Dispersion relations for $\gamma\gamma^* \to \pi\pi, \pi\eta$

Igor Danilkin

in coll. with O. Deineka & M. Vanderhaeghen

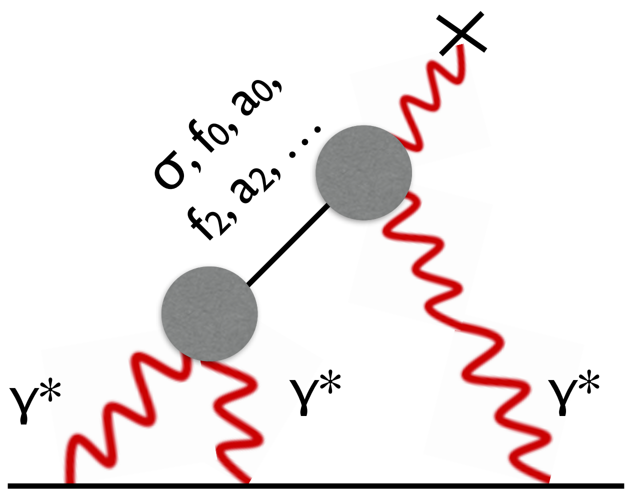
Second Workshop of the Muon g − 2 Theory Initiative Mainz,
June 18 - 22, 2018

SFB 1044
THE LOW-ENERGY FRONTIER
OF THE STANDARD MODEL

JOHANNES GUTENBERG UNIVERSITÄT MAINZ
JG|U

Multi-meson production

Important contributions beyond **pseudo-scalar** poles



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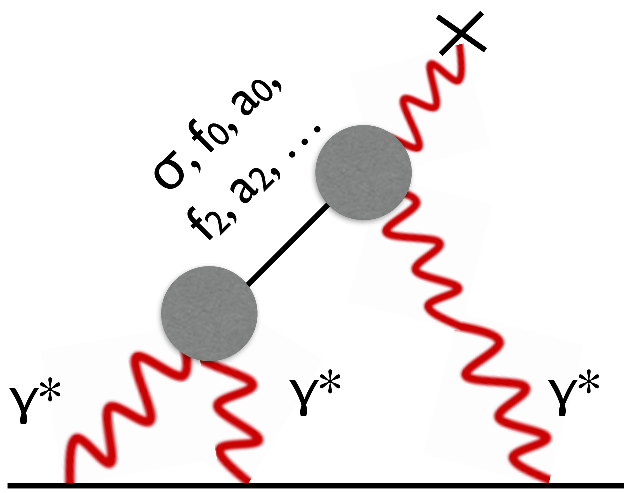
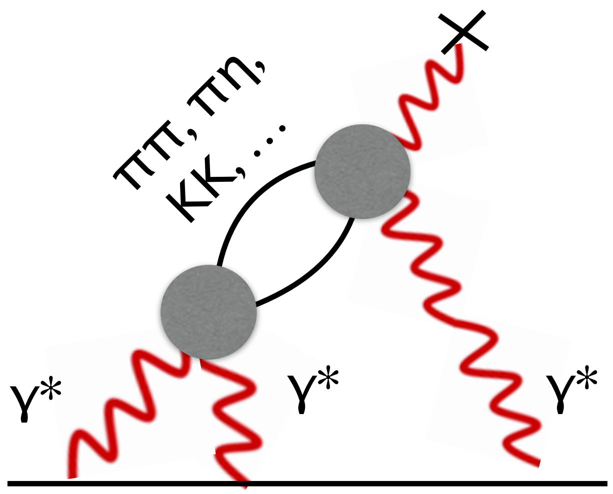


dispersive analysis for
ππ, πη, … loops



Pauk, Vanderhaeghen, (2014)

Colangelo, Hoferichter, Procura, Stoffer, (2017)

$$a_\mu^{\pi\text{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -2.4(0.1) \times 10^{-10}$$

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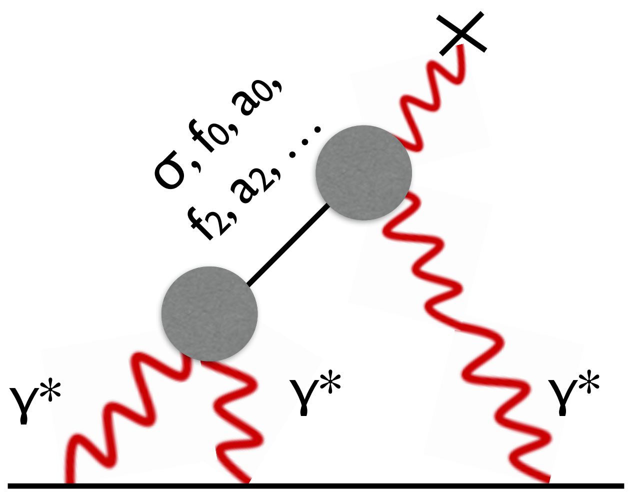

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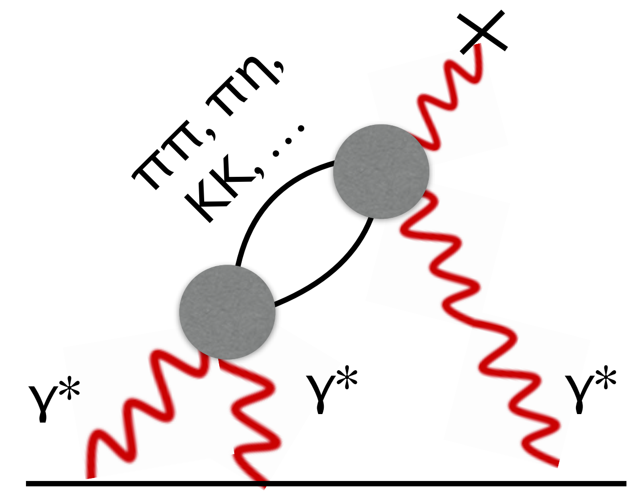

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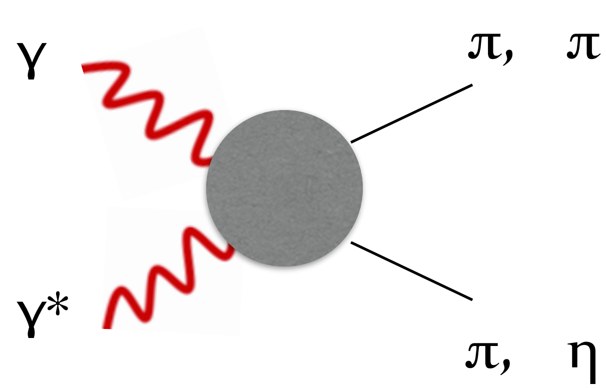
Important ingredient: γγ*→ππ, πη, …



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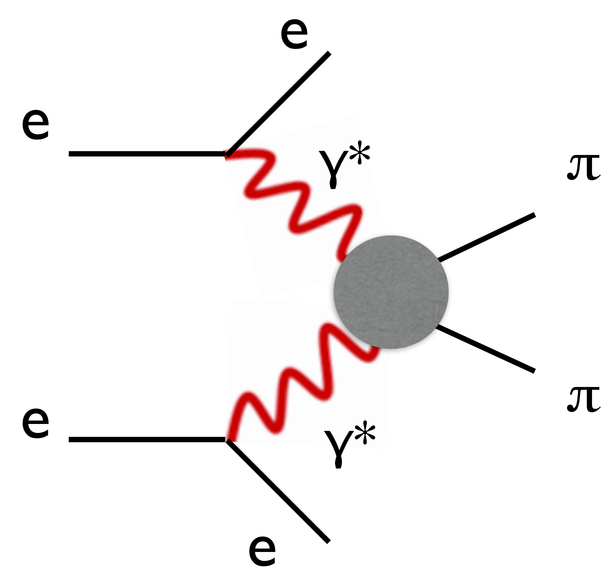
γγ→ππ, KK, ηη, πη (Belle: 07,08, 09,10, ..)
γγ*→ππ, πη (BESIII in progress)

Experiment

Observables in experiment $e^+e^- \to e^+e^-\pi\pi$



$$d\sigma = \frac{\alpha^2}{16\pi^4 Q_1^2 Q_2^2} \frac{2\sqrt{X}}{s(1-4m^2/s)^{1/2}} \cdot \frac{d^3\vec{p}_1'}{E_1'} \cdot \frac{d^3\vec{p}_2'}{E_2'}$$
$$\times \; \{4\,\rho_1^{++}\,\rho_2^{++}\,\sigma_{TT} + \rho_1^{00}\,\rho_2^{00}\,\sigma_{LL} + 2\,\rho_1^{++}\,\rho_2^{00}\,\sigma_{TL} + \ldots\}\,,$$

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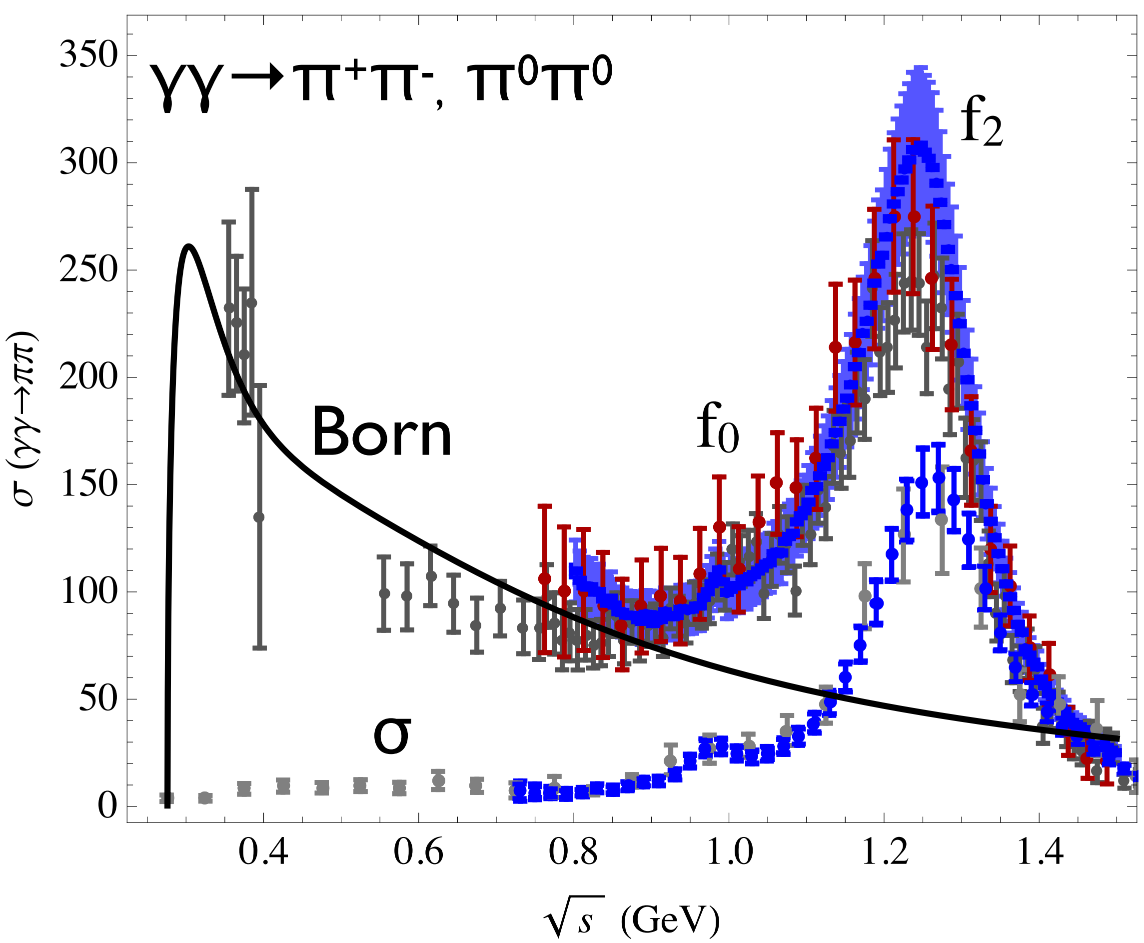

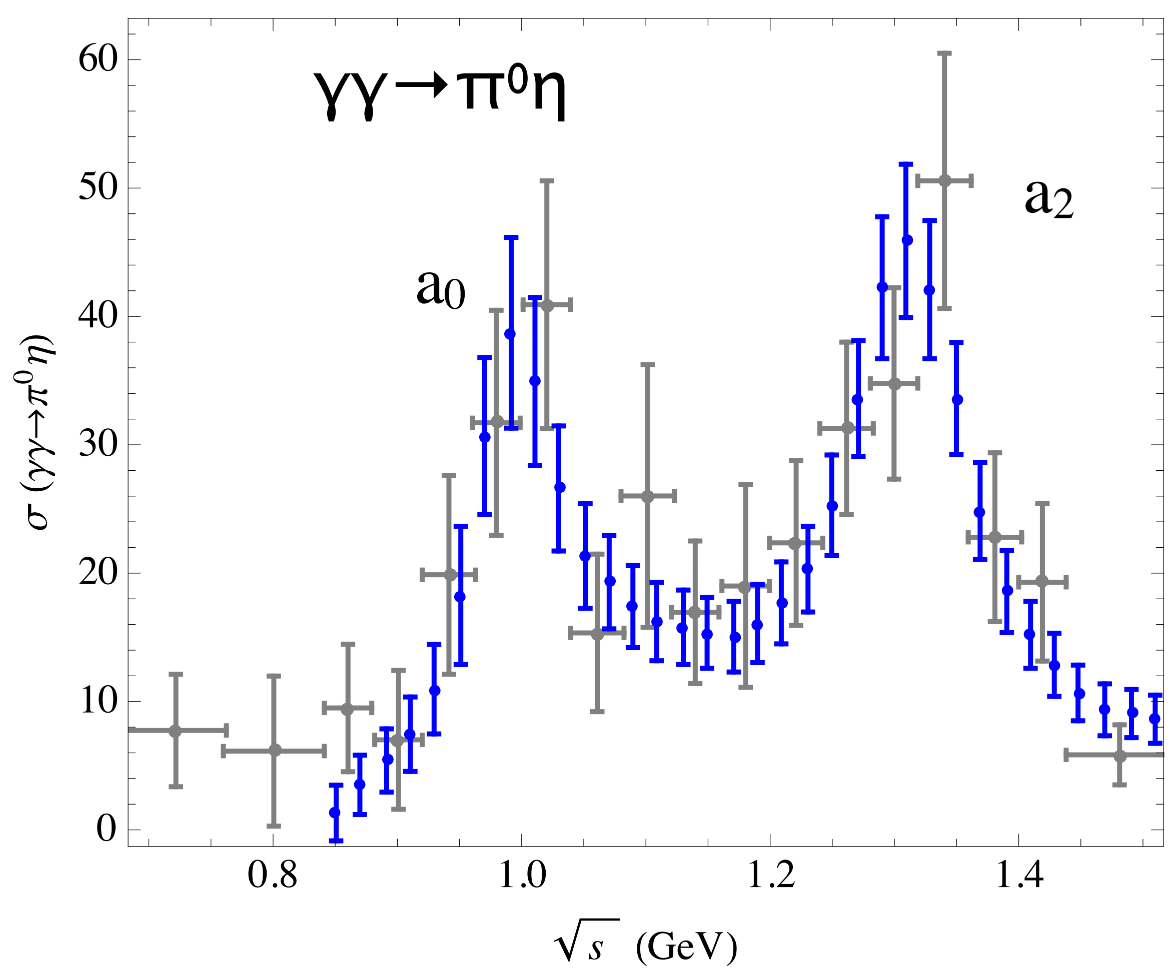

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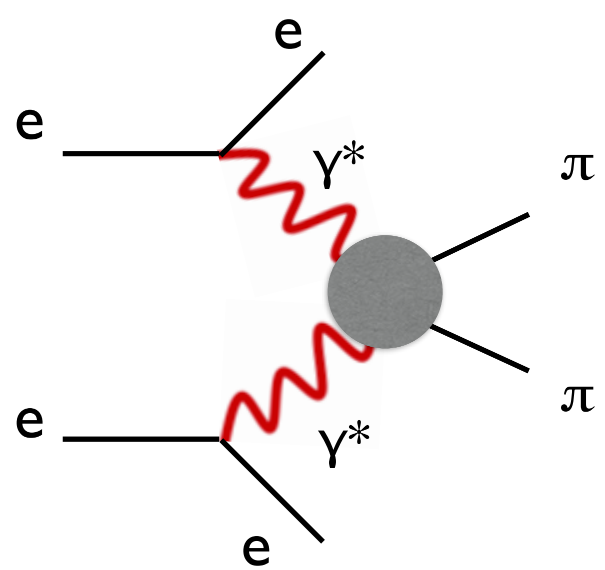

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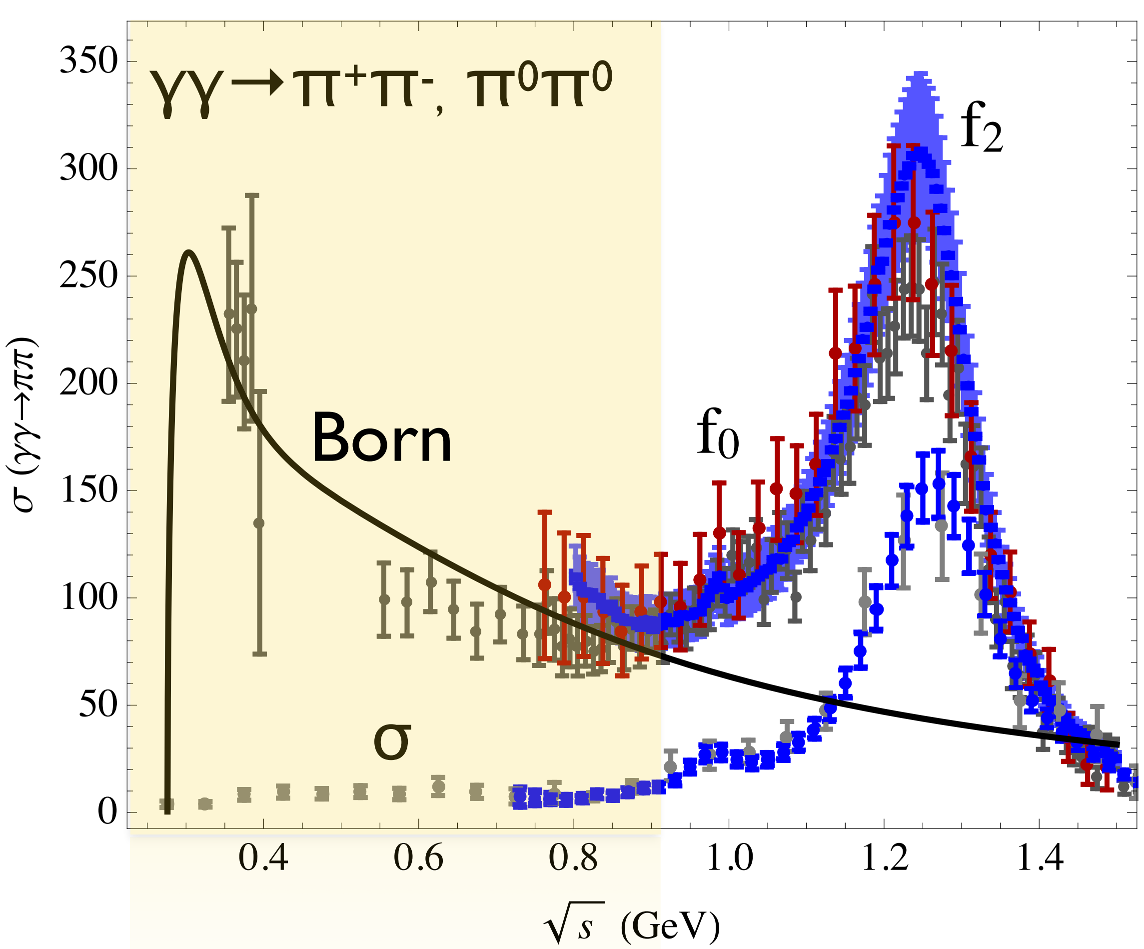

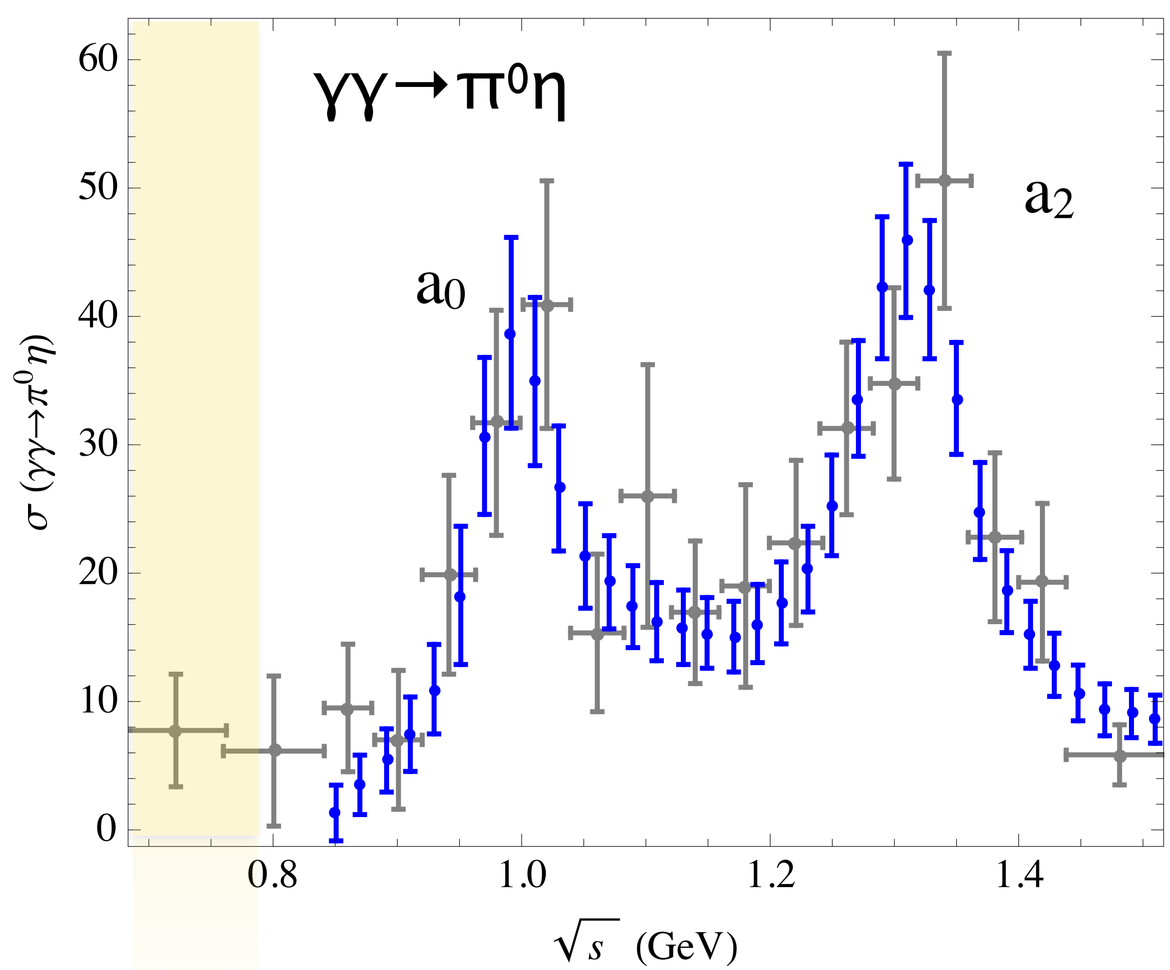

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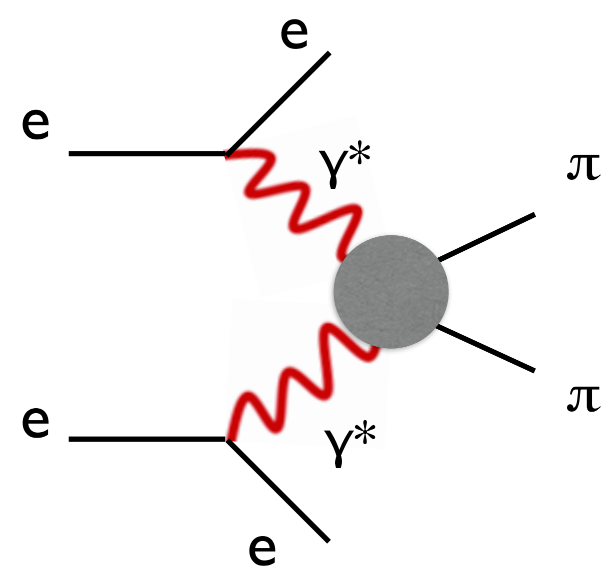

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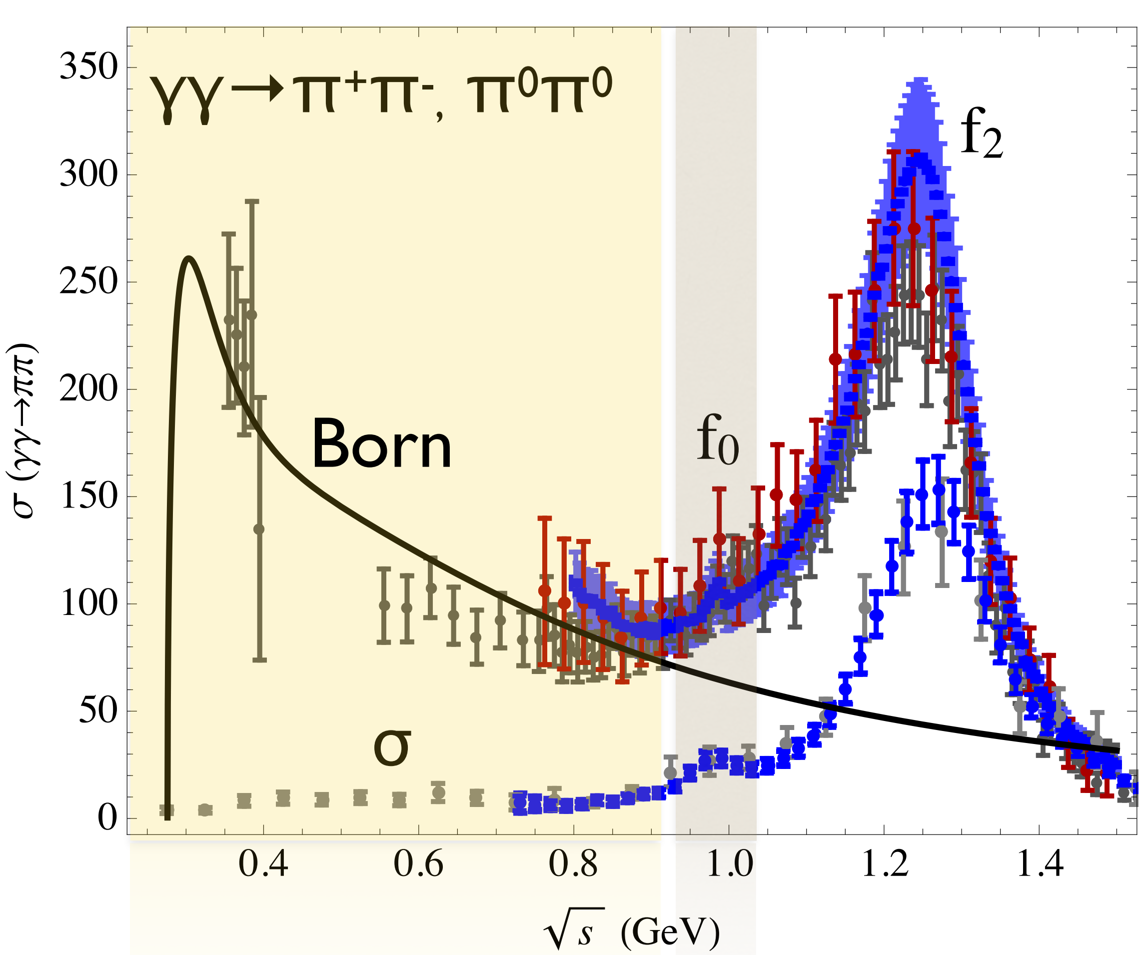

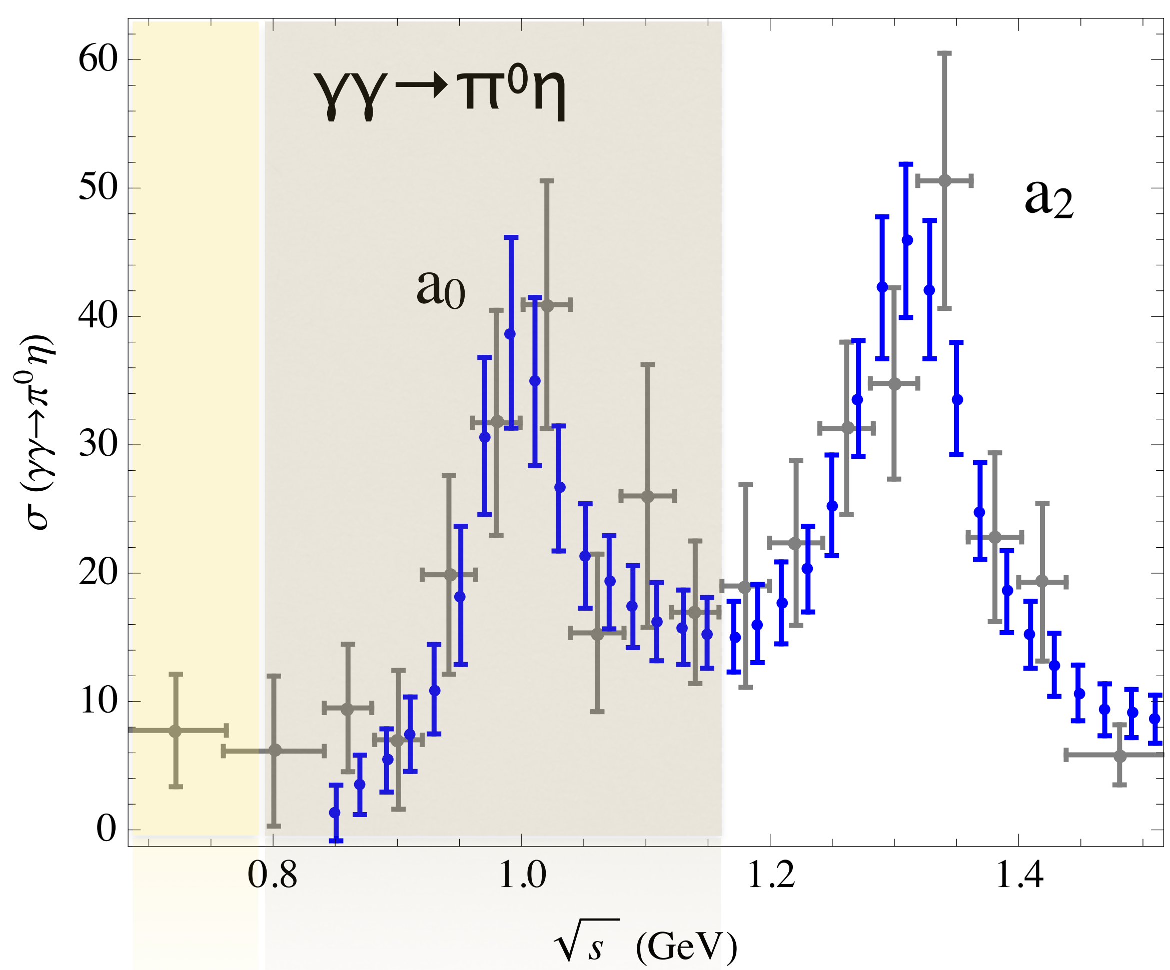

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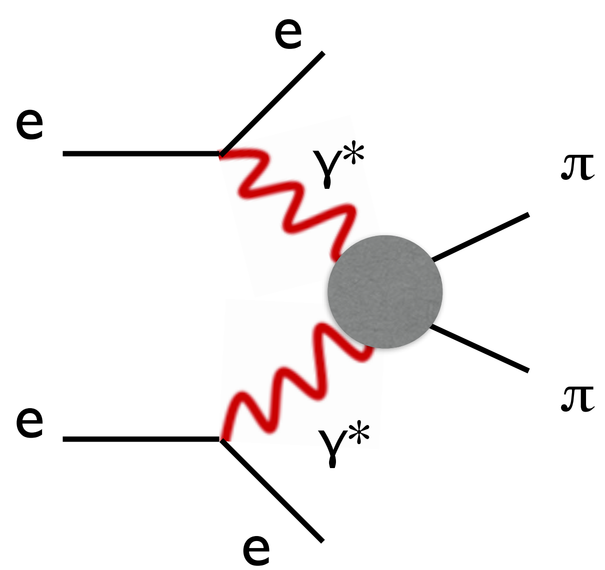

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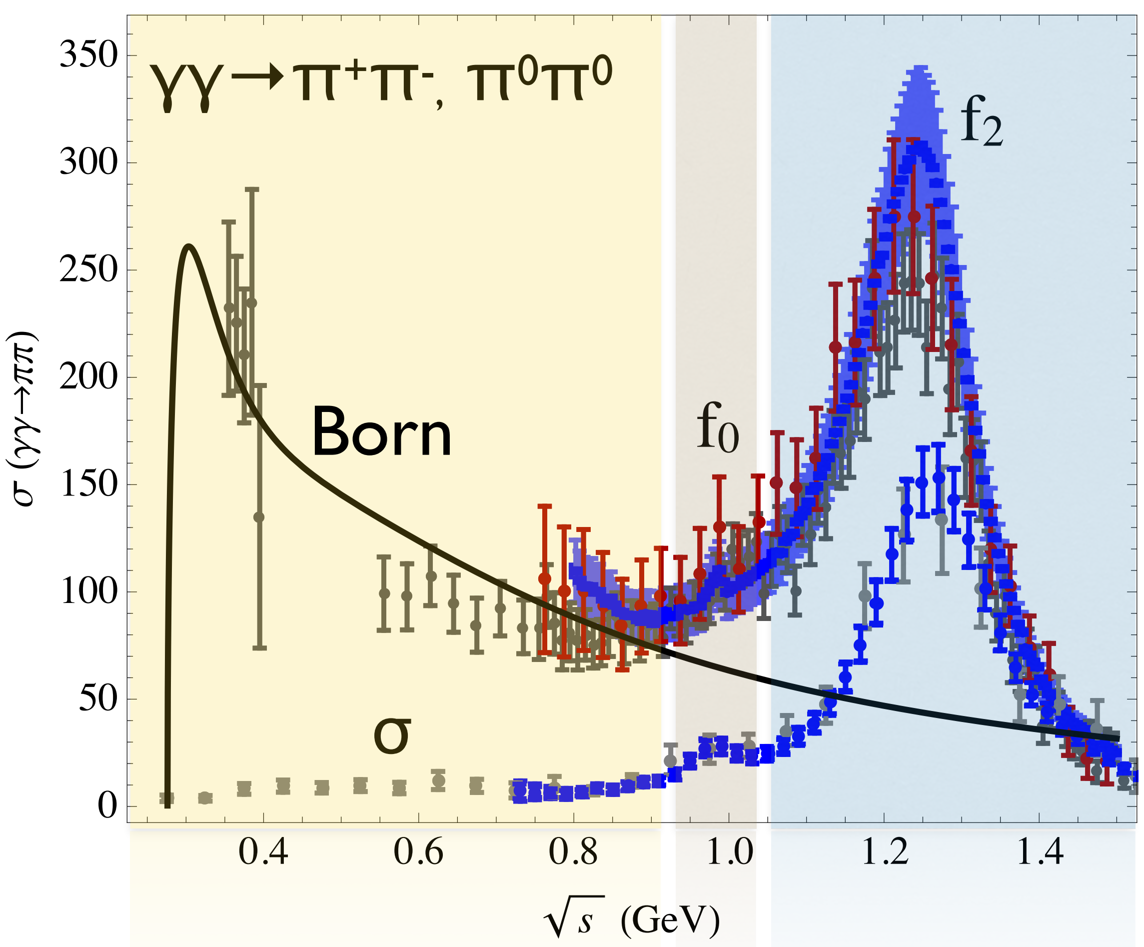
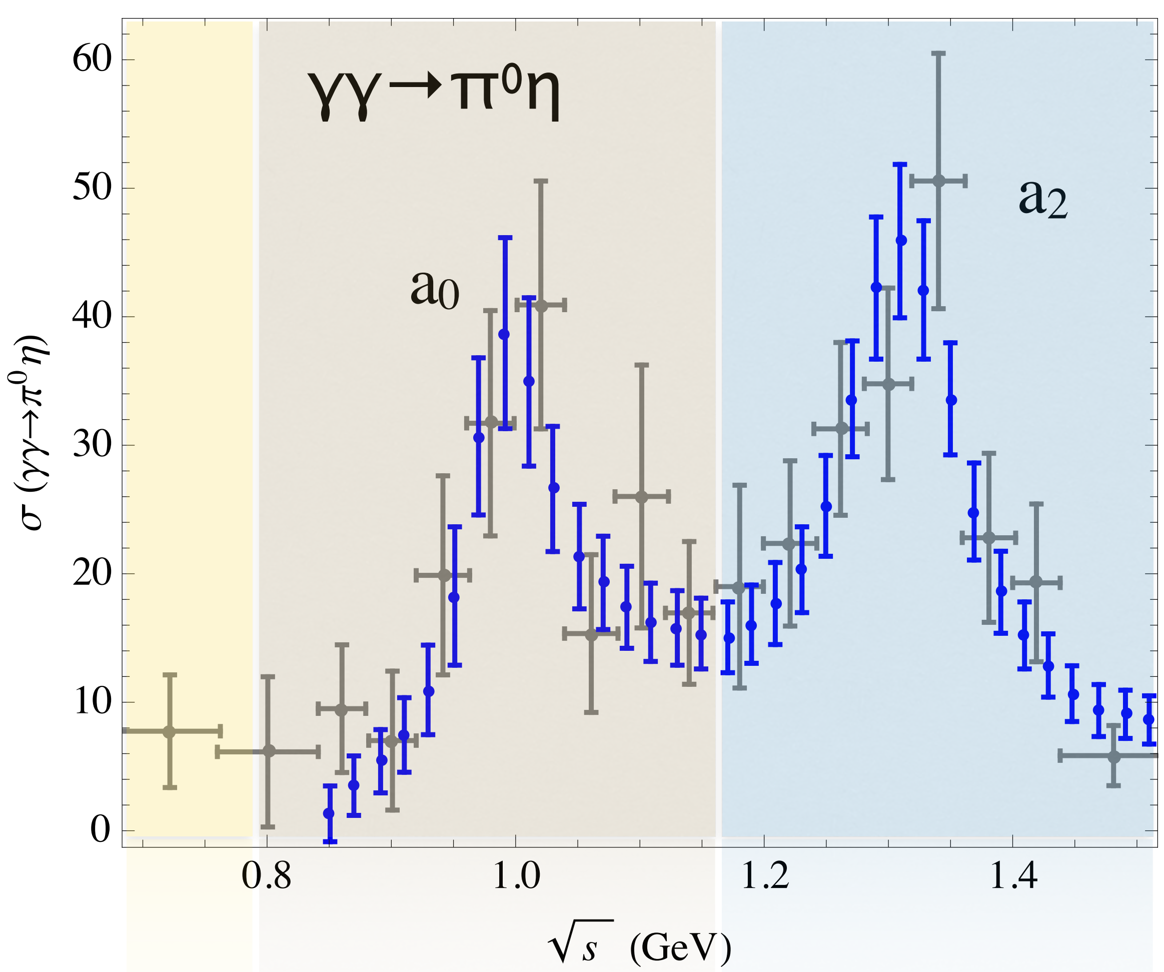
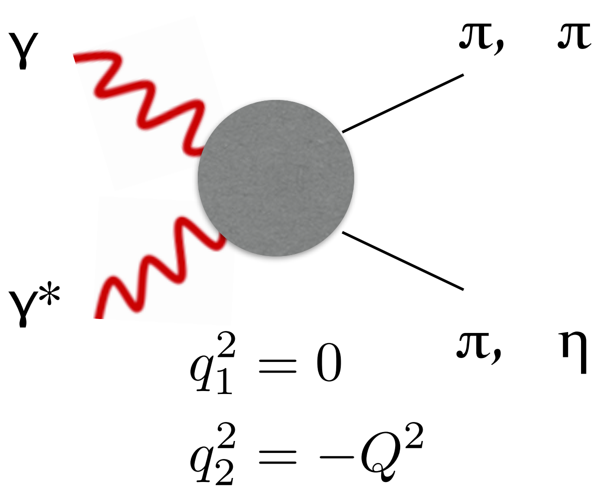




Cross section



C=+1:   J

Helicity amplitudes

$$\langle \pi(p_1)\pi(p_2)|T|\gamma(q_1,\lambda_1)\gamma(q_2,\lambda_2)\rangle = (2\pi)^4\,\delta^{(4)}(p_1+p_2-q_1-q_2)\,H_{\lambda_1\lambda_2}$$

$$H_{\lambda_1\lambda_2} = H^{\mu\nu}\epsilon_\mu(\lambda_1)\,\epsilon_\nu(\lambda_2), \quad \lambda_1 = \pm1,\ \lambda_2 = \pm1, 0$$

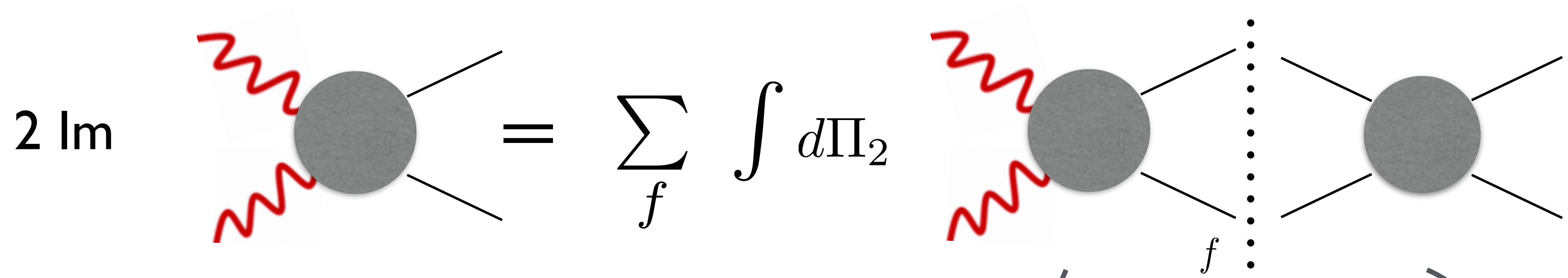
P symmetry: **6** → **3** independent amplitudes H_{++}, H_{+-}, H_{+0}

Cross sections

$$\sigma_{TT} = \pi\alpha^2 \frac{\rho(s)}{4\,(s+Q^2)} \int d\cos\theta\,\left(|H_{++}|^2 + |H_{+-}|^2\right)$$

$$\sigma_{TL} = \pi\alpha^2 \frac{\rho(s)}{2\,(s+Q^2)} \int d\cos\theta\,|H_{+0}|^2$$

Unitarity & Analyticity

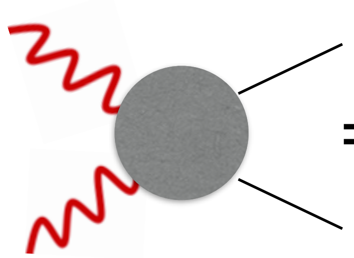
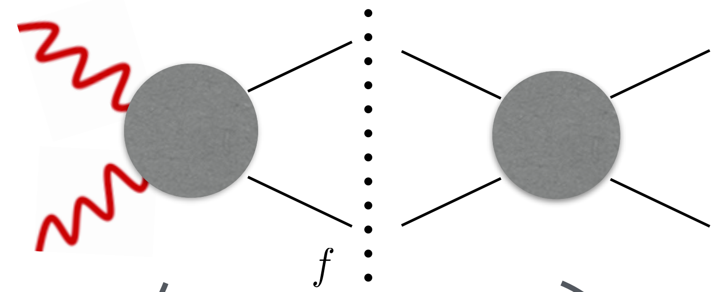


Partial wave expansion

$$H_{\lambda_1\lambda_2}(s,t) = \sum_{J=0}^{\infty}(2J+1)\, h_{J,\lambda_1\lambda_2}(s)\, d^J_{\lambda_1-\lambda_2,0}(\theta)$$

$$T(s,t) = \sum_{J=0}^{\infty}(2J+1)\, t_J(s)\, P_J(\theta)$$

Unitarity & Analyticity

2 Im  $= \sum_f \int d\Pi_2$ 

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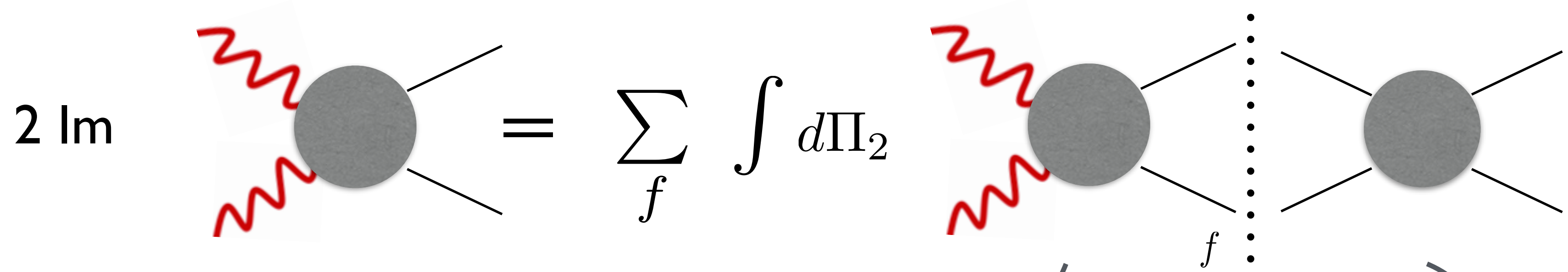

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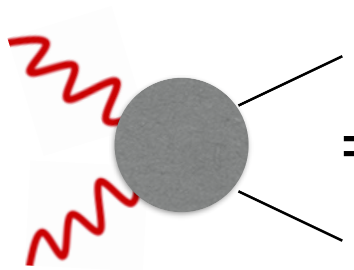
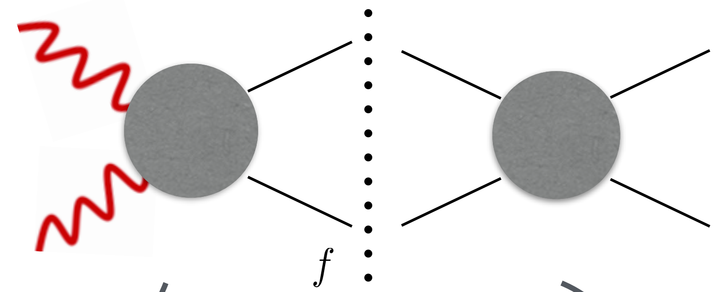
These "diagonalise unitarity" and contain resonance information

Definite: J, λ_1, λ_2

$$\mathrm{Im}\, h_{\gamma\gamma^*\to\pi\pi}(s) = h_{\gamma\gamma^*\to\pi\pi}(s)\, \rho_{\pi\pi}(s)\, t^*_{\pi\pi\to\pi\pi}(s)$$

$$\mathrm{Im}\, h_{\gamma\gamma^*\to\pi\pi}(s) = \rho_{\pi\pi}\, h_{\gamma\gamma^*\to\pi\pi}\, t^*_{\pi\pi\to\pi\pi} + \rho_{KK}\, h_{\gamma\gamma^*\to KK}\, t^*_{KK\to\pi\pi} + \dots$$

Unitarity & Analyticity

2 Im  $= \sum_f \int d\Pi_2$ 

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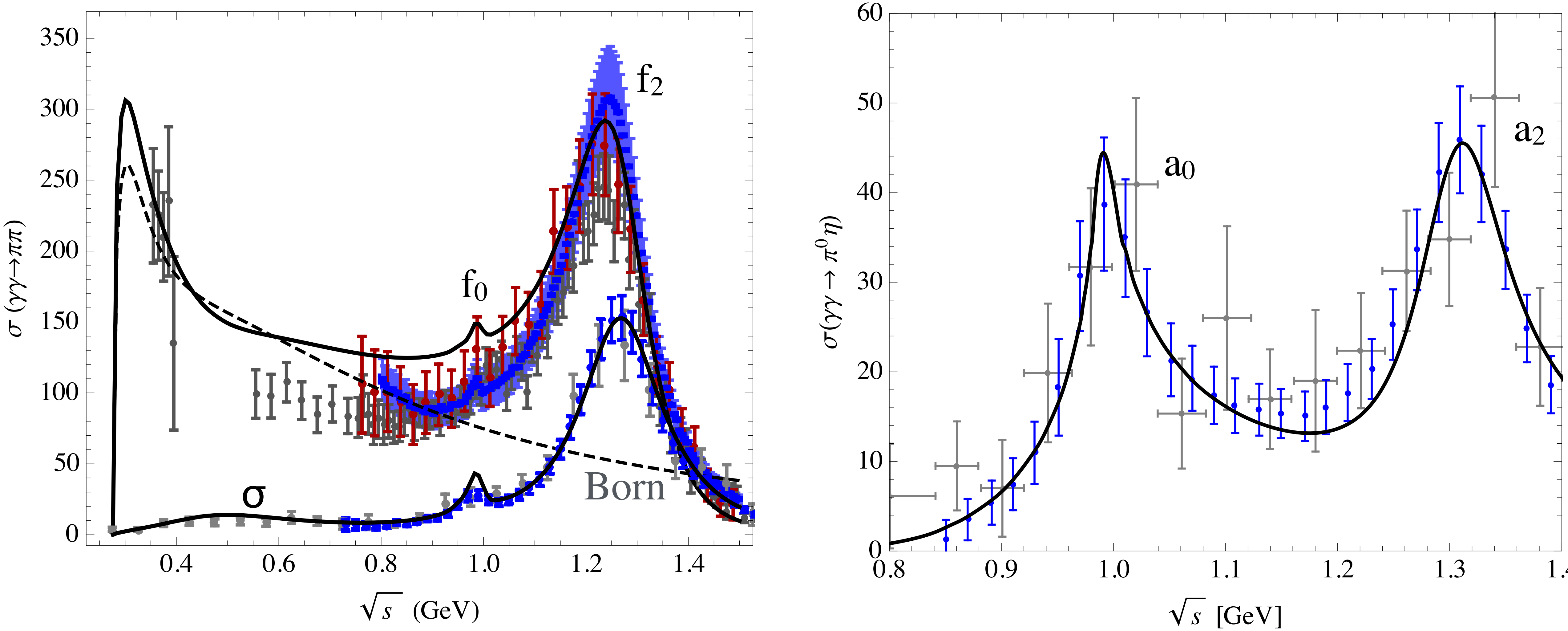
Analyticity relates scattering amplitude at different energies

$$h(s) = \frac{1}{2\pi i}\int_C ds' \frac{h(s')}{s'-s} = \int_{-\infty}^{s_L} \frac{ds'}{\pi}\frac{\mathrm{Im}\, h(s')}{s'-s} + \int_{s_{th}}^{\infty} \frac{ds'}{\pi}\frac{\mathrm{Im}\, h(s')}{s'-s}$$

Results for $Q^2=0$

$\gamma\gamma \to \pi^+\pi^-, \pi^0\pi^0$ $\gamma\gamma \to \pi^0\eta$



- **Coupled-channel** dispersive treatment of $f_0(980)$ and $a_0(980)$ is **crucial**
- $f_2(1270)$ described dispersively through Omnes function
- $a_2(1320)$ described as a Breit Wigner resonance

I.D., Deineka, Vanderhaeghen (2017)

I.D., Vanderhaeghen (work in progress)

What has been done so far?

$Q^2 = 0$	Approach	Inelasticity	Number of fitted parameters to $\sigma_{\gamma\gamma\to MM}$	Range of applicability
[Hoferichter et. al. 2011]	Roy-Steiner	$\pi\pi$	0	$\sqrt{s} < 0.98$ GeV
[Morgan et. al. 1998]	Disp, Omnes	$\pi\pi$	0	$\sqrt{s} \lesssim 0.6$ GeV
[Dai et. al. 2014]	Amplitude anal.	$\pi\pi$, KK	>20	$\sqrt{s} < 1.5$ GeV
[Garcia-Martin et.al. 2010]	Disp, Omnes	$\pi\pi$, KK	6	$\sqrt{s} < 1.3$ GeV
[Current work]	Disp, Omnes	$\pi\pi$, KK $\pi\eta$, KK	0	$\sqrt{s} < 1.4$ GeV
$Q^2 \neq 0$				
[Moussallam 2013]	Disp, Omnes	$\pi\pi$, J=0	0	$\sqrt{s} \lesssim 0.8$ GeV
[Colangelo et.al. 2017]	Roy-Steiner	$\pi\pi$, J=0	0	$\sqrt{s} \lesssim 0.8$ GeV
[Current work]	Disp, Omnes	$\pi\pi$, KK, J=0,2 $\pi\eta$, KK	0	$\sqrt{s} < 1.4$ GeV

Only dispersive analyses are shown

Kinematic constraints: $Q^2=0$

helicity amplitudes $$H_{\lambda_1,\lambda_2} = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2)\sum_{i=1}^{2} F_i(s,t)\, L_i^{\mu\nu}$$

- ✓ Ward identities
- ✓ free from kinematic constraints
- ✓ valid for $m_1 \neq m_2$

Bardeen, Tung (1968)

I.D, Lutz, Leupold, Terschlusen (2012)

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$$h^{(J)}_{\lambda_1\lambda_2} = \int \frac{d\cos\theta}{2}\, d^J_{\lambda_1-\lambda_2,0}(\theta)\, H_{\lambda_1\lambda_2}$$

<u>object free of kinematic constraints</u>

$$A_n^{(J)} = \frac{1}{(p\,q)^J}\int \frac{d\cos\theta}{2}\,P_J(\theta)\,F_n(s,t)$$

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Born subtracted ampl. at small energies:

$\Lambda=0, J\geq 0$ $$h^{(J)}_{++}(s) - h^{(J),Born}_{++}(s) \simeq s^{J/2+1}\, p^J$$

$\Lambda=2, J\geq 2$ $$h^{(J)}_{+-}(s) - h^{(J),Born}_{+-}(s) \simeq s^{J/2}\, p^J$$

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Soft photon theorem

Low, Gell-Mann, Goldberger (1954)

Born subtracted ampl. at small energies:

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Λ=2, J≥2 $$h^{(J)}_{+-}(s) - h^{(J),Born}_{+-}(s) \simeq s^{J/2}\, p^J$$

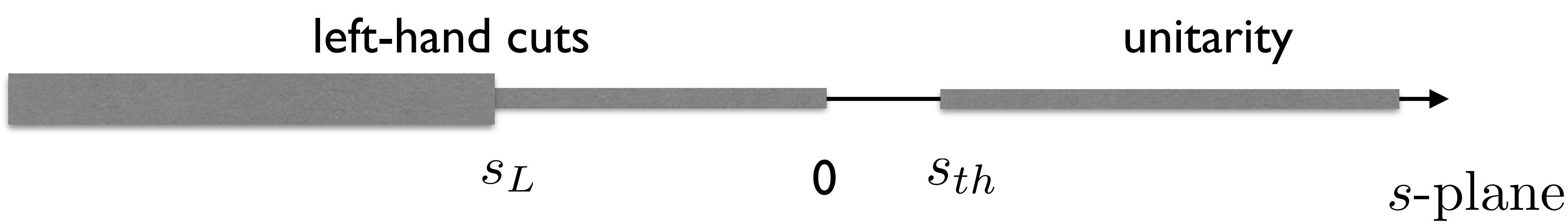
Dispersion relation: $Q^2=0$

Morgan et. al. (1998)
Garcia-Martin et. al. (2010)

Write unsubtracted dispersive representation for

$$\Omega_J^{-1}(s)\,\frac{(h_{++}^{(J)}(s)-h_{++}^{(J)\,Born}(s))}{s^{J/2+1}\,p^J}$$

$$\Omega_J^{-1}(s)\,\frac{(h_{+-}^{(J)}(s)-h_{+-}^{(J)\,Born}(s))}{s^{J/2}\,p^J}$$



Helicity - 0, s-wave

$$\begin{pmatrix} h_{++}^{(0)}(s) \\ k_{++}^{(0)}(s) \end{pmatrix} = \begin{pmatrix} h_{++}^{(0)Born}(s) \\ k_{++}^{(0)Born}(s) \end{pmatrix} + s\,\Omega_J(s)\left[\frac{1}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{s'}\,\frac{\Omega_J(s')^{-1}}{s'-s}\begin{pmatrix} \mathrm{Im}\,\bar{h}_{++}^{(0)}(s') \\ \mathrm{Im}\,\bar{k}_{++}^{(0)}(s') \end{pmatrix}\right.$$
$$\left. -\frac{1}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'}\,\frac{\mathrm{Im}\,\Omega_J(s')^{-1}}{s'-s}\begin{pmatrix} h_{++}^{(0)Born}(s') \\ k_{++}^{(0)Born}(s') \end{pmatrix}\right]$$

Unitarity: $s \geq s_{th}$

$$\mathrm{Im}\,h(s) = h(s)\,\rho(s)\,t^*(s)$$
$$\mathrm{Im}\,\Omega(s) = \Omega(s)\,\rho(s)\,t^*(s)$$

Kinematic constraints: $Q^2 \neq 0$

helicity amplitudes $$H_{\lambda_1,\lambda_2} = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2) \sum_{i=1}^{3} F_i(s,t)\, L_i^{\mu\nu}$$

Tarrach (1975)
Drechsel, Metz et al (1998)

$$\begin{aligned}
H_{++} &= (s+Q^2)\left(-\frac{1}{2}F_1 + \frac{2\,p^2}{s}Q^2\cos^2\theta\,(F_2 + (s+Q^2)\,F_3)\right) \\
H_{+-} &= \left(s+Q^2\right)\left(-2\sin^2\theta\,p^2\,F_2\right) \\
H_{+0} &= \left(s+Q^2\right)\sqrt{Q^2}\,\sin\theta\,\cos\theta\,p^2\sqrt{\frac{2}{s}}\left(-2\,F_2 - (s+Q^2)\,F_3\right)
\end{aligned}$$

Kinematic constraints: $Q^2 \neq 0$

helicity amplitudes $$H_{\lambda_1,\lambda_2} = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2) \sum_{i=1}^{3} F_i(s,t)\, L_i^{\mu\nu}$$

Tarrach (1975)
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H_{+0} &= \left(s+Q^2\right)\sqrt{Q^2}\,\sin\theta\,\cos\theta\,p^2\sqrt{\frac{2}{s}}\left(-2\,F_2 - (s+Q^2)\,F_3\right)
\end{aligned}$$

p.w. helicity amplitudes

$$h^{(J)}_{\lambda_1\lambda_2} = \int \frac{d\cos\theta}{2}\, d^J_{\lambda_1-\lambda_2,0}(\theta)\, H_{\lambda_1\lambda_2}$$

<u>object free of kinematic constraints</u>

$$A_n^{(J)} = \frac{1}{(p\,q)^J}\int \frac{d\cos\theta}{2}\, P_J(\theta)\, F_n(s,t)$$

Kinematic constraints: $Q^2 \neq 0$

helicity amplitudes $$H_{\lambda_1,\lambda_2} = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2) \sum_{i=1}^{3} F_i(s,t)\, L_i^{\mu\nu}$$

Tarrach (1975)
Drechsel, Metz et al (1998)

$$
\begin{aligned}
H_{++} &= (s+Q^2)\left(-\frac{1}{2}F_1 + \frac{2\,p^2}{s}Q^2\cos^2\theta\,(F_2 + (s+Q^2)\,F_3)\right)\\
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<u>object free of kinematic constraints</u>

$$A_n^{(J)} = \frac{1}{(p\,q)^J}\int \frac{d\cos\theta}{2}\, P_J(\theta)\, F_n(s,t)$$

$$
\begin{aligned}
h^{(J)}_{++}(s) &= (s+Q^2)\,p^J\,q^{J-2}\left[-\frac{1}{2}\,q^2\,A_1^J(s) + \alpha_J\frac{2\,Q^2}{s}\left(A_2^{J-2}(s) + (s+Q^2)A_3^{J-2}(s)\right) + \ldots\right]\\
h^{(J)}_{+-}(s) &= (s+Q^2)\,p^J\,q^{J-2}\left[-2\,\beta_J\,A_2^{J-2}(s) + \ldots\right]\\
h^{(J)}_{+0}(s) &= (s+Q^2)\,p^J\,q^{J-2}\sqrt{\frac{Q^2}{s}}\left[\sqrt{2}\,\gamma_J\,\left(-2\,A_2^{J-2}(s) - (s+Q^2)A_3^{J-2}(s)\right) + \ldots\right]
\end{aligned}
$$

Kinematic constraints: $Q^2 \neq 0$

Tarrach (1975)
Drechsel, Metz et al (1998)

helicity amplitudes $\quad H_{\lambda_1,\lambda_2} = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2) \sum_{i=1}^{3} F_i(s,t)\, L_i^{\mu\nu}$

$$
\begin{aligned}
H_{++} &= (s+Q^2)\left(-\frac{1}{2}F_1 + \frac{2\,p^2}{s}Q^2\cos^2\theta\,(F_2 + (s+Q^2)\,F_3)\right)\\
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$$
\begin{aligned}
h^{(J)}_{++}(s) &= (s+Q^2)\,p^J\,q^{J-2}\left[-\frac{1}{2}\,q^2\,A_1^J(s) + \alpha_J\frac{2\,Q^2}{s}\left(A_2^{J-2}(s) + (s+Q^2)A_3^{J-2}(s)\right) + \ldots\right]\\
h^{(J)}_{+-}(s) &= (s+Q^2)\,p^J\,q^{J-2}\left[-2\,\beta_J\,A_2^{J-2}(s) + \ldots\right]\\
h^{(J)}_{+0}(s) &= (s+Q^2)\,p^J\,q^{J-2}\sqrt{\frac{Q^2}{s}}\left[\sqrt{2}\,\gamma_J\,\left(-2\,A_2^{J-2}(s) - (s+Q^2)A_3^{J-2}(s)\right) + \ldots\right]
\end{aligned}
$$

Born subtracted ampl. at small energies:

$$
\begin{pmatrix}\hat h_1^{(J)}\\ \hat h_2^{(J)}\\ \hat h_3^{(J)}\end{pmatrix} = \frac{1}{(s+Q^2)\,p^J\,q^{J-2}}
\begin{pmatrix}
1 & 0 & 0\\
\frac{1}{\beta_J}\frac{1}{s+Q^2} & -\frac{1}{\sqrt{2}\,\gamma_J}\frac{1}{s+Q^2}\sqrt{\frac{Q^2}{s}} & 0\\
-\frac{\alpha_J}{\beta_J}\frac{Q^2}{s\,q^2} & \frac{\sqrt{2}\,\alpha_J}{\gamma_J}\frac{Q^2}{s\,q^2}\sqrt{\frac{Q^2}{s}} & \frac{1}{q^2}
\end{pmatrix}
\begin{pmatrix}h_{+-}^{(J)}\\ h_{+0}^{(J)}\\ h_{++}^{(J)}\end{pmatrix}
$$

Kinematic constraints: $Q^2 \neq 0$

Tarrach (1975)
Drechsel, Metz et al (1998)

helicity amplitudes $\quad H_{\lambda_1,\lambda_2} = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2) \sum_{i=1}^{3} F_i(s,t)\, L_i^{\mu\nu}$

$$
\begin{aligned}
H_{++} &= (s+Q^2)\left(-\frac{1}{2}F_1 + \frac{2\,p^2}{s}Q^2\cos^2\theta\,(F_2 + (s+Q^2)\,F_3)\right)\\
H_{+-} &= (s+Q^2)\,(-2\sin^2\theta\, p^2\, F_2)\\
H_{+0} &= (s+Q^2)\,\sqrt{Q^2}\,\sin\theta\,\cos\theta\, p^2\sqrt{\frac{2}{s}}\,(-2\,F_2 - (s+Q^2)\,F_3)
\end{aligned}
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p.w. helicity amplitudes

$$
h^{(J)}_{\lambda_1\lambda_2} = \int \frac{d\cos\theta}{2}\, d^J_{\lambda_1-\lambda_2,0}(\theta)\, H_{\lambda_1\lambda_2}
$$

<u>object free of kinematic constraints</u>

$$
A_n^{(J)} = \frac{1}{(p\,q)^J}\int \frac{d\cos\theta}{2}\, P_J(\theta)\, F_n(s,t)
$$

$$
\begin{aligned}
h^{(J)}_{++}(s) &= (s+Q^2)\,p^J\,q^{J-2}\left[-\frac{1}{2}\,q^2\,A_1^J(s) + \alpha_J\frac{2\,Q^2}{s}\,(A_2^{J-2}(s) + (s+Q^2)A_3^{J-2}(s)) + \ldots\right]\\
h^{(J)}_{+-}(s) &= (s+Q^2)\,p^J\,q^{J-2}\left[-2\,\beta_J\,A_2^{J-2}(s) + \ldots\right]\\
h^{(J)}_{+0}(s) &= (s+Q^2)\,p^J\,q^{J-2}\sqrt{\frac{Q^2}{s}}\left[\sqrt{2}\,\gamma_J\,(-2\,A_2^{J-2}(s) - (s+Q^2)A_3^{J-2}(s)) + \ldots\right]
\end{aligned}
$$

Born subtracted ampl. at small energies:

Λ=0, J=0 $\quad h^{(0)}_{++}(s) - h^{(0),Born}_{++}(s) \simeq (s+Q^2)$

Λ=2, J≥2 $\quad h^{(J)}_{+-}(s) - h^{(J),Born}_{+-}(s) \simeq (s+Q^2)\,p^J\,q^{J-2}$

Kinematic constraints: $Q^2 \neq 0$

Tarrach (1975)
Drechsel, Metz et al (1998)

helicity amplitudes $$H_{\lambda_1,\lambda_2} = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2) \sum_{i=1}^{3} F_i(s,t)\, L_i^{\mu\nu}$$

$$\begin{aligned}
H_{++} &= (s+Q^2)\left(-\frac{1}{2}F_1 + \frac{2\,p^2}{s}Q^2\cos^2\theta\,(F_2 + (s+Q^2)\,F_3)\right) \\
H_{+-} &= (s+Q^2)\,(-2\sin^2\theta\, p^2\, F_2) \\
H_{+0} &= (s+Q^2)\,\sqrt{Q^2}\,\sin\theta\,\cos\theta\, p^2\sqrt{\frac{2}{s}}\,(-2\,F_2 - (s+Q^2)\,F_3)
\end{aligned}$$

p.w. helicity amplitudes

$$h^{(J)}_{\lambda_1\lambda_2} = \int \frac{d\cos\theta}{2}\, d^J_{\lambda_1-\lambda_2,0}(\theta)\, H_{\lambda_1\lambda_2}$$

<u>object free of kinematic constraints</u>

$$A_n^{(J)} = \frac{1}{(p\,q)^J}\int \frac{d\cos\theta}{2}\, P_J(\theta)\, F_n(s,t)$$

$$\begin{aligned}
h^{(J)}_{++}(s) &= (s+Q^2)\,p^J\,q^{J-2}\left[-\frac{1}{2}\,q^2\,A_1^J(s) + \alpha_J\frac{2\,Q^2}{s}\,(A_2^{J-2}(s) + (s+Q^2)A_3^{J-2}(s)) + \ldots\right] \\
h^{(J)}_{+-}(s) &= (s+Q^2)\,p^J\,q^{J-2}\left[-2\,\beta_J\,A_2^{J-2}(s) + \ldots\right] \\
h^{(J)}_{+0}(s) &= (s+Q^2)\,p^J\,q^{J-2}\sqrt{\frac{Q^2}{s}}\left[\sqrt{2}\,\gamma_J\,(-2\,A_2^{J-2}(s) - (s+Q^2)A_3^{J-2}(s)) + \ldots\right]
\end{aligned}$$

Soft photon theorem

Low, Gell-Mann,
Goldberger (1954)

Born subtracted ampl. at small energies:

Λ=0, J=0 $$h^{(0)}_{++}(s) - h^{(0),Born}_{++}(s) \simeq (s+Q^2)$$

Λ=2, J≥2 $$h^{(J)}_{+-}(s) - h^{(J),Born}_{+-}(s) \simeq (s+Q^2)\,p^J\,q^{J-2}$$

Dispersion relation: $Q^2 \neq 0$

I.D., Vanderhaeghen
(work in progress)

Moussallam (2013)

Write unsubtracted dispersive representation for

$$\Omega_0^{-1}(s)\frac{(h_{++}^{(0)}(s) - h_{++}^{(0)\,Born}(s))}{s+Q^2}$$

Helicity - 0, s-wave

$$\begin{pmatrix} h_{++}^{(0)}(s) \\ k_{++}^{(0)}(s) \end{pmatrix} = \begin{pmatrix} h_{++}^{(0)Born}(s) \\ k_{++}^{(0)Born}(s) \end{pmatrix} + (s+Q^2)\,\Omega_0(s)\left[\frac{1}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{s'+Q^2}\,\frac{\Omega_0(s')^{-1}}{s'-s}\begin{pmatrix} \mathrm{Im}\,\bar{h}_{++}^{(0)}(s') \\ \mathrm{Im}\,\bar{k}_{++}^{(0)}(s') \end{pmatrix} - \frac{1}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'+Q^2}\,\frac{\mathrm{Im}\,\Omega_0(s')^{-1}}{s'-s}\begin{pmatrix} h_{++}^{(0)Born}(s') \\ k_{++}^{(0)Born}(s') \end{pmatrix}\right]$$

Coupled-channel Omnes function

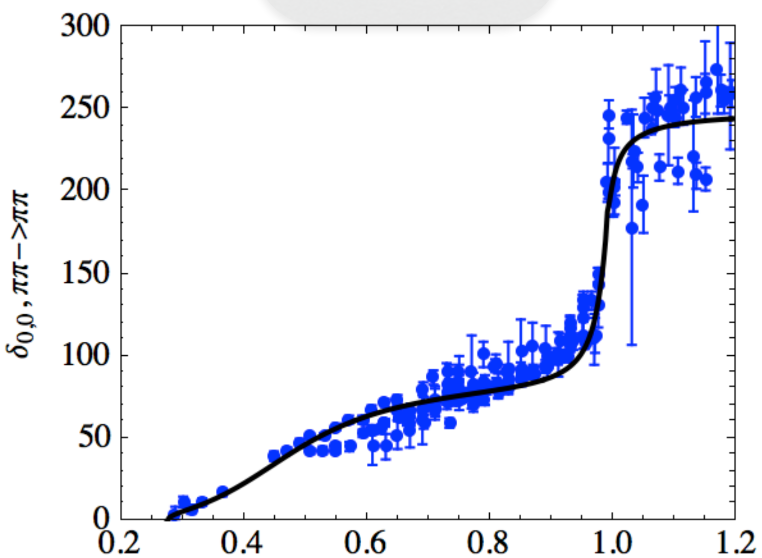
$$\Omega(s) = \begin{pmatrix} \Omega_{\pi\pi\to\pi\pi} & \Omega_{\pi\pi\to K\bar{K}} \\ \Omega_{K\bar{K}\to\pi\pi} & \Omega_{K\bar{K}\to K\bar{K}} \end{pmatrix}$$

Omnes function I=0, {ππ, KK}

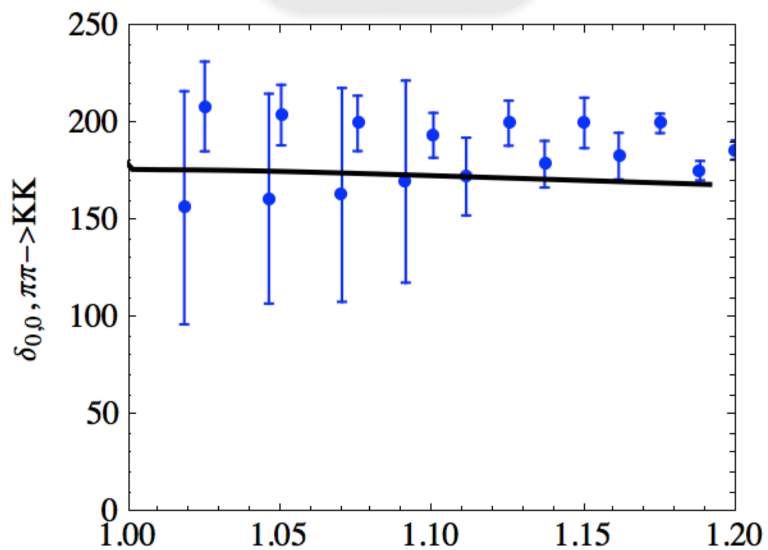
Coupled channel Omnes

$$\Omega(s) = \begin{pmatrix} \Omega_{\pi\pi\to\pi\pi} & \Omega_{\pi\pi\to K\bar{K}} \\ \Omega_{K\bar{K}\to\pi\pi} & \Omega_{K\bar{K}\to K\bar{K}} \end{pmatrix}$$

ππ→ππ

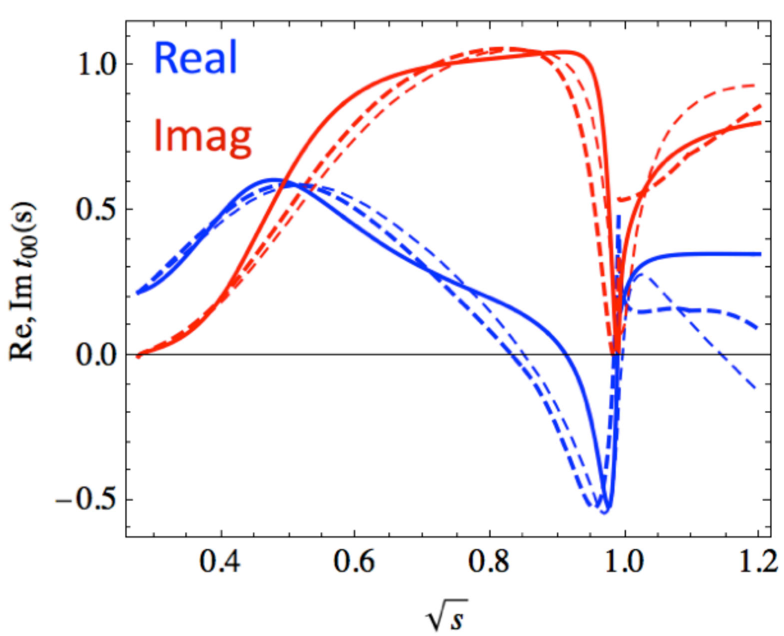


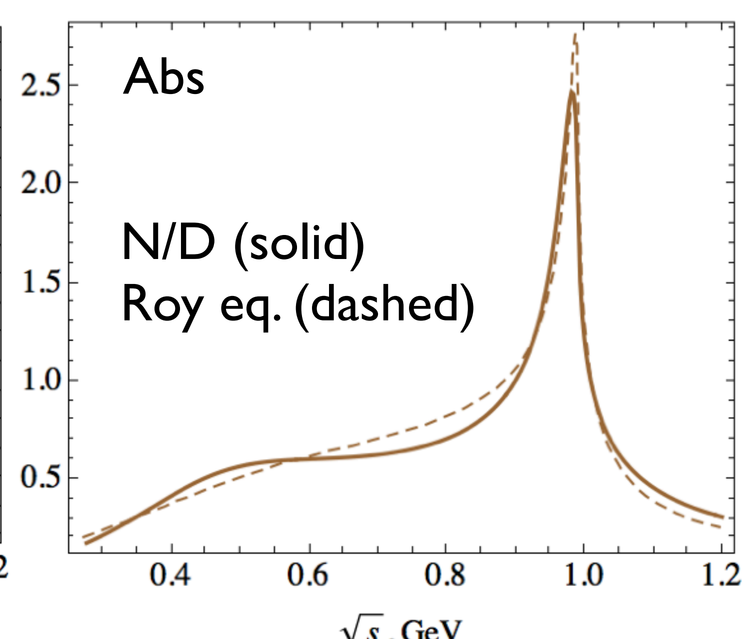
ππ→KK



Bounded p.w. amplitudes and Omnes at large energies

$$T(s) = \Omega(s)\, N(s)$$





$$N(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'} \frac{\rho(s')N(s')(U(s)-U(s'))}{s'-s}$$

$$\Omega^{-1}(s) = 1 - \frac{s}{\pi}\int_R \frac{ds'}{s'} \frac{\rho(s')N(s')}{s'-s}$$

$$U(s) = \sum_k C_k\, \xi(s)^k$$

Chew, Mandelstam
Lutz, I.D, Gasparyan

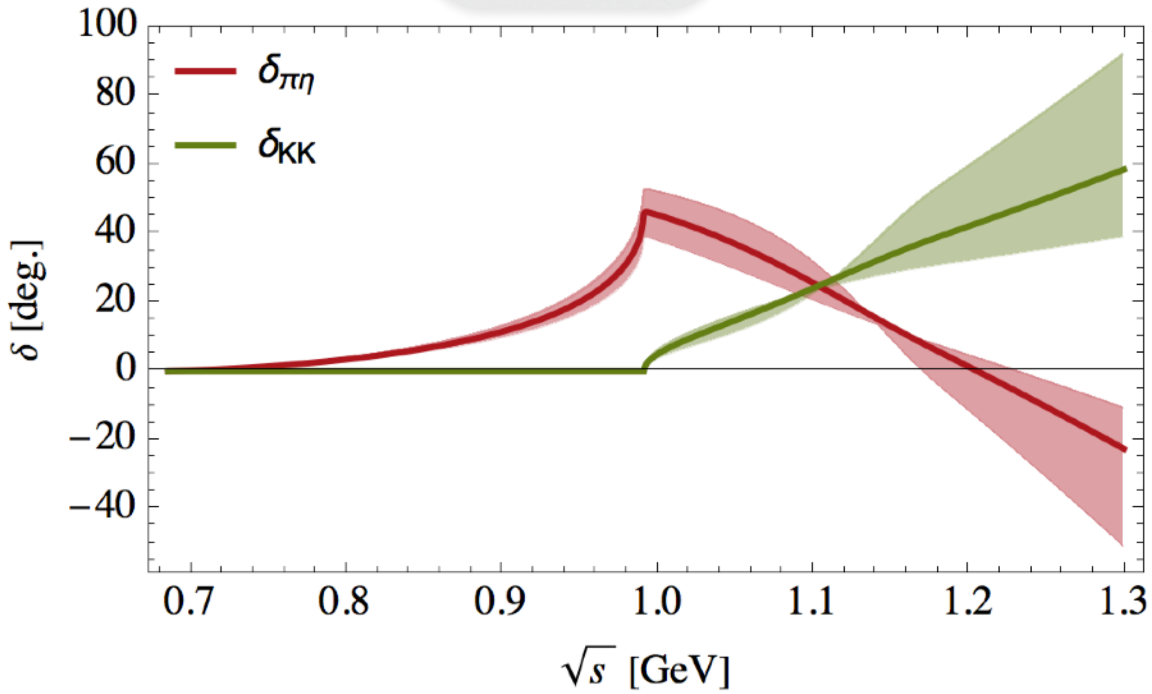
C_k **fitted** to Exp. **data** and Roy Eq. solutions

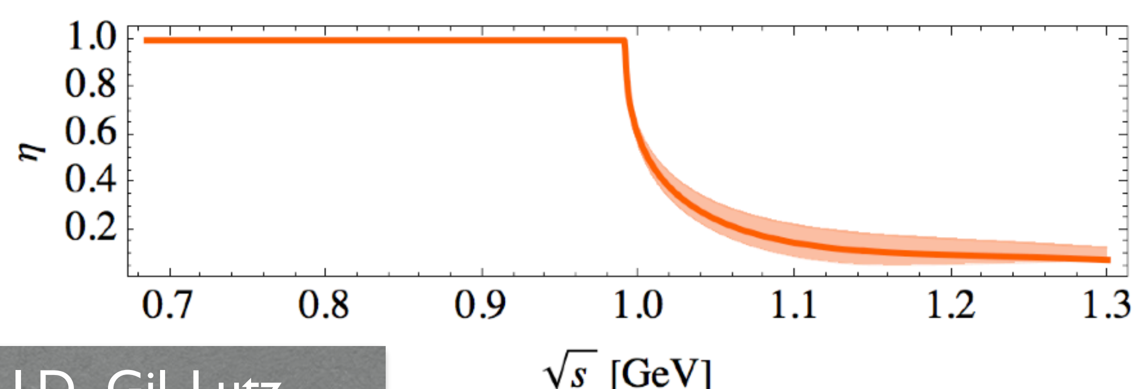
Omnes function I=1, {πη, KK}

Coupled channel Omnes

$$\Omega(s) = \begin{pmatrix} \Omega_{\pi\eta\to\pi\eta} & \Omega_{\pi\eta\to K\bar{K}} \\ \Omega_{K\bar{K}\to\pi\eta} & \Omega_{K\bar{K}\to K\bar{K}} \end{pmatrix}$$

πη→πη





I.D., Gil, Lutz (2011), (2013)

Bounded p.w. amplitudes and Omnes at large energies

$$T(s) = \Omega(s)\, N(s)$$

$$N(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'} \frac{\rho(s')N(s')(U(s)-U(s'))}{s'-s}$$

$$\Omega^{-1}(s) = 1 - \frac{s}{\pi}\int_R \frac{ds'}{s'} \frac{\rho(s')N(s')}{s'-s}$$

$$U(s) = \sum_k C_k\, \xi(s)^k$$

C_k **matched** to SU(3) **ChPT** at threshold

Poles in the complex plane

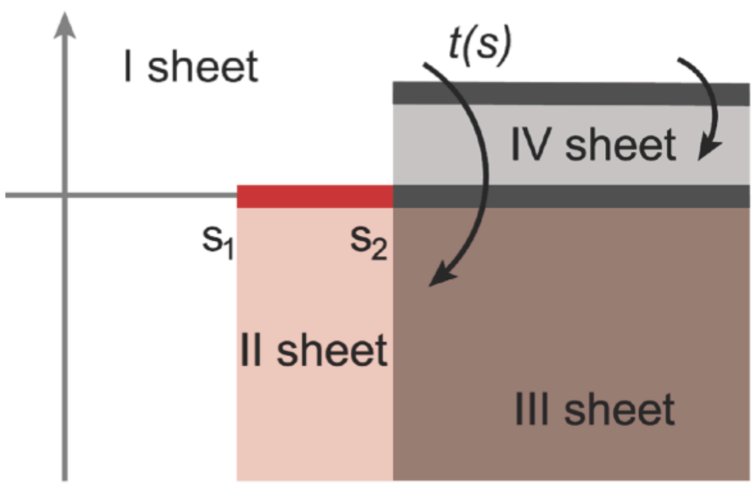
Unitarity:

$$t^I(s+i\epsilon) - t^I(s-i\epsilon) = 2i\rho(s)t^I(s+i\epsilon)t^I(s-i\epsilon)$$

$$t^I(s+i\epsilon) = \frac{t^I(s-i\epsilon)}{1-2i\rho(s)t^I(s-i\epsilon)}$$

$$t^{II}(s-i\epsilon) \overset{\epsilon\to 0}{=} t^I(s+i\epsilon)$$

$$t^{II}(s) = \frac{t^I(s)}{1-2i\rho(s)t^I(s)}$$



$$\rho_i(s) = 2k_i(s)/\sqrt{s}$$

Sheet	Im k_1	Im k_2
I	+	+
II	−	+
III	−	−
IV	+	−

sigma: $\sqrt{s_\sigma^{\mathrm{II}}} = 0.436(5) \pm \frac{i}{2}\, 0.357(40)\ \mathrm{GeV}$

$f_0(980)$: $\sqrt{s_{f_0}^{\mathrm{II}}} = 0.990(5) \pm \frac{i}{2}\, 0.033(20)\ \mathrm{GeV}$

$a_0(980)$: $\sqrt{s_{a_0}^{\mathrm{IV}}} = (1.12^{-0.07}_{+0.02}) \pm \frac{i}{2}\, (0.28^{+0.08}_{-0.13})\ \mathrm{GeV}$

Left-hand cuts

Dispersive integral for J=0

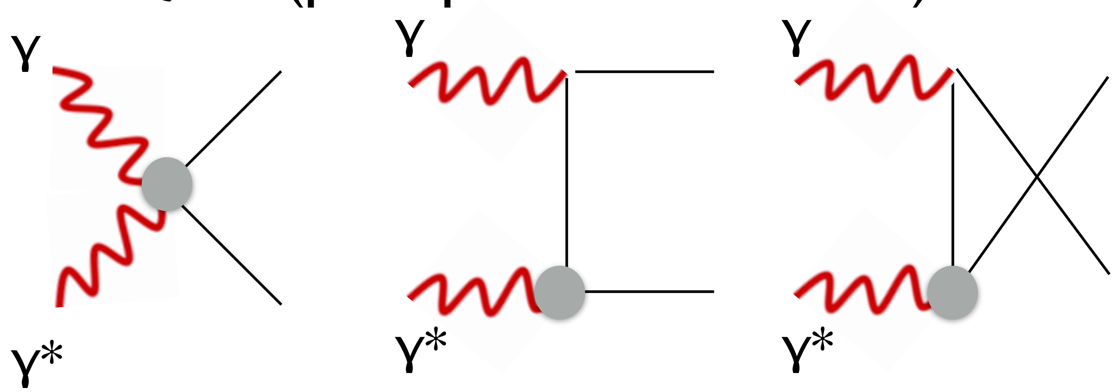
$$\left(\begin{array}{c} h_{++}^{(0)}(s) \\ k_{++}^{(0)}(s) \end{array}\right) = \left(\begin{array}{c} h_{++}^{(0)Born}(s) \\ k_{++}^{(0)Born}(s) \end{array}\right) + (s+Q^2)\,\Omega(s)\left[\left(\begin{array}{c} a \\ b \end{array}\right) + \frac{s+Q^2}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{(s'+Q^2)^2}\,\frac{\Omega(s')^{-1}}{s'-s}\left(\begin{array}{c} \mathrm{Im}\,\bar{h}_{++}^{(0)}(s') \\ \mathrm{Im}\,\bar{k}_{++}^{(0)}(s') \end{array}\right)\right.$$
$$\left. - \frac{s+Q^2}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{(s'+Q^2)^2}\,\frac{\mathrm{Im}\,\Omega(s')^{-1}}{s'-s}\left(\begin{array}{c} h_{++}^{(0)Born}(s') \\ k_{++}^{(0)Born}(s') \end{array}\right)\right]$$

Left-hand cuts

Dispersive integral for J=0

$$\begin{pmatrix} h^{(0)}_{++}(s) \\ k^{(0)}_{++}(s) \end{pmatrix} = \begin{pmatrix} h^{(0)Born}_{++}(s) \\ k^{(0)Born}_{++}(s) \end{pmatrix} + (s+Q^2)\,\Omega(s)\left[\begin{pmatrix} a \\ b \end{pmatrix} + \frac{s+Q^2}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{(s'+Q^2)^2}\frac{\Omega(s')^{-1}}{s'-s}\begin{pmatrix} \mathrm{Im}\,\bar{h}^{(0)}_{++}(s') \\ \mathrm{Im}\,\bar{k}^{(0)}_{++}(s') \end{pmatrix} \right.$$
$$\left. - \frac{s+Q^2}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{(s'+Q^2)^2}\frac{\mathrm{Im}\,\Omega(s')^{-1}}{s'-s}\begin{pmatrix} h^{(0)Born}_{++}(s') \\ k^{(0)Born}_{++}(s') \end{pmatrix}\right]$$

<u>Scalar QED (pion pole contribution)</u>



Fearing Scherer (1998)
Colangelo et.al. (2015)

Vertex ππγ*

$$\langle \pi^+ | j_\mu(0) | \pi^+(p') \rangle = (p+p')_\mu\, F_\pi(Q^2)$$

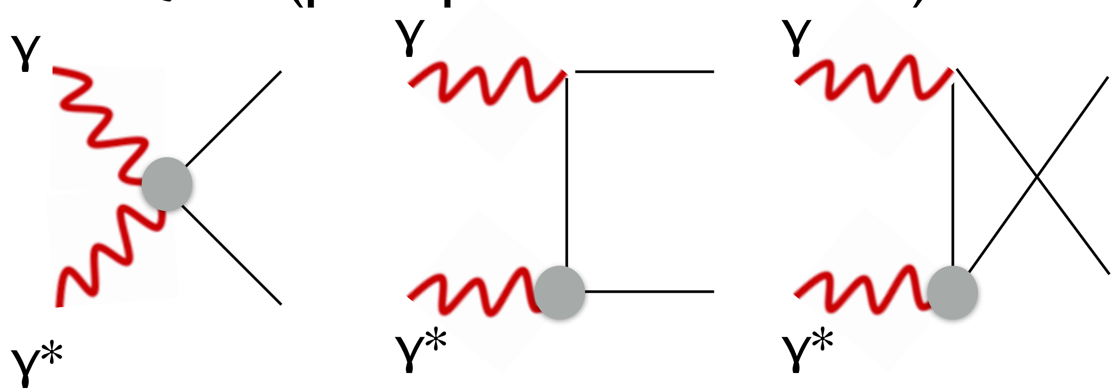
$$F_\pi(Q^2) = \frac{1}{1+Q^2/M_\rho^2}$$

Left-hand cuts

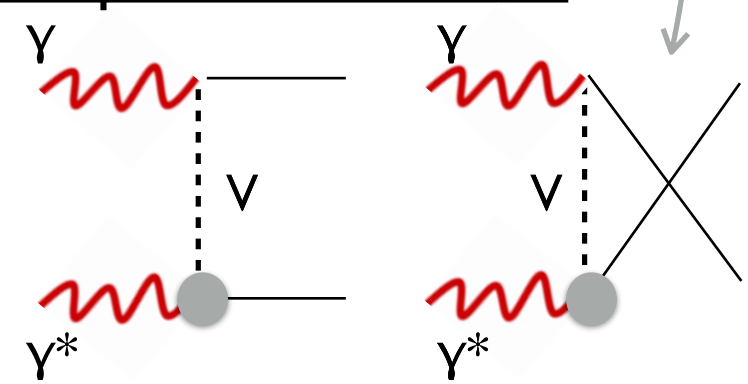
Dispersive integral for J=0

$$\begin{pmatrix} h^{(0)}_{++}(s) \\ k^{(0)}_{++}(s) \end{pmatrix} = \begin{pmatrix} h^{(0)Born}_{++}(s) \\ k^{(0)Born}_{++}(s) \end{pmatrix} + (s+Q^2)\,\Omega(s)\left[\begin{pmatrix} a \\ b \end{pmatrix} + \frac{s+Q^2}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{(s'+Q^2)^2}\,\frac{\Omega(s')^{-1}}{s'-s}\begin{pmatrix} \mathrm{Im}\,\bar{h}^{(0)}_{++}(s') \\ \mathrm{Im}\,\bar{k}^{(0)}_{++}(s') \end{pmatrix} - \frac{s+Q^2}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{(s'+Q^2)^2}\,\frac{\mathrm{Im}\,\Omega(s')^{-1}}{s'-s}\begin{pmatrix} h^{(0)Born}_{++}(s') \\ k^{(0)Born}_{++}(s') \end{pmatrix}\right]$$

Scalar QED (pion pole contribution)



Vector pole contribution



Fearing Scherer (1998)
Colangelo et.al. (2015)

Vertex ππγ*

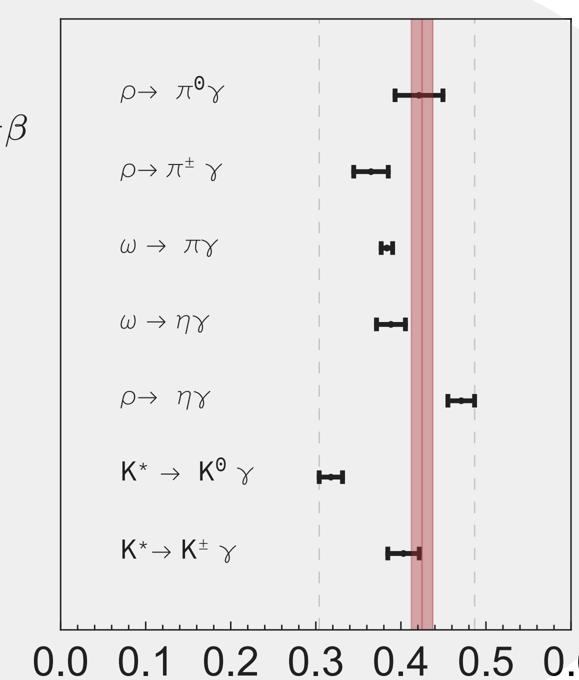
$$\langle \pi^+ | j_\mu(0) | \pi^+(p') \rangle = (p+p')_\mu\, F_\pi(Q^2)$$

$$F_\pi(Q^2) = \frac{1}{1+Q^2/M_\rho^2}$$

$$\mathcal{L} = e\, C_V\, \epsilon_{\mu\nu\alpha\beta}\, F^{\mu\nu}\, \partial^\alpha \phi\, V^\beta$$

$$F_{\pi\omega}(Q^2) = \frac{1}{1+Q^2/M_\rho^2}$$

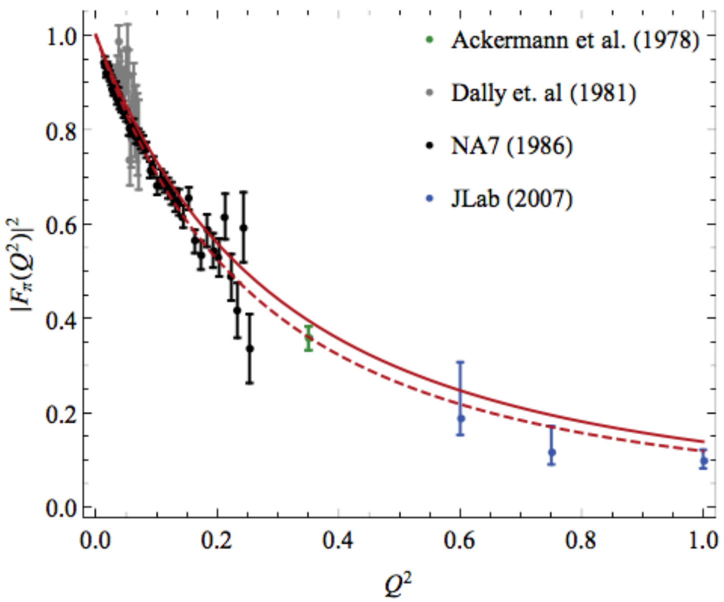
$$F_{\pi\rho}(Q^2) = \frac{1}{1+Q^2/M_\omega^2}$$

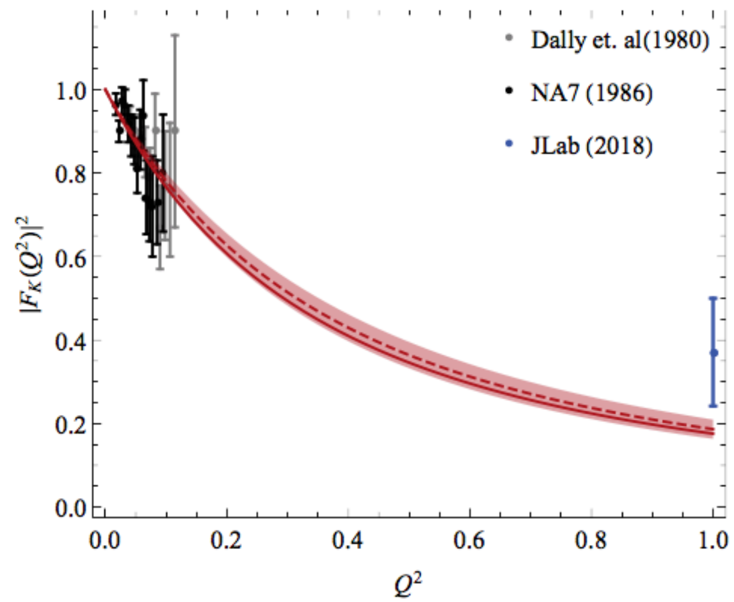


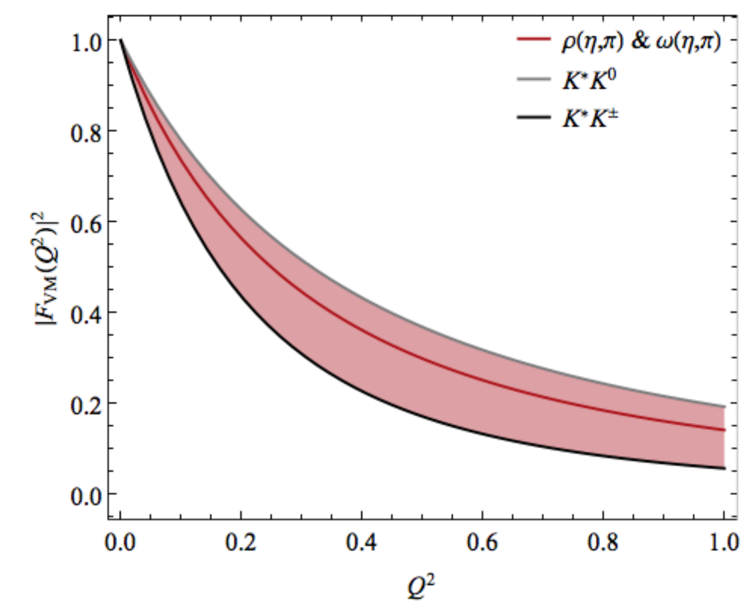
Form factors

Form factors (VMD):

$$F_\pi(Q^2) = \frac{1}{1+Q^2/m_\rho^2} \overset{\text{data}}{\Longrightarrow} \frac{1}{1+Q^2/\Lambda_\pi^2}\,,\ \Lambda_\pi^2 = 0.525\pm 0.008, \quad F_K(Q^2) = \frac{1/2}{1+Q^2/m_\rho^2} + \frac{1/6}{1+Q^2/m_\omega^2} + \frac{1/3}{1+Q^2/m_\phi^2} \overset{\text{data}}{\Longrightarrow} \frac{1}{1+Q^2/\Lambda_K^2}\,,\ \Lambda_K^2 = 0.760\pm 0.081$$





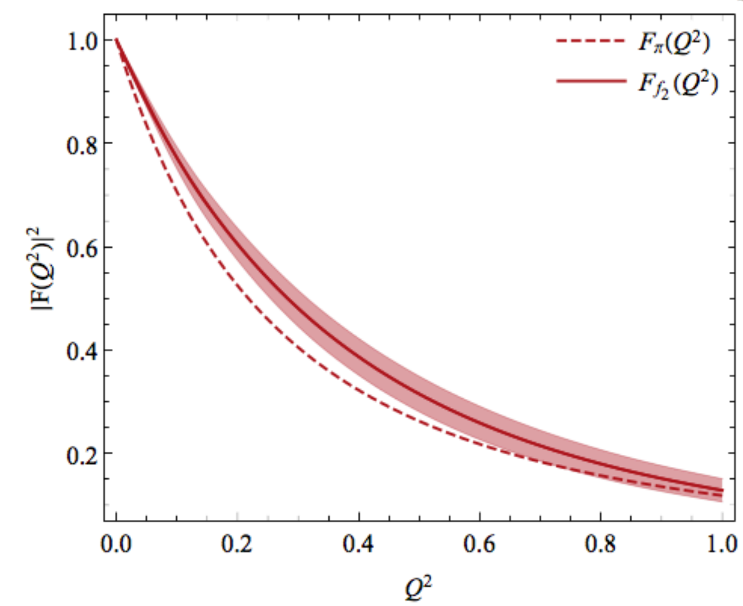


Form factor $a_2(1320)$:

$$F_{a_2}(Q^2) \approx F_{f_2}(Q^2) = \frac{1}{\left(1+Q^2/\Lambda_{f_2}^2\right)^2}\,,\Lambda_{f_2} = 1.222\pm 0.066$$

or from sum rules:

$$F_{a_2}(Q^2) \approx F_\pi(Q^2)$$



f_2(1270) form factor: M. Masuda *et al.* [Belle Collaboration], Phys. Rev. D **93** (2016) no.3, 032003
Light-by-light sum rules: V. Pascalutsa, V. Pauk and M. Vanderhaeghen, Phys. Rev. D **85** (2012) 116001

Subtraction constants

Dispersive integral for J=0

$$\begin{pmatrix} h_{++}^{(0)}(s) \\ k_{++}^{(0)}(s) \end{pmatrix} = \begin{pmatrix} h_{++}^{(0)Born}(s) \\ k_{++}^{(0)Born}(s) \end{pmatrix} + (s+Q^2)\,\Omega(s)\left[\begin{pmatrix} a \\ b \end{pmatrix} + \frac{s+Q^2}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{(s'+Q^2)^2}\,\frac{\Omega(s')^{-1}}{s'-s}\begin{pmatrix} \mathrm{Im}\,\bar{h}_{++}^{(0)}(s') \\ \mathrm{Im}\,\bar{k}_{++}^{(0)}(s') \end{pmatrix}\right.$$
$$\left. - \frac{s+Q^2}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{(s'+Q^2)^2}\,\frac{\mathrm{Im}\,\Omega(s')^{-1}}{s'-s}\begin{pmatrix} h_{++}^{(0)Born}(s') \\ k_{++}^{(0)Born}(s') \end{pmatrix}\right]$$

Unsubtracted dispersion relation (no VM)

{ππ,KK}

$$(\alpha_1-\beta_1)_{\pi^+} = 5.06\,(6.29)\,10^{-4}\,fm^3$$

$$(\alpha_1-\beta_1)_{\pi^0} = 8.47\,(9.71)\,10^{-4}\,fm^3$$

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Once-subtracted dispersion relation (with VM)

$$(\alpha_1-\beta_1)_{\pi^+}^{NLO} = 6.0\cdot 10^{-4}\,fm^3$$
$$(\alpha_1-\beta_1)_{\pi^0}^{NLO} = -1.0\cdot 10^{-4}\,fm^3$$

$4.0 \pm 1.2 \pm 1.4\cdot 10^{-4}\,fm^3$

COMPASS data on $(\alpha_1-\beta_1)_{\pi^+}$

Subtraction constants

Dispersive integral for J=0

$$\begin{pmatrix} h_{++}^{(0)}(s) \\ k_{++}^{(0)}(s) \end{pmatrix} = \begin{pmatrix} h_{++}^{(0)Born}(s) \\ k_{++}^{(0)Born}(s) \end{pmatrix} + (s+Q^2)\,\Omega(s)\left[\begin{pmatrix} a \\ b \end{pmatrix} + \frac{s+Q^2}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{(s'+Q^2)^2}\frac{\Omega(s')^{-1}}{s'-s}\begin{pmatrix} \mathrm{Im}\,\bar{h}_{++}^{(0)}(s') \\ \mathrm{Im}\,\bar{k}_{++}^{(0)}(s') \end{pmatrix} - \frac{s+Q^2}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{(s'+Q^2)^2}\frac{\mathrm{Im}\,\Omega(s')^{-1}}{s'-s}\begin{pmatrix} h_{++}^{(0)Born}(s') \\ k_{++}^{(0)Born}(s') \end{pmatrix}\right]$$

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$4.0 \pm 1.2 \pm 1.4 \cdot 10^{-4}\,fm^3$

COMPASS data on $(\alpha_1-\beta_1)_{\pi^+}$

For space like photons: generalized polarizabilities

$$\pm\frac{2\alpha}{m_\pi}\frac{H_{+\pm}^{n}}{s+Q^2} = (\alpha_1\mp\beta_1)_{\pi^0}+\ldots$$
$$\pm\frac{2\alpha}{m_\pi}\frac{(H_{+\pm}^{c}-H_{+\pm}^{Born})}{s+Q^2} = (\alpha_1\mp\beta_1)_{\pi^+}+\ldots$$

Single tagged BES-III data for $\pi^+\pi^-$, $\pi^0\pi^0$ in range 0.1 GeV^2 < Q^2 < 2 GeV^2 under analysis

$f_2(1270)$ and $a_2(1320)$ contributions

Watson theorem (for elastic unitarity) J=2: $\phi(\gamma\gamma \to \pi\pi) = \phi(\pi\pi \to \pi\pi) = \delta(\pi\pi \to \pi\pi)$

Roy analysis (2011)
R. Garcia-Martin at.al.

$$\Omega(s) = \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\phi_{\gamma\gamma\to\pi\pi}(s)}{s'-s}\right)$$

$f_2(1270)$ and $a_2(1320)$ contributions

Watson theorem (for elastic unitarity) J=2: $\phi(\gamma\gamma \to \pi\pi) = \phi(\pi\pi \to \pi\pi) = \delta(\pi\pi \to \pi\pi)$

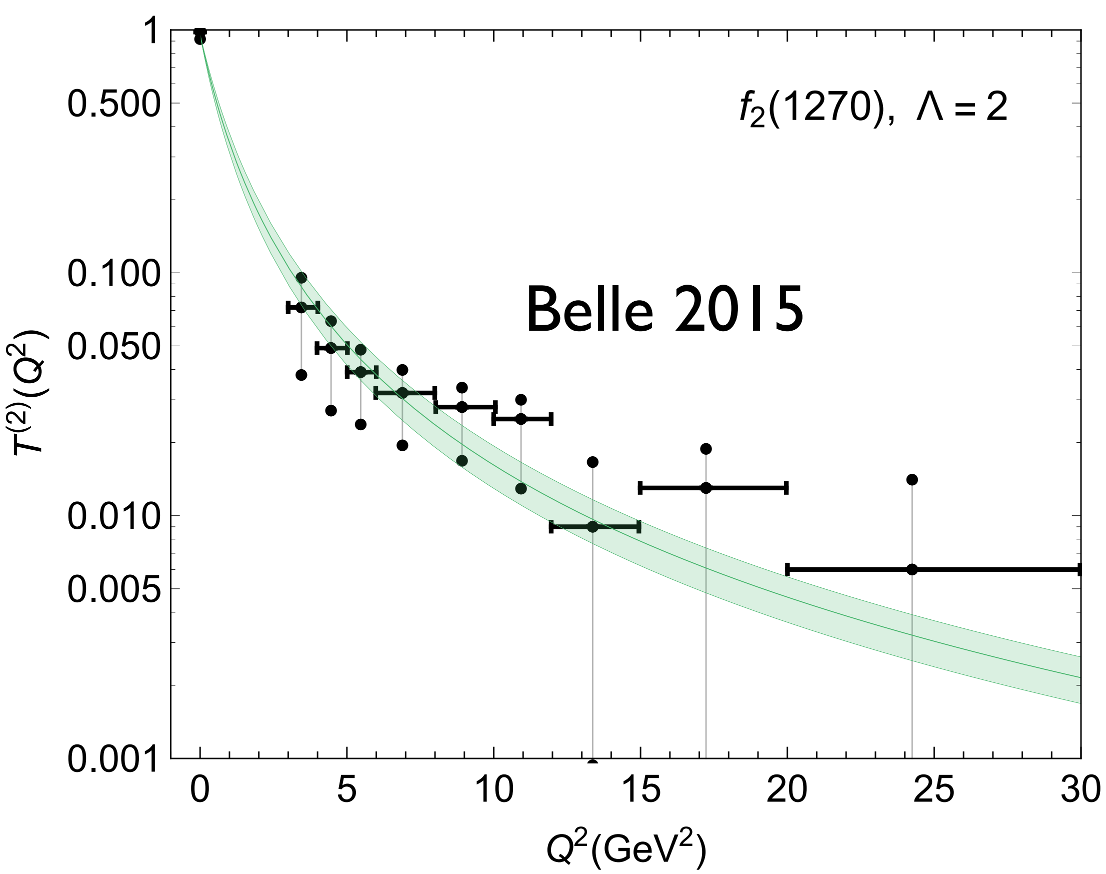
$$\Omega(s) = \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\phi_{\gamma\gamma\to\pi\pi}(s)}{s'-s}\right)$$

Roy analysis (2011)
R. Garcia-Martin at.al.

Unitarized Breit Wigner + Background

$$h_{+-}^{(2)f_2}(s) = \frac{C_{f_2\to\gamma\gamma}C_{f_2\to\pi\pi}}{10\sqrt{6}}\,\frac{s(s+Q^2)\beta(s)}{s-M^2+i\,M\,\Gamma(s)}\,T_{f_2}^{(\Lambda=2)}(Q^2)$$

$$h_{+-}^{(2)} = h_{+-}^{(2)Born} + h_{+-}^{(2)f_2}\,e^{i\phi_0} = |h_{+-}^{(2)}|e^{i\delta(\pi\pi\to\pi\pi)}$$



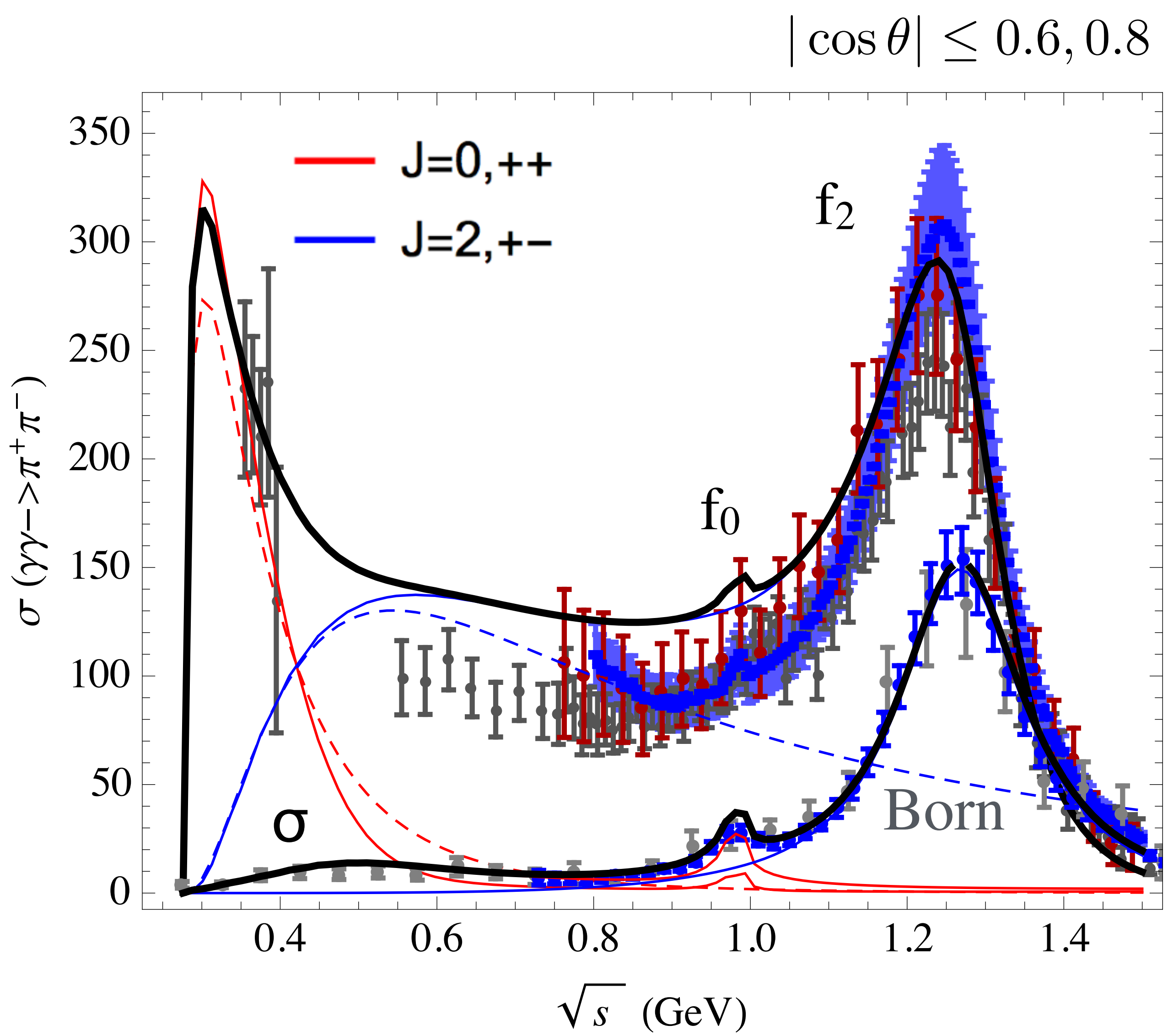
$$T_{f_2}^{(\Lambda=2)}(Q^2) = \sqrt{r^{(2)}}\,\frac{1}{\left(1+Q^2/\Lambda_{f_2}^2\right)^2}$$

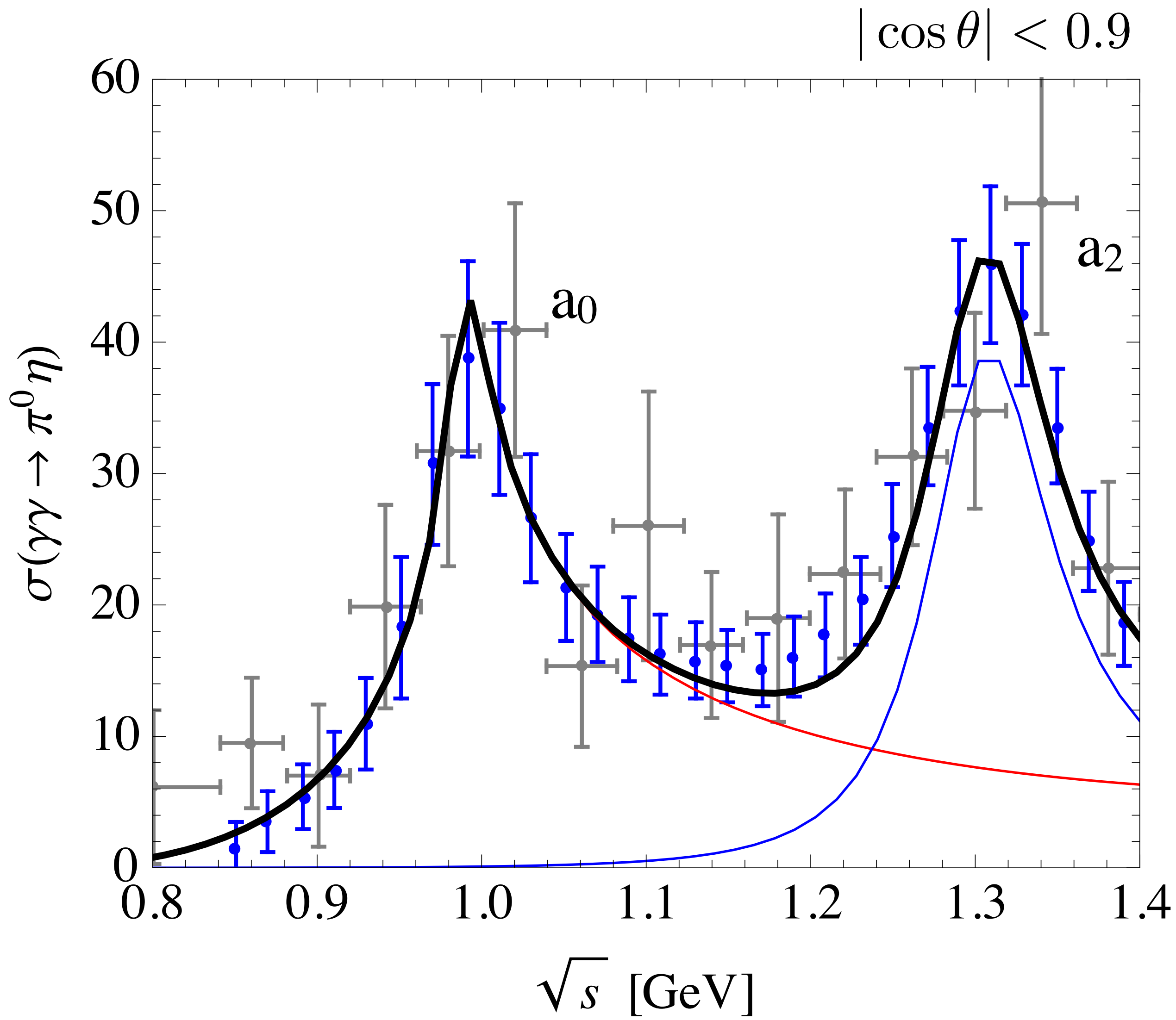
$$\Lambda_{f_2} = 1.222 \pm 0.066\,\mathrm{GeV}$$

pQCD for the tensor FF:
leading FF (using quark DA) is helicity 0 ~1/Q^2, whereas helicity 2 is twist-4, so goes as 1/Q^4.

Kivel et al. (2016)

Results for Q^2=0





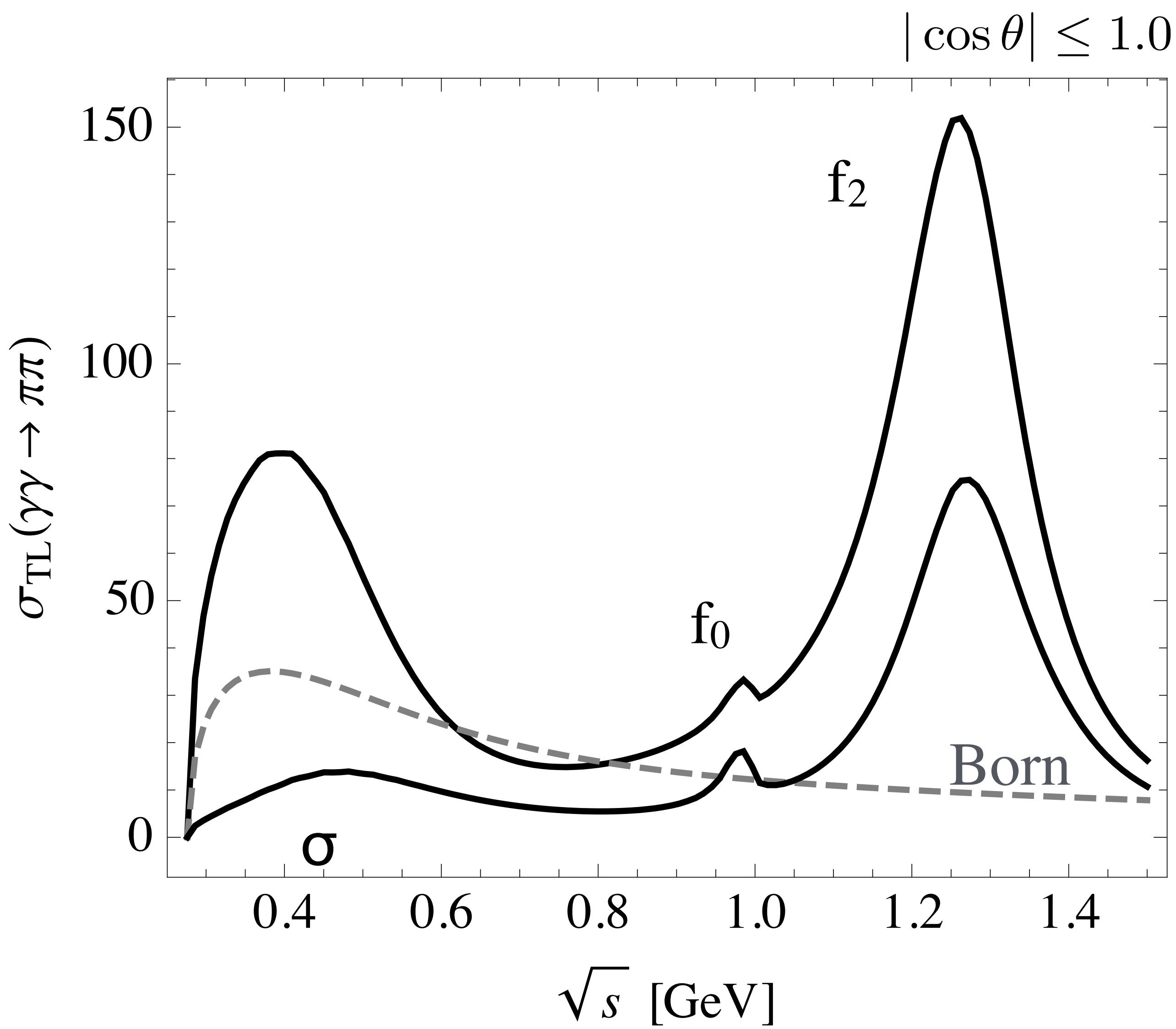
- ✓ **Coupled-channel** dispersive treatment of $f_0(980)$ and $a_0(980)$
- ✓ $f_2(1270)$ described dispersively through Omnes function
- ✓ $a_2(1320)$ described as a Breit Wigner resonance

I.D., Deineka, Vanderhaeghen (2017)

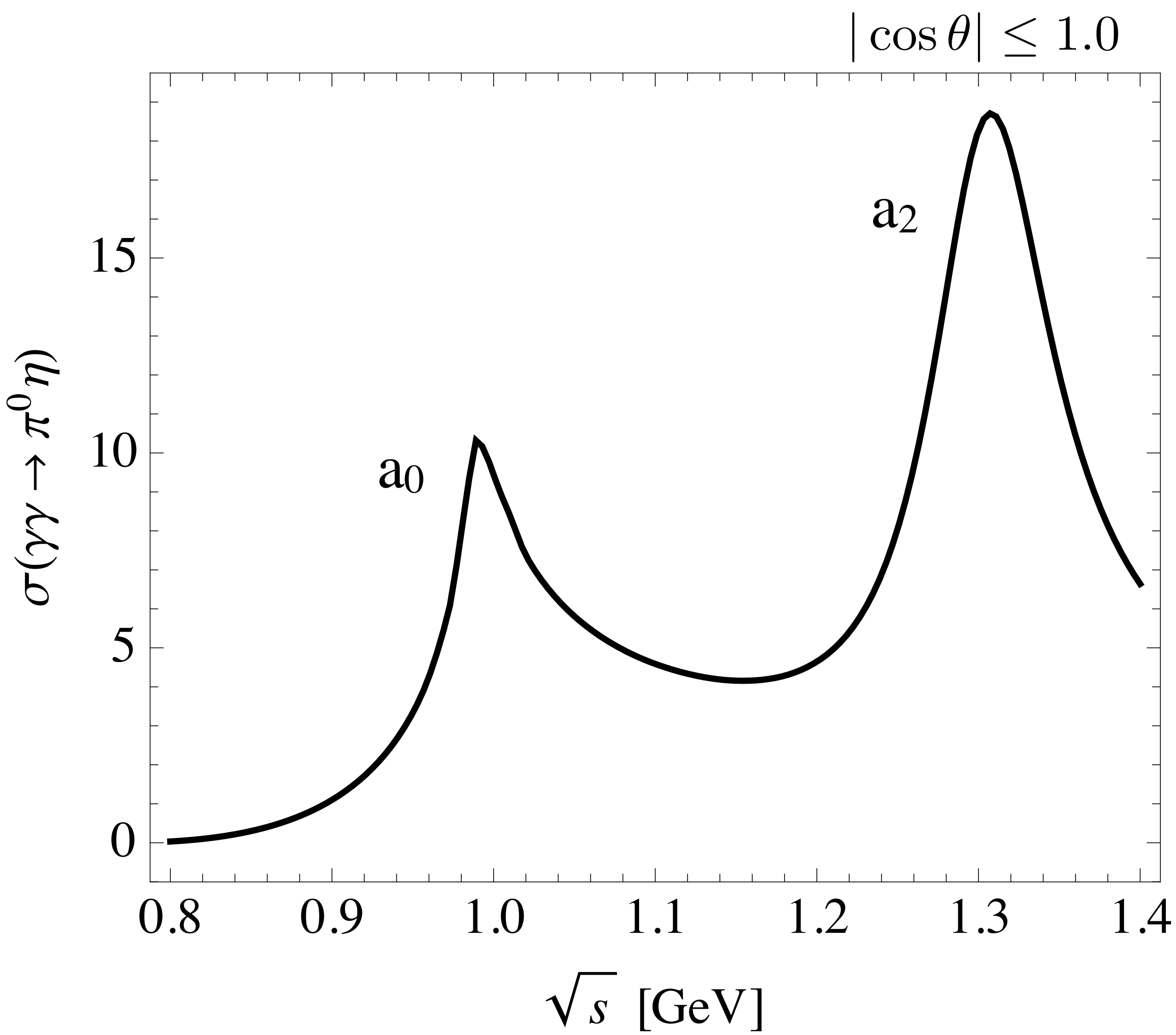
I.D., Vanderhaeghen (work in progress)

Results for Q^2=0.5 (prediction)

$\gamma\gamma^* \to \pi^+\pi^-, \pi^0\pi^0$



$\gamma\gamma^* \to \pi^0\eta$

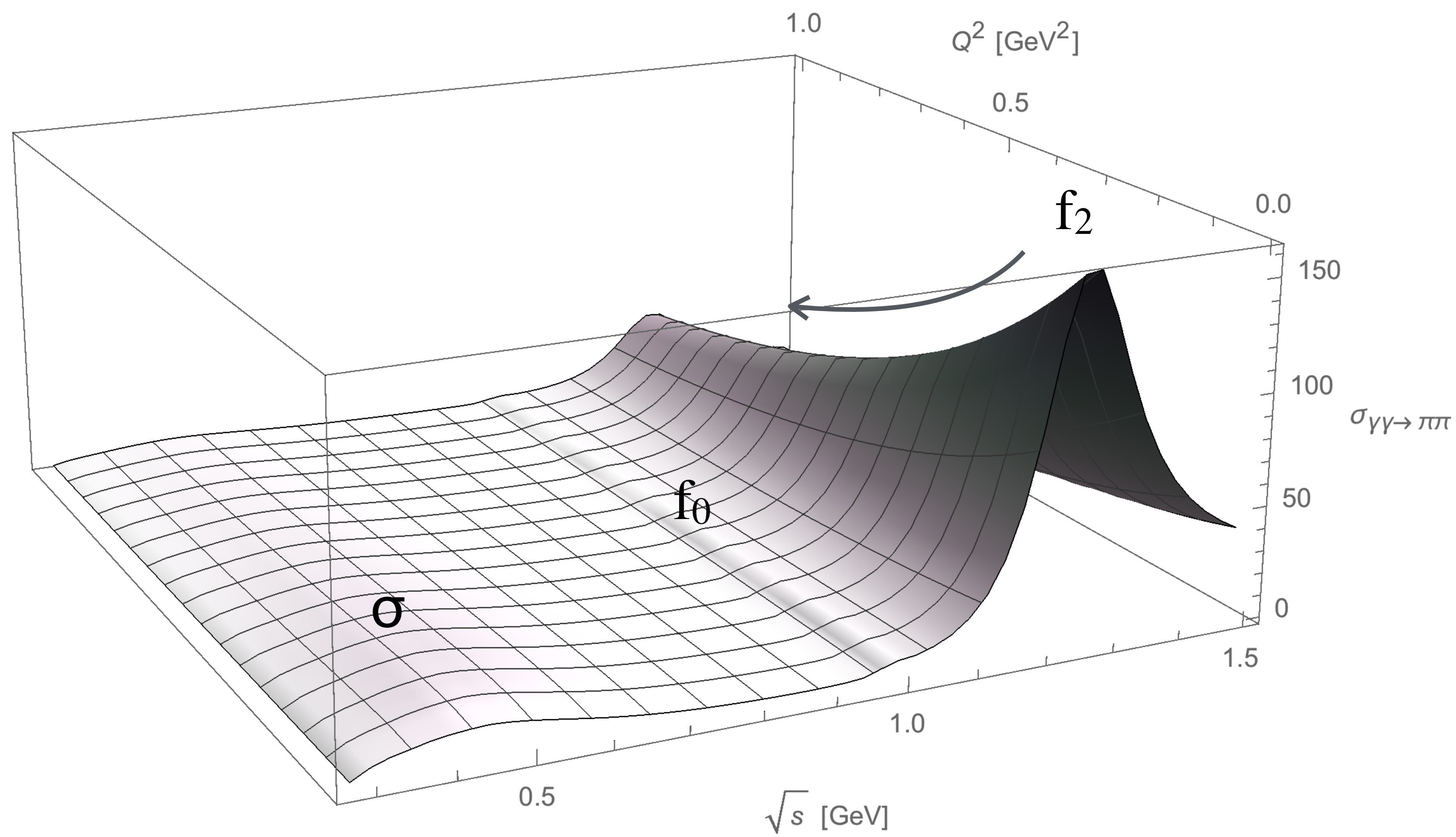


- ✓ **Coupled-channel** dispersive treatment of $f_0(980)$ and $a_0(980)$
- ✓ $f_2(1270)$ described dispersively through Omnes function
- ✓ $a_2(1320)$ described as a Breit Wigner resonance with TFF from $f_2(1270)$ Belle 2015 data

I.D., Deineka, Vanderhaeghen
(work in progress)

Results for $f_2(1270)$ TFF

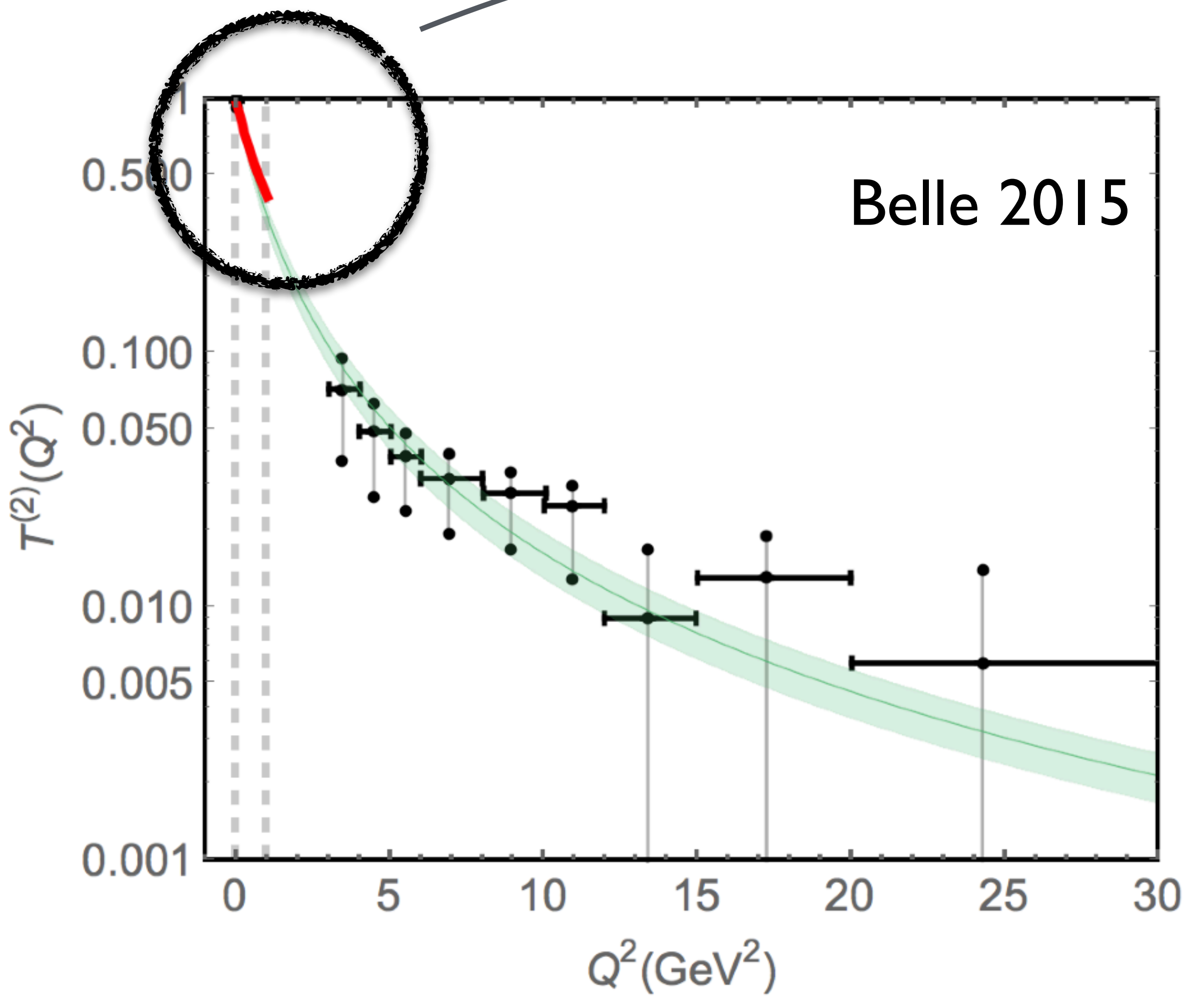
$\gamma\gamma^* \rightarrow \pi^0\pi^0$ dispersive $f_2(1270)$

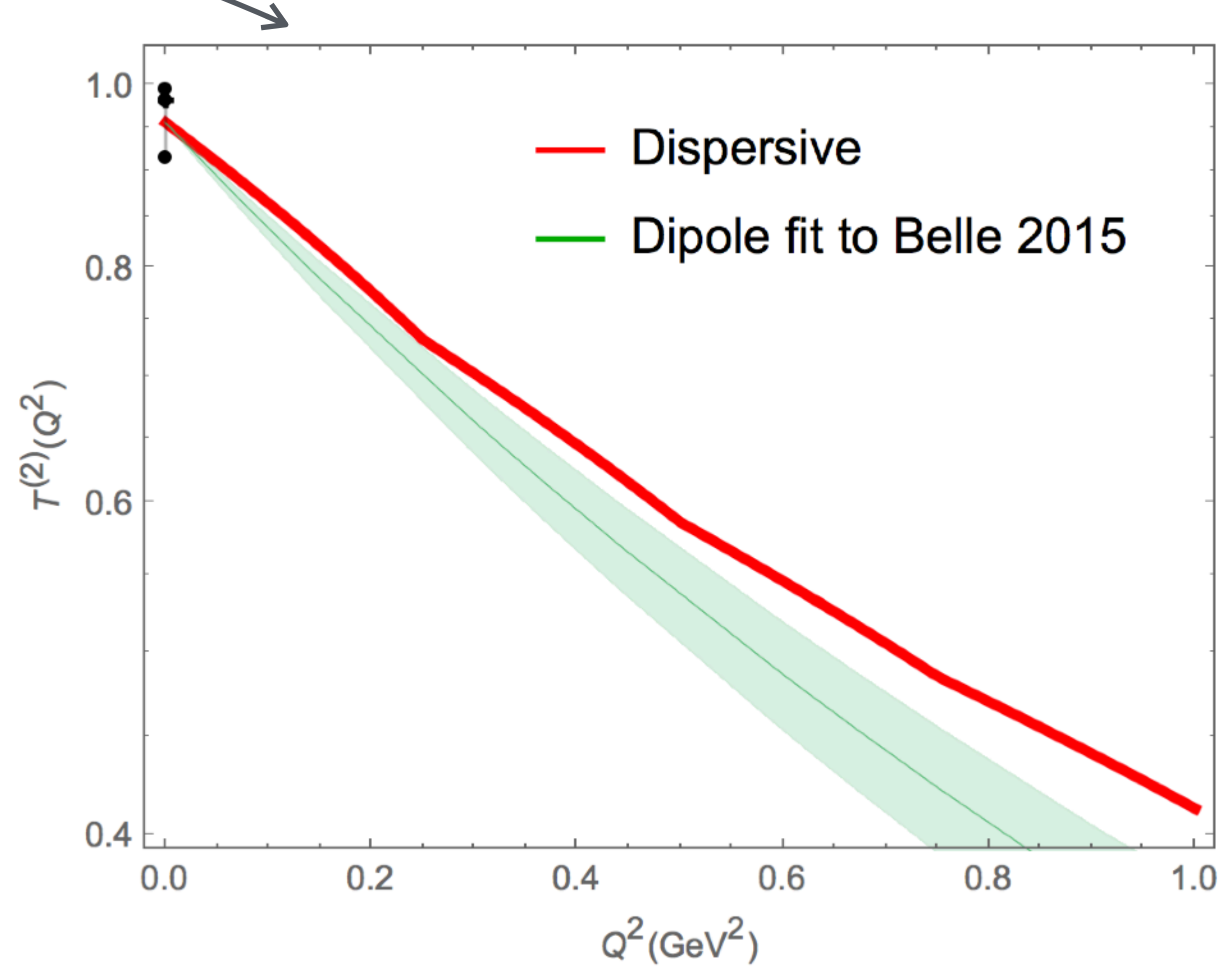


One can predict the TFF for $f_2(1270)$ from dispersive analysis

$$|T_{f_2}^{(\Lambda=2)}(Q^2)|^2 \approx r^{(2)} \left(\frac{\sigma_{TT}(s,Q^2)}{\sigma_{TT}(s,0)} \left(1 + \frac{Q^2}{s}\right)^{-1} \right)_{s=M_{f_2}^2}$$

Results for f₂(1270) TFF





Fit to Belle 2015 data

$$T_{f_2}^{(\Lambda=2)}(Q^2) = \sqrt{r^{(2)}}\frac{1}{\left(1+Q^2/\Lambda_{f_2}^2\right)^2}$$

$$\Lambda_{f_2} = 1.222 \pm 0.066\,\mathrm{GeV}$$

Summary and Outlook

- Need to take into account $f_0(500)$, $f_0(980)$, $a_0(980)$, $f_2(1270)$, $a_2(1320)$ and non resonant contributions in a dispersive approach to (g-2)

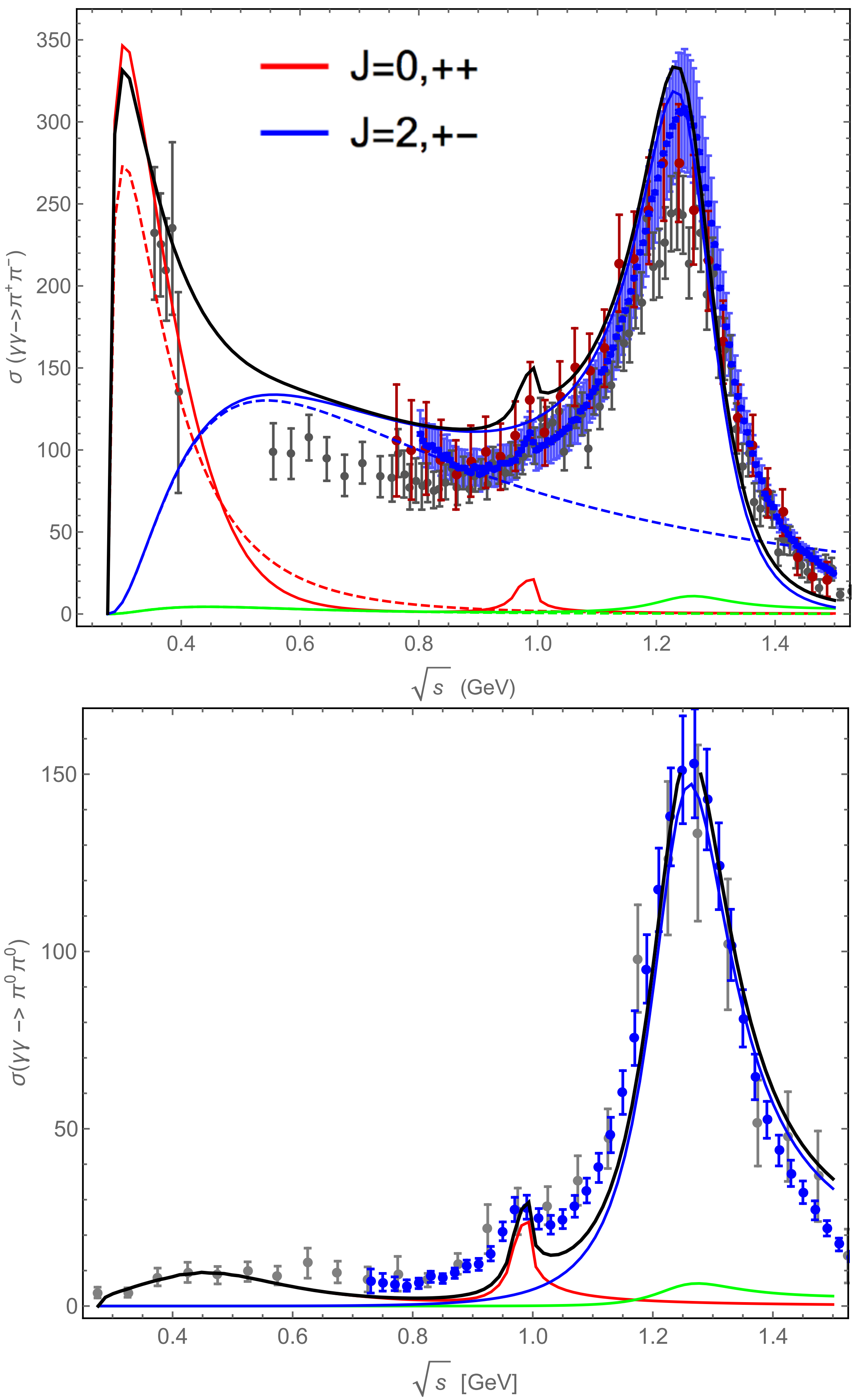
- Main ingredients: $\boldsymbol{\gamma^*\gamma^*\to\pi\pi, \pi\eta, KK}$… (work in progress). Can be used in **different** (g-2) dispersive approaches.

- It is important to **validate** dispersive treatment of $\boldsymbol{\gamma\gamma^*\to\pi\pi, \pi\eta, KK}$… with upcoming BES III data

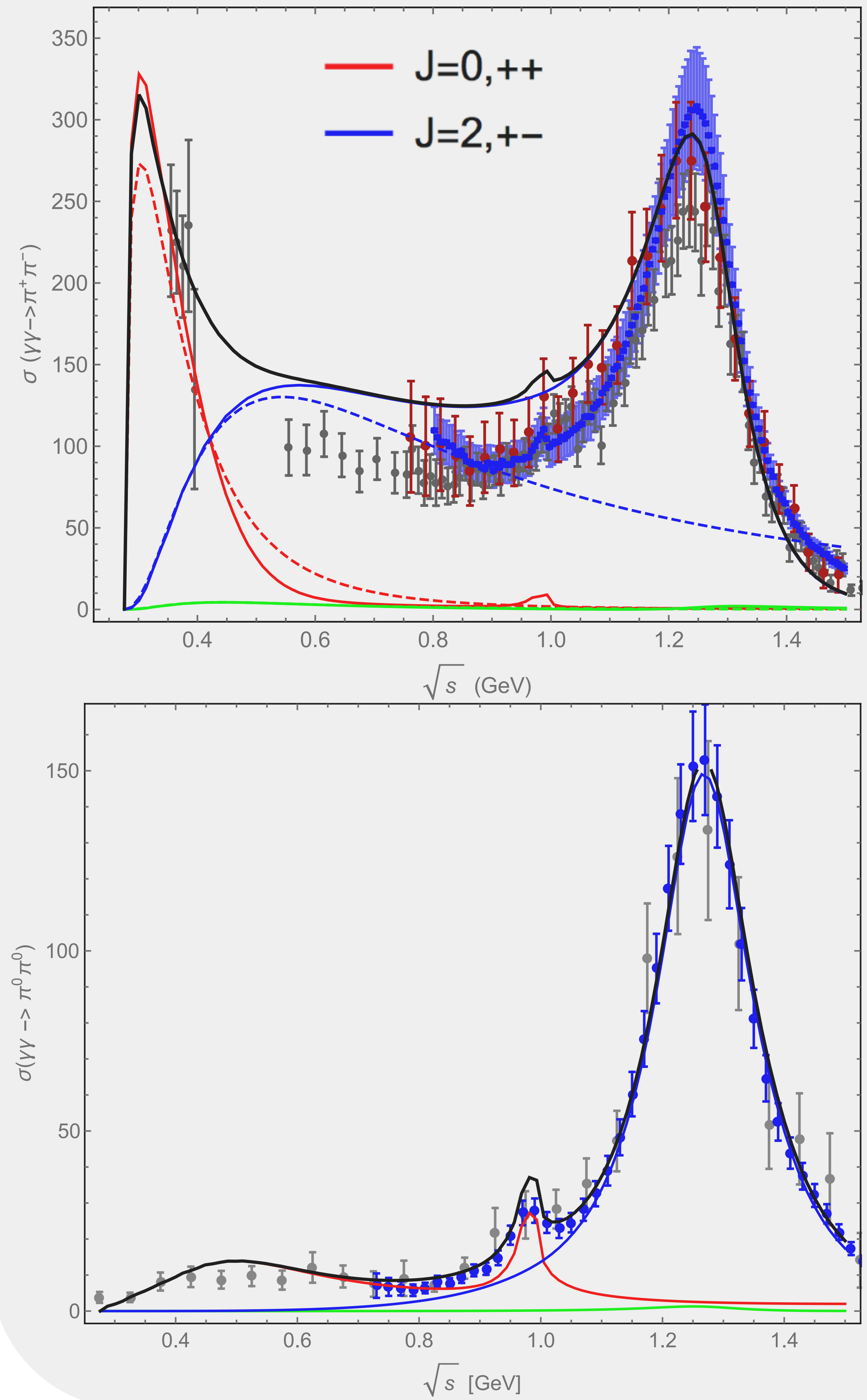
Thank you!

Extra slides

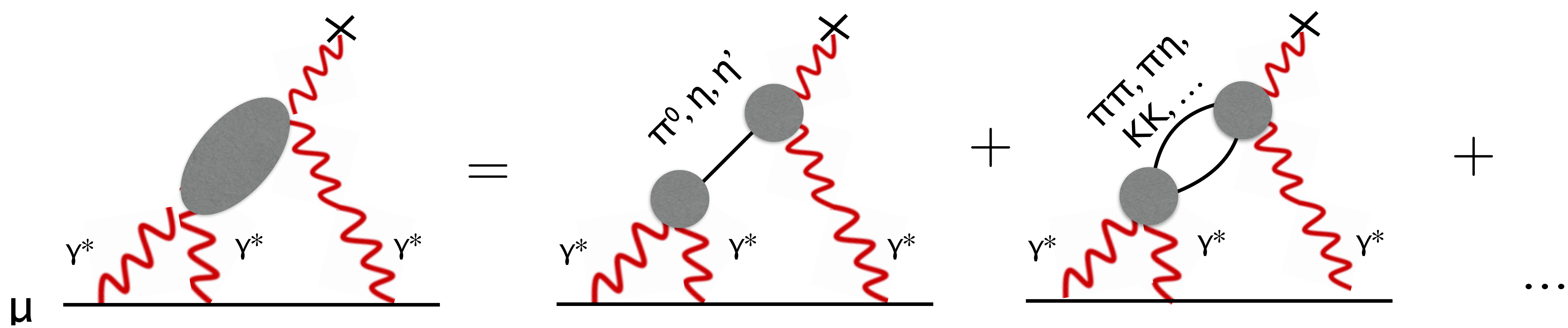
no VM, S wave: unsubtracted, D wave: BW+Born



with VM, S wave: once-subtracted, D wave: disp



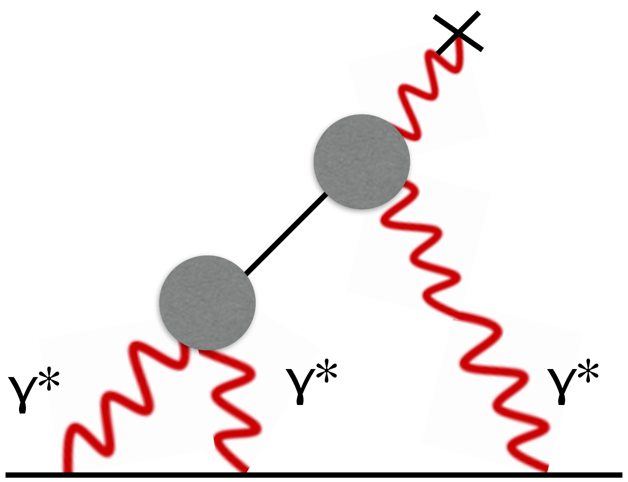
HLbL contributions to (g-2) in units 10^{-10}



Authors	π^0, η, η'	$\pi\pi$, KK	scalars	axial vectors	quark loops	Total
BPaP(96)	8.5(1.3)	-1.9(1.3)	-0.68(0.20)	0.25(0.10)	2.1(03)	8.3(3.2)
HKS(96)	8.3(0.6)	-0.5(0.8)	—	0.17(0.17)	1.0(1.1)	9.0(1.5)
KnN(02)	8.3(1.2)	—	—	—	—	8.0(4.0)
MV(04)	11.4(1.0)	—	—	2.2(0.5)	—	13.6(2.5)
PdRV(09)	11.4(1.3)	-1.9(1.9)	-0.7(0.7)	1.5(1.0)	0.23	**10.5(2.6)**
N/JN(09)	9.9(1.6)	-1.9(1.3)	-0.7(0.2)	2.2(0.5)	2.1(0.3)	11.6(3.9)
J(15)	9.9(1.6)	-1.9(1.3)	-0.7(0.2)	**0.75(0.27)**	2.1(0.3)	**10.2(3.9)**

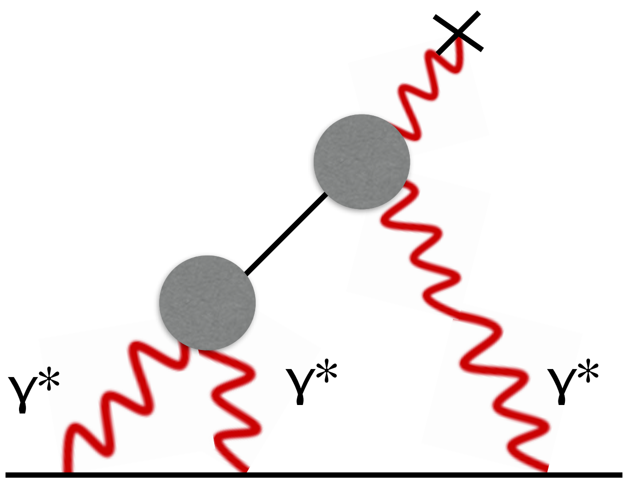
B=Bjnens, Pa=Pallante, P=Prades, H=Hayakawa, K=Kinoshita, S=Sanda, Kn=Knecht, N=Nyffeler, M=Melnikov, V=Vainshtein, dR=de Rafael, J=Jegerlehner

Meson contributions to (g-2)



$$a_\mu^{LbL} = \lim_{k\to 0} ie^6 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4}\, T^{\mu\nu\lambda\sigma}(q_1, k-q_1-q_2, q_2)\, \Pi_{\mu\nu\lambda\sigma}(q_1, k-q_1-q_2, q_2)$$
$$\frac{1}{q_1^2}\,\frac{1}{q_2^2}\,\frac{1}{(k-q_1-q_2)^2}\,\frac{1}{(p+q_1)^2-m^2}\,\frac{1}{(p'-q_2)^2-m^2}$$

Meson contributions to (g-2)



Lepton tensor: well known

$$a_\mu^{LbL} = \lim_{k\to 0} ie^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} T^{\mu\nu\lambda\sigma}(q_1, k-q_1-q_2, q_2)\, \Pi_{\mu\nu\lambda\sigma}(q_1, k-q_1-q_2, q_2)$$
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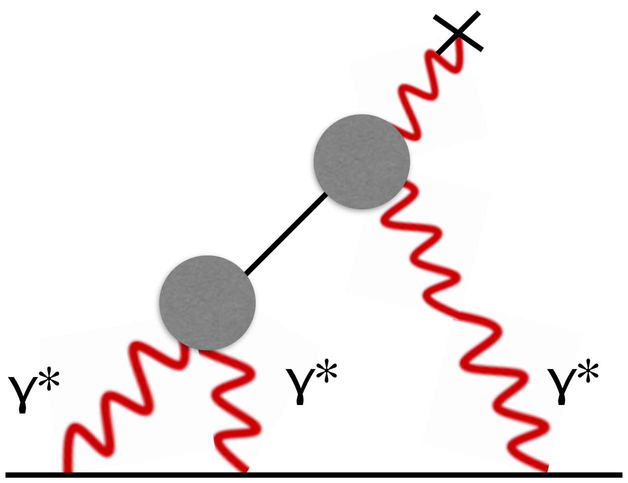
Meson contributions to (g-2)



Lepton tensor: well known

Hadron tensor: requires input from **TFFs**

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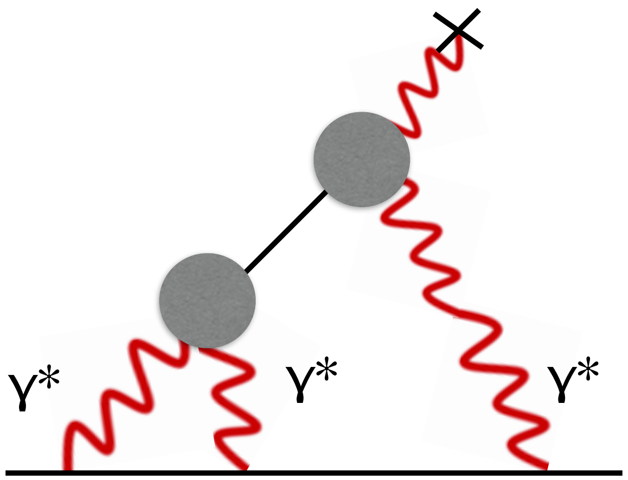
Meson contributions to (g-2)



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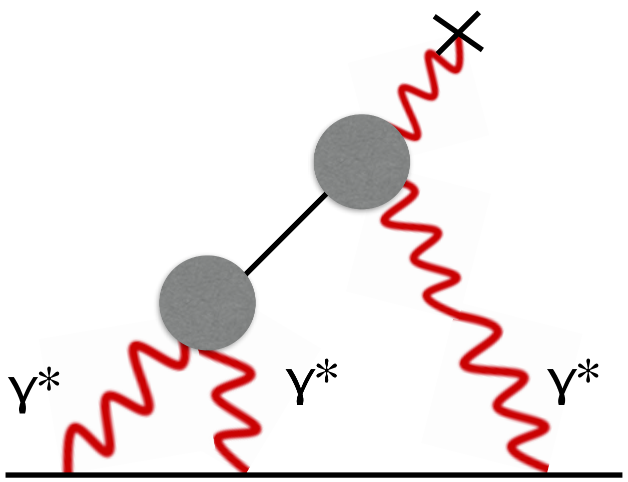
Results (excluding low energy region):

$$a_\mu[f_2(1270), f_2(1565)] = (0.1 \pm 0.01) \times 10^{-10}$$

Meson contributions to (g-2)



Lepton tensor: well known

Hadron tensor: requires input from **TFFs**

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$$\frac{1}{q_1^2}\frac{1}{q_2^2}\frac{1}{(k-q_1-q_2)^2}\frac{1}{(p+q_1)^2-m^2}\frac{1}{(p'-q_2)^2-m^2}$$

Results (excluding low energy region):

$$a_\mu[f_2(1270), f_2(1565)] = (0.1 \pm 0.01) \times 10^{-10}$$

New evaluation of axial vector contributions (satisfying Landau-Yang theorem)

$$a_\mu[f_1(1285), f_1(1420)] = (0.64 \pm 0.20) \times 10^{-10}$$
$$= (0.75 \pm 0.27) \times 10^{-10}$$

Pauk, Vdh (2013)
Jegerlehner (2015)

$$\delta a_\mu^{exp} = 1.6 \times 10^{-10}$$

FNAL, J-PARC experiments