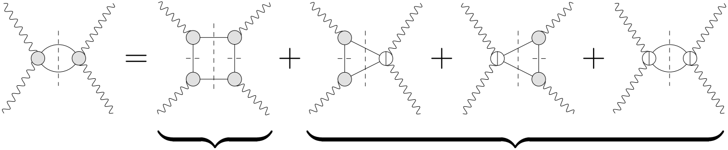
Topologies in the rescattering contribution

Our S-wave solution for $\gamma^*\gamma^* \to \pi\pi$:



recursive PWE, no LHC

Two-pion contributions to HLbL:



pion box rescattering contribution

The subprocess

Omnès solution of unitarity relation for $\gamma^*\gamma^* \to \pi\pi$ helicity partial waves:

$$h_i(s) = \Delta_i(s) + \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{K_{ij}(s,s') \sin\delta_0(s') \Delta_j(s')}{|\Omega_0(s')|}$$

- $\Delta_i(s)$: inhomogeneity due to left-hand cut
- $\Omega_0(s)$: Omnès function with $\pi\pi$ S-wave phase shifts $\delta_0(s)$ as input
- $K_{ij}(s,s')$: integration kernels
- S-waves: kernels emerge from a 2×2 system for $h_{0,++}$ and $h_{0,00}$ and two scalar functions $A_{1,2}$

S-wave rescattering contribution

- pion-pole approximation to left-hand cut
  $\Rightarrow q^2$-dependence given by F_π^V
- phase shifts based on modified inverse-amplitude method ($f_0(500)$ parameters accurately reproduced)
- result for S-waves: $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

Pion polarisabilities

- definition of polarisabilities:

$$\frac{2\alpha}{M_\pi s}\hat{h}_{0,++}(s) = (\alpha_1 - \beta_1) + \frac{s}{12}(\alpha_2 - \beta_2) + \mathcal{O}(s^2)$$

- $\hat{h}_{0,++}$: Born-term subtracted helicity partial wave
- from the Omnès solution: sum rule for polarisabilities, e.g. for pion-pole LHC

$$\frac{M_\pi}{2\alpha}(\alpha_1 - \beta_1) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{|\Omega_0(s')|s'^2}$$

Pion polarisabilities

	sum rule	ChPT → Gasser et al. (2005, 2006)
$(\alpha_1 - \beta_1)^{\pi^\pm}$ $\left[10^{-4}\,\mathrm{fm}^3\right]$	$5.4\ldots5.8$	$5.7(1.0)$
$(\alpha_1 - \beta_1)^{\pi^0}$ $\left[10^{-4}\,\mathrm{fm}^3\right]$	$11.2\ldots8.9$	$-1.9(2)$

- $\pi^\pm$ polarisabilities accurately reproduced (also in agreement with COMPASS measurement)
- π^0 polarisabilities require inclusion of higher intermediate states in the LHC, especially ω
- relation to $(g-2)_\mu$ only indirect (different kinematic region)

Extension to D-waves

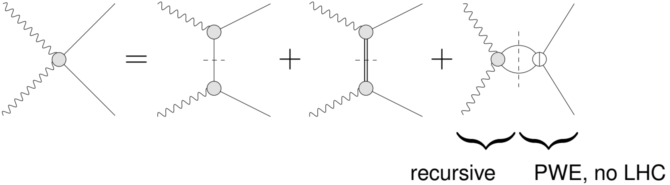
- D-waves describe $f_2(1270)$ resonance in terms of $\pi\pi$ rescattering
- inclusion of higher left-hand cuts (ρ, ω resonances) necessary to reproduce observed $f_2(1270)$ resonance peak in on-shell $\gamma\gamma \to \pi\pi$
- NWA for vector resonance LHC with $V\pi\gamma$ interaction

$$\mathcal{L} = eC_V \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} \partial_\lambda \pi V_\sigma$$

- coupling C_V related to decay width $\Gamma(V \to \pi\gamma)$
- off-shell behaviour described by resonance transition form factors $F_{V\pi}(q^2)$

Topologies in the Omnès solution

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ with higher left-hand cuts provides the following:



Modified Omnès representation

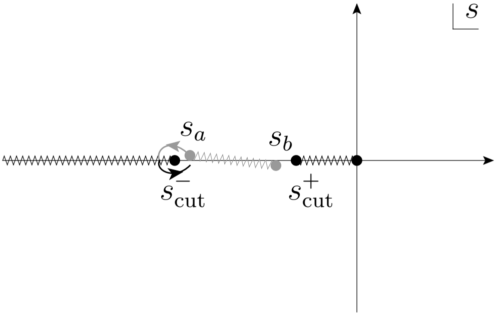
→ García-Martín, Moussallam 2010

$$h_i(s) = N_i(s) + \frac{\Omega(s)}{\pi}\left\{ \int_{-\infty}^{0} ds' \frac{K_{ij}(s,s')\mathrm{Im}h_j(s')}{\Omega(s')} + \int_{4M_\pi^2}^{\infty} ds' \frac{K_{ij}(s,s')\sin\delta(s')N_j(s')}{|\Omega(s')|} \right\}$$

- $N_i(s)$: only Born term as inhomogeneity
- higher left-hand cuts in first dispersion integral: fix polynomial ambiguities of Lagrangian formulation
- $\Omega(s)$: Omnès function with $\pi\pi$ phase $\delta(s)$ as input
- $K_{ij}(s,s')$: integration kernels from the full 5×5 D-wave Roy–Steiner system

"Anomalous thresholds" for large space-like q_i^2

Left-hand cut structure of resonance partial waves:



- two logarithmic branch cuts $(-\infty, s_{\rm cut}^-]$, $[s_{\rm cut}^+, 0]$
- square-root branch cut on second sheet, but extends into the physical sheet for $q_1^2 q_2^2 > (M_R^2 - M_\pi^2)^2$

“Anomalous thresholds” for large space-like q_i^2

- deformation of integration contour for $q_1^2 q_2^2 > (M_R^2 - M_\pi^2)^2$
- anomalous singularity s_a behaves for some D-wave contributions like $(s_a - s)^{-7/2}$
- contour integral around s_a does not vanish and makes result finite
- cancellations require careful numerical implementation

Higher left-hand cuts and pion polarisabilities

consider quadrupole polarisabilities:

	sum rule	ChPT → Gasser et al. (2005, 2006)
$(\alpha_2-\beta_2)^{\pi^\pm}$ $[10^{-4}\,\text{fm}^5]$	$19.9\ldots20.1$	16.2 $[21.6]$
$(\alpha_2-\beta_2)^{\pi^0}$ $[10^{-4}\,\text{fm}^5]$	$26.3\ldots27.1$	$37.6(3.3)$

- π^0 polarisabilities again in bad agreement
- add ρ,ω left-hand cut contribution:

$$(\alpha_2-\beta_2)^{\pi^\pm}_V = 0.9\times10^{-4}\,\text{fm}^5\,,$$
$$(\alpha_2-\beta_2)^{\pi^0}_V = 10.3\times10^{-4}\,\text{fm}^5$$

- π^0 polarisabilities restored, $\pi^\pm$ barely affected

Higher intermediate states

- in the limit of narrow widths, resonance contributions reduce to pole contributions with resonance transition form factors as input

  $\rightarrow$ Pauk, Vanderhaeghen (2014); Danilkin, Vanderhaeghen (2017)

- compare to dispersive treatment and use $f_2(1270)$ as a test case
- BTT Lorentz decomposition for scalar, axial, and tensor resonances $\Rightarrow$ avoid kinematic singularities
- dispersive treatment requires residue in HLbL basis (differences to Lagrangian model formulation)

Higher intermediate states

- our 9 HLbL sum rules for $(g-2)_\mu$ kinematics allow different dispersive representations $\rightarrow$ JHEP **04** (2017) 161
- single resonance states not uniquely defined unless sum rules are fulfilled
- for forward scattering, one sum rule reduces to known forward sum rule $\rightarrow$ Pascalutsa, Pauk, Vanderhaeghen (2012)

Conclusion and outlook

- precise prediction for S-wave $\pi\pi$-rescattering contribution with pion-pole left-hand cut:

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

- D-wave contribution work in progress: requires inclusion of higher left-hand cuts
- compare to narrow-width approximation of $f_2(1270)$