Update on Mellin-Barnes Approximants to HVP

Eduardo de Rafael

Aix-Marseille Université, Univ. Toulon,
CNRS, CPT, Marseille, France

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Talk Based on:
E.de R. Phys. Rev. (2017),
J. Charles, D. Greynat, E.de R. Phys.Rev. (2018): ArXiv:1712.02202v3.

Work in progress with *Jérôme Charles and David Greynat.*

Hadronic Spectral Function Representation

$$a_\mu^{\rm HVP} = \frac{\alpha}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dt}{t}\int_0^1 dx\,\frac{x^2(1-x)}{x^2+\frac{t}{m_\mu^2}(1-x)}\,\frac{1}{\pi}{\rm Im}\Pi(t)$$

$$\sigma(t)_{[e^+e^-\to(\gamma)\to{\rm Hadrons}]} = \frac{4\pi^2\alpha}{t}\,\frac{1}{\pi}{\rm Im}\Pi(t)$$

Euclidean Hadronic Self-Energy Representation

B.E. Lautrup-E. de Rafael '69 , EdeR '94 , T. Blum '03

$$a_\mu^{\rm HVP} = \frac{\alpha}{\pi}\int_0^1 dx\,(1-x)\underbrace{\int_{4m_\pi^2}^{\infty}\frac{dt}{t}\,\frac{\frac{x^2}{1-x}m_\mu^2}{t+\frac{x^2}{1-x}m_\mu^2}\,\frac{1}{\pi}{\rm Im}\Pi(t)}_{\rm Dispersion\ Relation},$$

$$= -\frac{\alpha}{\pi}\int_0^1 dx(1-x)\,\underbrace{\Pi\left(Q^2\equiv\frac{x^2}{1-x}m_\mu^2\right)}_{\rm Accessible\ via\ LQCD}.$$

$$a_\mu^{\rm HVP} = \frac{\alpha}{\pi}\int_{4m_\pi^2}^\infty \frac{dt}{t}\int_0^1 dx \frac{x^2(1-x)}{x^2+\frac{t}{m_\mu^2}(1-x)}\frac{1}{\pi}{\rm Im}\Pi(t)$$

$$= \frac{\alpha}{\pi}\int_{4m_\pi^2}^\infty \frac{dt}{t}\frac{m_\mu^2}{t}\int_0^1 dx\, x^2 \frac{1}{1+\frac{\frac{x^2}{1-x}m_\mu^2}{t}}\frac{1}{\pi}{\rm Im}\Pi(t)\,,$$

Inserting $\frac{1}{1+\frac{\frac{x^2}{1-x}m_\mu^2}{t}} = \frac{1}{2\pi i}\int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{\frac{x^2}{1-x}m_\mu^2}{t}\right)^{-s}\Gamma(s)\Gamma(1-s)$ and integrating over x

Mellin-Barnes Representation

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right)\frac{m_\mu^2}{t_0}\frac{1}{2\pi i}\int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{m_\mu^2}{t_0}\right)^{-s}\mathcal{F}(s)\,\mathcal{M}(s)\,,\quad c_s\equiv{\rm Re}(s)\in]0,1[$$

$$\mathcal{F}(s) = -\Gamma(3-2s)\,\Gamma(-3+s)\,\Gamma(1+s)\,,\qquad t_0 = 4m_\pi^2\,;$$

$$\mathcal{M}(s) = \underbrace{\int_{t_0}^\infty \frac{dt}{t}\left(\frac{t}{t_0}\right)^{s-1}\frac{1}{\pi}{\rm Im}\Pi(t)}_{\text{Mellin Transform of the Spectral Function}}$$

Properties of the Mellin Transform of the Spectral Function

$$\mathcal{M}(s)=\int_{t_0}^{\infty}\frac{dt}{t}\left(\frac{t}{t_0}\right)^{s-1}\frac{1}{\pi}\mathrm{Im}\Pi(t)\,,\qquad \frac{1}{\pi}\mathrm{Im}\Pi(t)\geq 0\quad\text{for}\quad t\geq t_0=4m_{\pi^\pm}^2\,.$$

Complete Monotonicity

The Positivity of $\frac{1}{\pi}\mathrm{Im}\Pi(t)$ implies that $\mathcal{M}(s)$ and all its derivatives are Monotonically Increasing functions for $-\infty<s<1$, with extension to the full complex s-plane by Analytic Continuation.

Spectral Function Moments: $\mathcal{M}(s=0,-1,-2,\cdots)$

$$\underbrace{\int_{t_0}^{\infty}\frac{dt}{t}\left(\frac{t_0}{t}\right)^{1+n}\frac{1}{\pi}\mathrm{Im}\Pi(t)}_{Experiment}=\frac{(-1)^{n+1}}{(n+1)!}(t_0)^{n+1}\underbrace{\left(\frac{\partial^{n+1}}{(\partial Q^2)^{n+1}}\Pi(Q^2)\right)_{Q^2=0}}_{LQCD\ and\,/\,or\ Dedicated\ Experiment}\,,\qquad n=0,1,2,\cdots$$

The Leading Moment is an upper bound to a_μ^{HVP} (*J.S. Bell-EdeR '69*)

$$a_\mu^{\mathrm{HVP}}\leq\frac{\alpha}{\pi}\frac{1}{3}\frac{m_\mu^2}{t_0}\underbrace{\int_{4m_\pi^2}^{\infty}\frac{dt}{t}\frac{t_0}{t}\frac{1}{\pi}\mathrm{Im}\Pi(t)}_{\mathcal{M}(0)}=\left(\frac{\alpha}{\pi}\right)\frac{1}{3}\frac{m_\mu^2}{t_0}\underbrace{\left(-t_0\frac{\partial}{\partial Q^2}\Pi(Q^2)\right)_{Q^2=0}}_{\mathrm{LQCD}}$$

Mellin-Barnes Approximants

- **Ramanujan's Master Theorem** *(-G.H. Hardy's proof-)*

$$\int_0^\infty d\left(\frac{Q^2}{t_0}\right)\left(\frac{Q^2}{t_0}\right)^{s-1}$$

$$\left\{\left(-\frac{t_0}{Q^2}\Pi(Q^2)\right) \underset{Q^2\to 0}{\equiv} \mathcal{M}(0) - \frac{Q^2}{t_0}\mathcal{M}(-1) + \left(\frac{Q^2}{t_0}\right)^2 \mathcal{M}(-2) + \cdots \right\} = \Gamma(s)\Gamma(1-s)\,\mathcal{M}(s)$$

Convergence of **Discrete Moments** $\mathcal{M}(-n)$ to the **Full Mellin Transform** $\mathcal{M}(s)$ $(\mathbf{-n} \Rightarrow \mathbf{s})$.

- **Marichev's Class of Mellin Transforms**

Superpositions of *Standard Products* of gamma functions of the type:

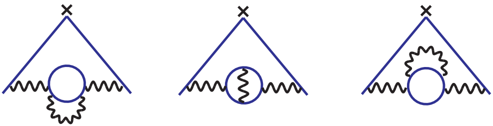
$$\mathcal{M}(s) = \sum_n \lambda_n \prod_{i,j,k,l} \frac{\Gamma(a_i - s)\Gamma(c_j + s)}{\Gamma(b_k - s)\Gamma(d_l + s)}\,, \qquad \lambda_n\,, a_i\,, b_k\,, c_j\,, d_l \quad \textit{constants}$$

Practically all functions in Mathematical Physics have Mellin transforms of this type.

We propose to consider **Mellin-Approximants** to $\mathcal{M}^{\rm HVP}(s)$ of this type, restricted by **QCD-properties** to the subclass:

$$\mathcal{M}_N(s) = \sum_n \lambda_n \prod_{k=1}^{N} \frac{\Gamma(a_k - s)}{\Gamma(b_k - s)}$$

with $\lambda_n\,, a_i\,, b_k\,, c_j\,, d_l$ constrained by Monotonicity, and fixed by Matching to Input Moments.



$$a_{\mu}^{\mathrm{VP}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \frac{673}{108} - \frac{41}{81}\pi^2 - \frac{4}{9}\pi^2 \log(2) - \frac{4}{9}\pi^2 \log^2(2) + \frac{4}{9}\log^4(2) - \frac{7}{270}\pi^4 + \frac{13}{18}\zeta(3) + \frac{32}{3}\mathrm{PolyLog}\left[4\,,\frac{1}{2}\right] \right\} = \left(\frac{\alpha}{\pi}\right)^3 0.0528707\cdots$$

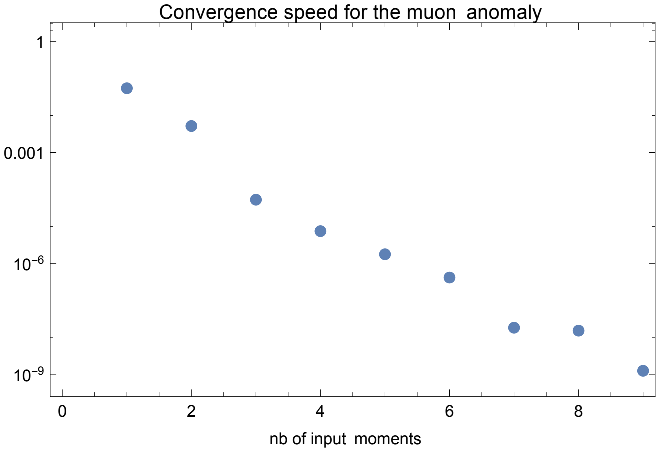
Results from Mellin Approximants $\mathcal{M}_N(s)$ in units of $\left(\frac{\alpha}{\pi}\right)^3$

Input Moments	Numerical result	Accuracy
$\mathcal{M}(0)$	0.0500007	5%
$\mathcal{M}(0)\,,\mathcal{M}(-1)$	0.0531447	0.5%
$\mathcal{M}(0)\,,\mathcal{M}(-1)\,,\mathcal{M}(-2)$	0.0528678	0.004%
$\mathcal{M}(0)\,,\mathcal{M}(-1)\,,\mathcal{M}(-2)\,,\mathcal{M}(-3)$	0.0528711	0.00075%
$\mathcal{M}(0)\,,\mathcal{M}(-1)\,,\mathcal{M}(-2)\,,\mathcal{M}(-3)\,,\mathcal{M}(-4)$	0.0528706	0.00018%

Convergence of Mellin-Approximants tested numerically up to $N = 9$

QED (fourth order) Fast Convergence for a_μ^{VP}

Logarithmic Plot of $\left|\frac{a_\mu^{VP}(N)-a_\mu^{VP}(\mathrm{exact})}{a_\mu^{VP}(\mathrm{exact})}\right|$ versus number of input moments



$$a_\mu^{\rm HVP}(\text{exp.}) = (6.933 \pm 0.025) \times 10^{-8}$$

A. Keshavarzi, D. Nomura, T. Teubner, arXiv:1802.02995v1 [hep-ph]

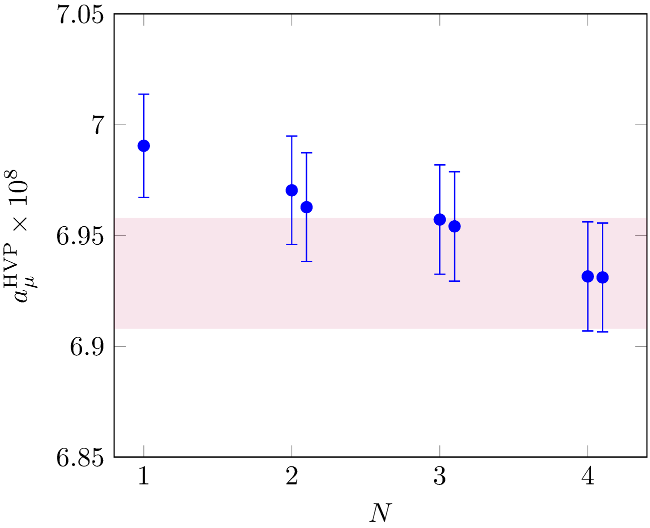
$\mathcal{M}(s)$ Moments and Errors in 10^{-3} units

Moment	Experimental Value	Relative Error
$\mathcal{M}(0)$	0.7176 ± 0.0026	0.36%
$\mathcal{M}(-1)$	0.11644 ± 0.00063	0.54%
$\mathcal{M}(-2)$	0.03041 ± 0.00029	0.95%
$\mathcal{M}(-3)$	0.01195 ± 0.00017	1.4%
$\mathcal{M}(-4)$	0.00625 ± 0.00011	1.8%
$\mathcal{M}(-5)$	0.003859 ± 0.000078	2.0%
...	...	...

$a_\mu^{\rm HVP}$ Results from Mellin Approximants in 10^{-8} units

Input Moments	Type of Approximant	Central Value	Stat. Uncert.
$s = 0$	$N = (1)$	6.991	0.023
$s = 0, -1$	$N = (2)$	6.970	0.024
$s = 0, -1, -2$	$N = (2) + (1)$	6.957	0.025
$s = 0, -1, -2, -3$	$N = (2) + (1) + (1)$	6.932	0.025

Results for a_μ^{HVP} with Errors



Beta-Function Approximants to the **Mellin Transform** of the **Spectral Function**

$$\mathcal{M}_N(s) = \frac{\alpha}{\pi}\frac{5}{3}\sum_{n=1}^{N}\lambda_n \underbrace{\Gamma(b_n - n)\frac{\Gamma(n-s)}{\Gamma(b_n - s)}}_{\textbf{Beta}(n-s,\, b_n-n)}, \qquad \lambda_1 = 1\,, \quad b_n \geq n+1\,.$$

They have simple Hadronic Self-Energy Approximants:

$$\Pi_N(Q^2) = -\frac{\alpha}{\pi}\frac{5}{3}\frac{Q^2}{t_0}\sum_{n=1}^{N}\lambda_n \frac{\Gamma(b_n - n)}{\Gamma(b_n)}\Gamma(n) \underbrace{{}_2F_1\left(\begin{array}{cc} 1 & n \\ & b_n \end{array}\middle| -\frac{Q^2}{t_0}\right)}_{\textit{Gauss Hypergeometric Function}}$$

and *Equivalent* simple Spectral Functions:

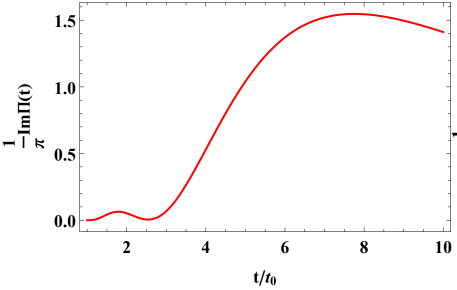
$$\frac{1}{\pi}\mathrm{Im}\Pi_N(t) = \frac{\alpha}{\pi}\frac{5}{3}\sum_{n=1}^{N}\lambda_n \left(\frac{4m_\pi^2}{t}\right)^{n-1}\left(1 - \frac{4m_\pi^2}{t}\right)^{b_n - n - 1}\theta(t - 4m_\pi^2)\,,$$

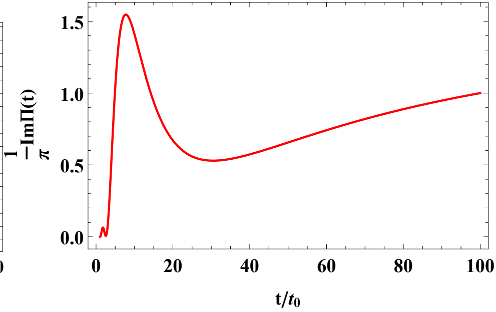
with the matching solutions for λ_n and $b_n \geq n+1$ constrained by the positivity of $\frac{1}{\pi}\mathrm{Im}\Pi_N(t)$.

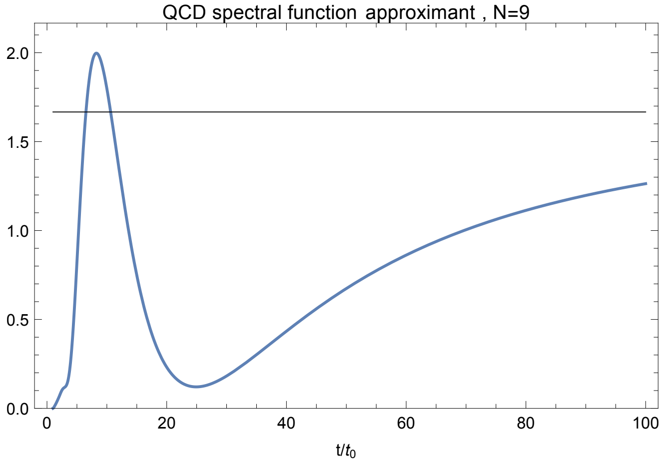
Using the central values of the first five moments from experiment:

$$a_\mu^{\rm HVP}(N=3) = 6.9335 \times 10^{-8}\,.$$

Shape of the "Equivalent" Spectral Function in $\frac{\alpha}{\pi}$ units:

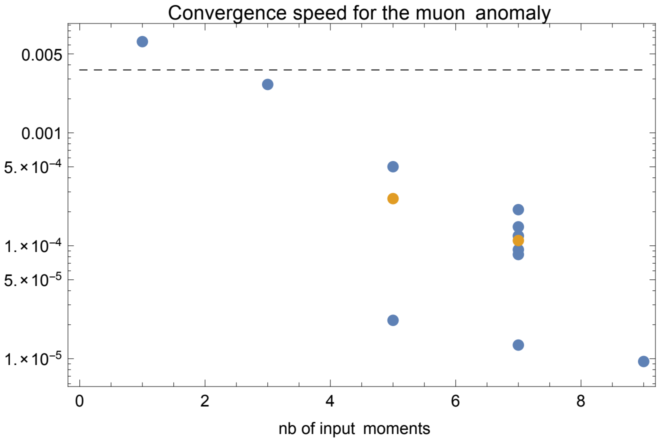





QCD spectral function approximant , N=9
2.0
1.5
1.0
0.5
0.0
0
20
40
60
80
100
t/t_0

Convergence Test

Log Plot of $\left| \frac{a_\mu^{\mathrm{HVP}}(N) - a_\mu^{\mathrm{HVP}}(\mathrm{exp.})}{a_\mu^{\mathrm{HVP}}(\mathrm{exp.})} \right|$ versus N



- We claim that, from an **Accurate LQCD Determination**, of the first few moments, one could reach an evaluation of a_{μ}^{HVP} with competitive precision -or even higher- than the present experimental determinations.

- **Accurate** determination of the **First Moment** is an excellent **Test**

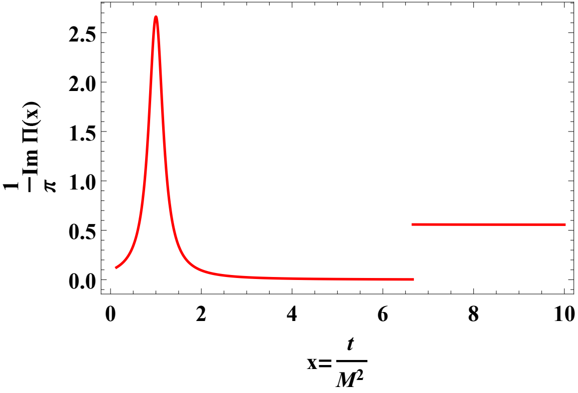
$$a_{\mu}^{\mathrm{HVP}} \leq \frac{\alpha}{\pi}\frac{1}{3}\underbrace{\int_{4m_{\pi}^{2}}^{\infty}\frac{dt}{t}\frac{m_{\mu}^{2}}{t}\frac{1}{\pi}\mathrm{Im}\Pi(t)}_{\mathcal{M}(0)\ \textit{from experiment}} = \left(\frac{\alpha}{\pi}\right)\frac{1}{3}\underbrace{\left(-m_{\mu}^{2}\frac{\partial}{\partial Q^{2}}\Pi(Q^{2})\right)_{Q^{2}=0}}_{\text{LQCD and/or } \textit{DEDICATED EXPERIMENT}}$$

- The fact that the $\Pi(Q^2)$ **Beta-Function Approximants** are simple superpositions of simple **Gauss-Hypergeometric-Functions** offers the possibility of using **LQCD** information on values of $\Pi(Q^2)$ **at fixed** Q^2**-values**, or an **Alternative Input** to Moments.

Breit-Wigner plus Theta-like Spectral Function

$$\frac{1}{\pi}\mathrm{Im}\Pi(t) = \left(\frac{\alpha}{\pi}\right)\left\{ f_V^2 M^2 \underbrace{\frac{\Gamma M}{(t-M^2)^2+\Gamma^2 M^2}}_{\Rightarrow\ \pi\delta(t-M^2)\quad \text{for}\quad \Gamma\to 0} \theta(t-t_0) + \sum_f q_f^2 \frac{N_c}{3}\theta(t-t_{\text{pQCD}}) \right\}$$

Shape of this Spectral Function ($M = M_\rho$, $\Gamma = \Gamma_\rho$, $t_0 = 4m_\pi^2$, $f_V^2 = 0.51$).



Mellin Transform of Breit-Wigner plus Continuum

Mellin Transform of Breit-Wigner infinite superposition of Beta-Functions:

$$\mathcal{M}_{BW}(s) = \sum_f q_f^2 \frac{N_c}{3}\left(\frac{t_0}{t_{\rm pQCD}}\right)^{1-s}\frac{1}{1-s} + f_V^2 \frac{M^2}{t_0}\sum_{n=1}^{\infty} \underbrace{\left(\frac{M^2}{t_0}\sqrt{1+\frac{\Gamma^2}{M^2}}\right)^{n-1} \sin\left[(n-1)\arctan\frac{\Gamma}{M}\right]}_{\lambda_n} \mathbf{Beta}(1+n-s,1)$$

In the Narrow-Width limit this sum collapses to

$$\mathcal{M}_{BW}(s) \underset{\Gamma\to 0}{\sim} \sum_f q_f^2 \frac{N_c}{3}\left(\frac{t_0}{t_{\rm pQCD}}\right)^{1-s}\frac{1}{1-s} + \underbrace{f_V^2 \pi \left(\frac{M^2}{t_0}\right)^{s-1}}_{\text{Mellin Transform of a Delta}-\text{Function}}$$

Padé Approximants are a very particular limit of Mellin-Barnes Approximants.

Question of α_s Corrections

Behaviour the Spectral Function (pQCD at one loop):

$$\frac{1}{\pi}\mathrm{Im}\Pi(t) \underset{t\to\infty}{\sim} \left(\frac{\alpha}{\pi}\right)\left(\sum_i q_i^2\right)\frac{1}{3}N_c \underbrace{\left[1+\frac{\alpha_s(\mu^2)}{\pi}\frac{1}{1+B\alpha_s(\mu^2)\log\frac{t}{\mu^2}}\right]}_{B\equiv\frac{-\beta_1}{2\pi}\quad \text{and} \quad \beta_1=\frac{1}{6}\left(-11\mathrm{N_c}+2\mathrm{n_f}\right)}$$

Mellin Transform

$$\mathcal{M}(s) \underset{\text{pQCD}}{\sim} \left(\frac{\alpha}{\pi}\right)\frac{5}{3}\,N_c\,\frac{1}{3}\,\frac{1}{1-s}\left\{1+\frac{\alpha_s(\mu^2)}{\pi}\left(\frac{\mu^2}{t_0}\right)^{s-1}\int_0^\infty d\omega\; e^{-\omega}\,\frac{1-s}{1+\omega(B\alpha_s)-s}\right\}$$

- It generates a singularity at $s=1+\omega(B\alpha_s)$ to be integrated over ω.
- The new singularity, for $s<1$, is suppressed by a factor $\frac{\alpha_s(\mu^2)}{\pi}\left(\frac{\mu^2}{t_0}\right)^{s-1}$.
- All this information is in the Input Moments.

Magic Moments

Standard Momets of the Spectral Function

$$\mathcal{M}(-n) = \underbrace{\int_{t_0}^{\infty} \frac{dt}{t}\left(\frac{t_0}{t}\right)^{1+n}\frac{1}{\pi}\mathrm{Im}\Pi(t)}_{\textit{Experiment}} = \frac{(-1)^{n+1}}{(n+1)!}(t_0)^{n+1}\underbrace{\left(\frac{\partial^{n+1}}{(\partial Q^2)^{n+1}}\Pi(Q^2)\right)_{Q^2=0}}_{\textit{LQCD and/or Dedicated Experiment}}\,, \quad n = 0, 1, 2, \cdots$$

Magic **Moments of the Spectral Function** (set $z = t/t_0$)

$$\Sigma(N) = \underbrace{\int_0^1 dz(1-z)^N \frac{1}{\pi}\mathrm{Im}\Pi(\frac{1}{z}t_0)}_{\textit{Experiment}} = \underbrace{\left[\left(1 - t_0\frac{\partial}{\partial Q^2}\right)^N\left(-t_0\frac{\partial\Pi(Q^2)}{\partial Q^2}\right)\right]_{Q^2=0}}_{\textit{LQCD and/or Dedicated Experiment}}\,, \quad N = 0, 1, 2, \cdots$$

The $\Sigma(N)$-Moments decrease much more slowly than the $\mathcal{M}(-n)$-Moments