Hybrid QCD sum rule for $(g\text{-}2)_\mu$

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based on:

C. A. Dominguez, **H. Horch**, **B. Jäger**, **N. F. Nasrallah**, **H. Spiesberger**, **H. Wittig, K. S. Phys. Rev. D96 (2017)**

and

S. Bodenstein, C. A. Dominguez, K. S. Phys. Rev. D85 (2012)

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1 The muon anomaly

$$a_\mu = \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2}\int_{s_{\mathrm{thr}}}^{\infty}\frac{ds}{s}K(s)R(s)$$
$$= \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2}\int_{4m_\pi^2}^{\infty}\frac{ds}{s}K(s)12\pi\ \mathrm{Im}\,\Pi_{\mathrm{EM}}(s)\,,$$

We split the integral into a low- and into a high-energy part

$$a_\mu = \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2}\int_{4m_\pi^2}^{s_0}\frac{ds}{s}K(s)12\pi\ \mathrm{Im}\,\Pi_{\mathrm{EM}}(s) + \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}\,K(s)R(s)$$

$$\Pi_{\mathrm{EM}}^{\mu\nu}(q^2) = i\int d^4x\, e^{iqx}\langle 0|T\left(j_{\mathrm{EM}}^{\mu}(x)\,j_{\mathrm{EM}}^{\nu}(0)\right)|0\rangle$$
$$= (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi_{\mathrm{EM}}(q^2)$$

$$j^{\mu}_{\mathrm{EM}}(x) = \sum_f Q_f \bar{q}_f(x) \gamma^{\mu} q_f(x)$$

Integration kernel $K(s)$

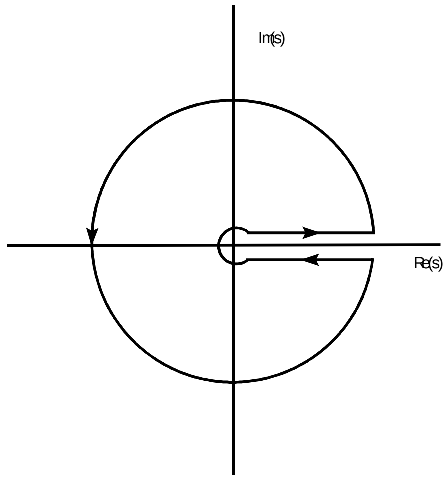
$$K(s) = \int_0^1 dx \, \frac{x^2(1-x)}{x^2 + \frac{s}{m_\mu^2}(1-x)}\,, \tag{1}$$

$sK(s)$ increases monotonically from 2.36×10^{-3} GeV^2 at $s = 4m_\pi^2$ to $m_\mu^2/3 \simeq 3.72 \times 10^{-3}$ at $s = \infty$.

$$K(s)/s \approx 1/s^2$$

Thus the kernel gives a very large weight to the low-energy region.

Cauchy-Theorem



Finite Energy Sum Rule

$$2\int_{s_{\mathrm{thr}}}^{s_0} ds f(s)\,\mathrm{Im}(s) = i\oint_{|s|=s_0} f(s)\Pi(s)ds + 2\pi\sum_{\mathrm{residues}}\mathrm{Res}[f(s)\Pi(s)]$$

for arbitrary for mermorphic $f(s)$.

Approximate $K(s)$ by a meromorphic function $K_1(s)$. As $K(s)/s$ behaves approximately like $1/s^2$, we choose,

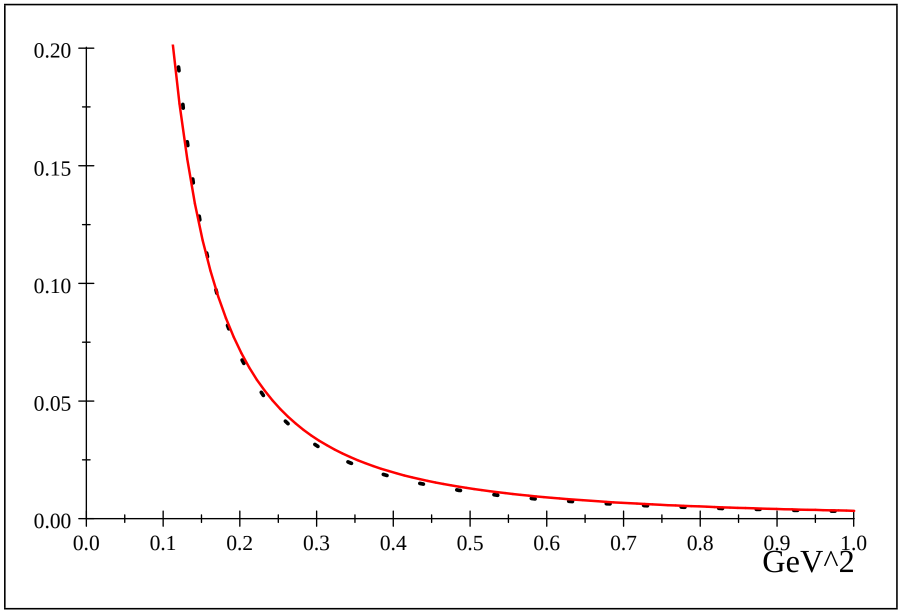
$$\frac{K_1(s)}{s} = \frac{c_{-2}}{s^2} + c_0 + c_1 s \qquad \text{for } 4m_\pi^2 \le s \le s_0 \tag{2}$$

where s_0 should be in the region where PQCD is valid. We choose $s_0 = 4$ GeV^2. The constants a_{-2}, a_0, and a_1 are determined by the conditions

$$\int_{0.078\ \text{GeV}^2}^{4\ \text{GeV}^2} \frac{K(s)}{s} ds = \int_{0.078}^{4} \frac{K_1(s)}{s} ds$$
$$\int_{0.078}^{4} s\frac{K(s)}{s} ds = \int_{0.078}^{4} s\frac{K_1(s)}{s} ds$$
$$\int_{0.078}^{4\ \text{GeV}^2} s^2\frac{K(s)}{s} ds = \int_{0.078}^{4} s^2\frac{K_1(s)}{s} ds$$

with the result:

$$a_{-2} = 2.762\,283 \times 10^{-3}, a_0 = 4.136\,099 \times 10^{-4}, a_1 = -9.913\,527 \times 10^{-5} \,.$$



The functions $K(s)/s$ (black,dotted) and $K_1(s)/s$ (red) for $4m_\pi^2 \leq s \leq 1$ GeV^2

Sum Rule

We add and subtract the approximation $K_1(s)$

$$a_\mu^{\text{uds}} = \frac{\alpha_{\text{EM}}^2}{3\,\pi^2}\int_{4m_\pi^2}^{s_0}\frac{ds}{s}\;[K(s)-K_1(s)+K_1(s)]\;12\pi\;\text{Im}\,\Pi_{\text{EM}}(s)$$
$$+\frac{\alpha_{\text{EM}}^2}{3\,\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}\,K(s)R(s)$$

FESR:

$$\int_{4m_\pi^2}^{s_0}\frac{ds}{s}K_1(s)\;\text{Im}\,\Pi_{\text{EM}}(s)$$
$$= -\frac{1}{2i}\oint_{|s|=s_0}\frac{ds}{s}K_1(s)\Pi_{\text{EM}}(s)+\;\pi\text{Res}\left[\frac{1}{s}K_1(s)\Pi_{\text{EM}}(s)\right]_{s=0}$$
$$= -\frac{1}{2i}\oint_{|s|=s_0}\frac{ds}{s}\,K_1(s)\Pi_{\text{EM}}(s)+\pi c_{-2}\Pi'_{\text{EM}}(0)\,, \qquad (3)$$

with

$$\Pi_{\rm EM}(s) = \left[\frac{5}{9}\Pi_{\rm ud}(s) + \frac{1}{9}\Pi_{\rm s}(s)\right]$$

Cauchy's Theorem leads to the sum rule:

$$a_\mu^{\rm uds} = \frac{\alpha_{EM}^2}{3\,\pi^2} 12^2 \pi a_{-2} \Pi'_{\rm uds}(0) - \frac{1}{2\pi i}\frac{\alpha_{EM}^2}{3\,\pi^2}\oint_{|s|=s_0} \frac{ds}{s}\, K_1(s) 12\pi\, \Pi_{\rm EM}^{\rm QCD}(s)$$
$$+ \frac{\alpha_{EM}^2}{3\,\pi^2}\int_{4m_\pi^2}^{s_0} \frac{ds}{s}\,[K(s) - K_1(s)]\, R(s) + \frac{\alpha_{EM}^2}{3\,\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}\, K(s)\, R(s) \tag{4}$$

It turns out that the data-dependent contribution is only a small correction. Experimental uncertainties are consequently considerably suppressed.

The sum rule Eq. (4) is exact

In the integral around the circle of radius s_0 we use PQCD at five-loop level

$$a_\mu^{\mathrm{uds}} = a_\mu^{(1)} + a_\mu^{(2)} + a_\mu^{(3)} + a_\mu^{(4)}$$

$$\begin{aligned}
a_\mu^{(1)} &= \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2}\, 6\pi i \oint_{|s|=s_0} \frac{ds}{s}\; K_1(s)\Pi_{\mathrm{EM}}(s)\,, \\
a_\mu^{(2)} &= \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2}\, 12\pi^2 c_{-2}\Pi'_{\mathrm{EM}}(0)\,, \\
a_\mu^{(3)} &= \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2} \int_{4m_\pi^2}^{s_0} \frac{ds}{s}\; [K(s) - K_1(s)]\; R(s)\,, \\
a_\mu^{(4)} &= \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2} \int_{s_0}^{\infty} \frac{ds}{s}\; K(s)\; R(s)\,.
\end{aligned}$$

$a_\mu^{(3)}$ from experimental data

$a_\mu^{(1)}, a_\mu^{(2)}, a_\mu^{(4)}$ obtained from theory

The precision of the approximation $K_1(s)$ is not relevant for the total value of a_μ^{uds} because of the difference $K(s) - K_1(s)$. A good approximation to the kernel will, however, reduce the impact of data uncertainties entering $a_\mu^{(3)}$.

In principle it is possible to include higher inverse powers of s. Then higher derivatives of Π_{EM} would contribute:

$$\frac{K_1(s)}{s} = \sum_{n\geq 2} \frac{c_{-n}}{s^n} \longrightarrow \; a_\mu^{(2)} = \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2}\, 12\pi^2 \sum_{n\geq 2} \frac{c_{-n}}{(n-1)!} \left(\frac{d}{ds}\right)^{n-1} \Pi_{\mathrm{EM}}(s)\bigg|_{s=0} \; .$$

If powers of s^{-n} up to $n = 3$ are included, the data driven contribution is less than one permille of the total.

Evaluation of the Light Quark Contribution

Π^{PQCD} is usually defined for the vector current of a single quark type, so that for three light flavors and $3\sum Q_f^2 = 2$, one has $\Pi_{\mathrm{EM}} = (2/3)\Pi^{\mathrm{PQCD}}$. We calculate the integral containing Π^{PQCD} using moments defined by

$$M_N(s_0) = 4\pi^2 \int_0^{s_0} \frac{ds}{s_0} \left[\frac{s}{s_0}\right]^N \frac{1}{\pi} \mathrm{Im}\, \Pi^{\mathrm{PQCD}}(s) = \frac{-1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s_0} \left[\frac{s}{s_0}\right]^N 4\pi^2 \Pi^{\mathrm{PQCD}}(s)\,,$$

$$4\pi^2 \Pi^{\mathrm{PQCD}}(q^2) = -\sum_{n=0}^{\infty} \alpha_s^n \sum_{k=1}^{n+1} c_{nk}^k \left[\ln \frac{-q^2}{\mu^2}\right]^k$$

$$I_{l,n} \equiv \frac{1}{2\pi i} \oint_{|z|=1} dz\, (z)^n \ln^l(-z) = \frac{1}{2\pi} \frac{(-1)^l 2l!}{(n+1)^{l+2}} \sum_{k=1}^{[\frac{l+1}{2}]} (-1)^k \frac{(n+1)^{2k} \pi^{2k-1}}{(2k-1)!}$$

The result for the low-energy PQCD contribution to the anomaly is

$$a_\mu^{(1)} = \frac{\alpha_{\rm EM}^2}{3\,\pi^2}\, 2 s_0 \left[\frac{c_{-2}}{s_0^2} M_{-2}(s_0) + c_0 M_0(s_0) + c_1 s_0 M_1(s_0) \right] .$$

Result

$$a_\mu^{(1)} = (9.56 \pm 0.21) \times 10^{-10} \qquad \text{(small)}$$

using contour-improved perturbation theory

$$a_\mu^{(1)} = 9.44 \times 10^{-10}$$

Compare with the experimental value

$$a_\mu^{\rm had} = (693.1 \pm 3.4) \times 10^{-10} \;\; \text{[Davier et al.2017]}$$

Uncertainty due to scale variations or the uncertainty from α_s is negligible for the total. Higher-dimensional operators, e.g. the gluon condensate, and due to duality violations.

$$\Pi_V^{\mathrm{GG}}(s) = \frac{1}{s^2} \sum_{f=u,d,s} Q_f^2 \frac{1}{12} \left(1 + \frac{7}{6} a\right) \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle ,$$

$$a_\mu^{(1)\mathrm{GG}} = \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2} \frac{-1}{2\pi i} \oint_{|s|=s_0} ds\, c_1 8\pi^2 \Pi_V^{\mathrm{GG}}(s)\,.$$

$$a_\mu^{(1)\,\mathrm{GG}} = (0.12 \pm 0.12) \times 10^{-10} \quad \text{for} \quad \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = 0.015 \pm 0.015 \;\; GeV^4$$

Duality violations even smaller due to their expected exponential fall-off.

Quark mass effects

$$a_\mu^{(1)\,\mathrm{strange}} = 0.05 \times 10^{-10} \; for \; m_s = 0.1 GeV$$

Main Contribution: the pole residue part $a_\mu^{(2)}$

$$a_\mu^{(2)} = \frac{\alpha_{EM}^2}{3\,\pi^2} 12\pi^2 c_{-2} \Pi'_{\rm EM}(0)\,.$$

$\Pi'_{\rm EM}(0)$	Collab.	a [fm]
0.0883(59)	Mainz/CLS [22]	C.L.
0.0959(30)	BMW [21]	C.L.
0.0889(16)	HPQCD [20]	0.15
0.0892(14)	HPQCD [20]	0.12

BMW also give the second derivative $\Pi''_{\rm EM}(0) = -0.181 \pm 0.013$ which would allow a much better fit to $K(s)$.

Data integral contribution: $a_\mu^{(3)}$

$$a_\mu^{(3)} = \frac{\alpha_{EM}^2}{3\,\pi^2} \int_{4m_\pi^2}^{4\text{GeV}^2} \frac{ds}{s} \left[K(s) - K_1(s)\right] R(s)\,.$$

For $R(s)$ we use our compilation of data

$$a_\mu^{(3)} = (55.5 \pm 0.6) \times 10^{-10}\,.$$

The data contribution to the anomaly is small, i.e. around 8 % of the total hadronic contribution. Data errors are reduced correspondingly.

Asymptotic Contribution

$K(s)$ can be safely approximated by its asymptotic form $K(s) \simeq \frac{m_\mu^2}{3s}$, with a precision of better than 2 %

$$
\begin{aligned}
a_\mu^{(4)} &= \left(\frac{\alpha_{EM} m_\mu}{3\,\pi}\right)^2 2 \int_{s_0}^{\infty} \frac{ds}{s^2}\, 4\pi^2 \frac{1}{\pi} \,\mathrm{Im}\,\Pi^{\mathrm{PQCD}}(s) \\
&= -\left(\frac{\alpha_{EM} m_\mu}{3\,\pi}\right)^2 \frac{2}{s_0} M_{-2}(s_0) \\
&= (36.05 \pm 0.40) \times 10^{-10} \qquad (5)
\end{aligned}
$$

The use of PQCD at squared energies above $s_0 = 4$ GeV^2 is well justified. The excellent agreement of $R(s)$ with PQCD is supported by the recent KEDR data in the range $\sqrt{s} = 1.84 - 3.05$ GeV. Moreover, since $a_\mu^{(4)}$ is only a small contribution to the total a_μ, one can expect corrections due to condensates and duality violations to be completely negligible.

Combined Results

$$a_\mu^{(u,d,s)} = (620.1 \pm 34.6) \times 10^{-10} \qquad \text{Mainz}$$
$$a_\mu^{(u,d,s)} = (664.9 \pm 17.7) \times 10^{-10} \qquad \text{BMW}$$

The slight disagreement between the two values shows that a careful assessment of the error contributions to the LQCD results for $\Pi'_{\mathrm{EM}}(0)$ is necessary.

The error in the final result is completely dominated by the uncertainty in the LQCD determination of the slope $\Pi'_{\mathrm{EM}}(0)$.

Within errors, the final result for $a_\mu^{(u,d,s)}$ does not depend on the specific choice of the approximate kernel $K_1(s)$ or on the divisor s_0. The changes of individual contributions to $a_\mu^{(u,d,s)}$ compensate each other in the total and the final result varies little within the given uncertainties.

Case	$a_\mu^{(1)}$	$a_\mu^{(2)}$ ([21], [22])	$a_\mu^{(3)}$	$a_\mu^{(4)}$	$a_\mu^{(u,d,s)}$ ([21], [22])
0	9.51	$(564.5 \pm 17.7, \quad 519.7 \pm 34.6)$	55.5 ± 0.6	35.37	$(664.9 \pm 17.7, \quad 620.1 \pm 34.6)$
1	36.48	$(555.7 \pm 17.4, \quad 511.5 \pm 34.2)$	56.6 ± 0.6	15.64	$(664.4 \pm 17.4, \quad 620.2 \pm 34.2)$
2	−47.13	$(482.3 \pm 15.1, \quad 443.9 \pm 29.6)$	195.0 ± 2.0	35.37	$(665.5 \pm 15.2, \quad 627.1 \pm 29.7)$
3	−57.26	$(586.0 \pm 18.4, \quad 539.3 \pm 36.0)$	96.9 ± 1.0	35.37	$(661.0 \pm 18.4, \quad 614.3 \pm 36.0)$

Case 1: $s_0 = 4$ GeV^2 , $\frac{K_1(s)}{s} = \frac{c_{-2}}{s^2} + c_0 + c_1 s$ for $4m_\pi^2 \leq s \leq s_0$

Case 2: Same as case 1, but for $s_0 = 9$ GeV^2 (instead of $s_0 = 4$ GeV^2)

Case 3: $K_1(s) = \frac{c_{-2}}{s}\left(1 - \frac{s^2}{s_0^2}\right)$, $s_0 = 4$ GeV^2

Case 4: Same as case 3, but using 0.2 GeV^2 for the lower limit of the fit (instead of $4m_\pi^2 = 0.078$ GeV^2)

- With future improved determination of $\Pi'(0)$ one can use

$$a_\mu^{\text{uds}} = (183.2 \pm 2.1 + 5027\Pi'_{\text{uds}}(0)\text{GeV}^2) \times 10^{-10}$$

Contribution of Heavy Quarks

Charm quark

$$K_1 = \frac{\alpha}{3\pi^2 s}\left[\frac{c_1}{s} + \frac{c_2}{s^2}\right] \qquad \text{fitted for } M^2_{J/\Psi} < s < 5GeV^2$$

with $c_1 = 0.003712$ GeV^2 and $c_2 = 0.0005133$ GeV^4. $|(K(s)-K_1(s))/K(s)| < 0.02\%$

Idea:

$$\underbrace{\text{quark mass}}_{\text{LQCD}} \Longleftrightarrow \text{integral over } R(s)$$

Sum rule

$$a_\mu^{\text{c}} = -\frac{\alpha_{EM}^2}{3\,\pi^2} 12^2 \text{Res}\left[K_1(s)\Pi_{\text{OPE}}^c(s), s=0\right] + \frac{\alpha_{EM}^2}{3\,\pi^2} 6\pi i \oint\limits_{|s|=s_0} \frac{ds}{s}\, K_1(s)\,\Pi_{\text{QCD}}^c(s)$$

$$+ \frac{\alpha_{EM}^2}{3\,\pi^2} \int_{s_0}^{\infty} \frac{ds}{s}\, K(s)\, R(s)$$

$$= (14.4 \pm 0.2) \times 10^{-10}$$

$$\frac{\alpha_{EM}^2}{3\,\pi^2} \int_{M_{J/\Psi}^2}^{s_0} \frac{ds}{s}\, [K(s) - K_1(s)]\, R(s) \sim \mathcal{O}(10^{-12})$$

$$\Pi_{\text{QCD}}^f(s) = \frac{3Q_f^2}{16\pi^2} \sum_{n\geq 0 n} C_n \left(\frac{s}{4M_Q^2}\right)^n$$

Similarly:

$$a_\mu^b = (0.29 \pm 0.01)$$