Pseudoscalar-Meson Contributions

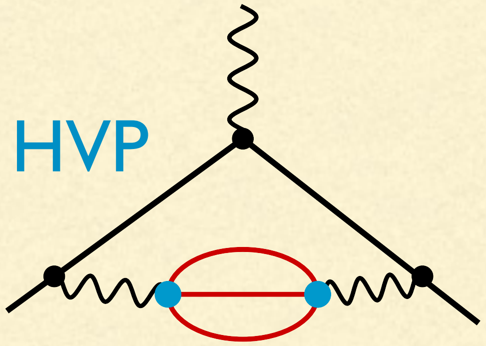
via Schwinger's Sum Rule

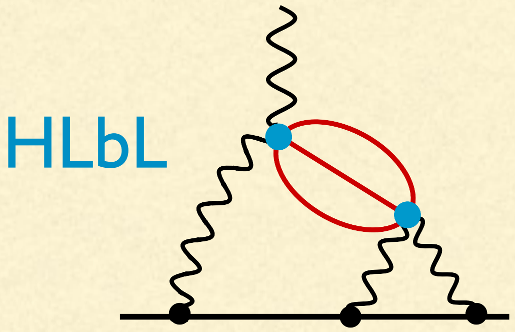
Franziska Hagelstein (AEC Bern)

in coll. with
Vladimir Pascalutsa (JGU Mainz)

MOTIVATION

- Uncertainty of the SM prediction for $(g-2)_\mu$ is dominated by hadronic contributions:





- HVP is calculated with a systematic data-driven dispersive approach:

$$\boxed{a^{\mathrm{HVP}} = \frac{\alpha}{\pi^2}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{d}s}{s}\,\mathrm{Im}\,\Pi^{\mathrm{had}}(s)\,K(s/m^2)}$$

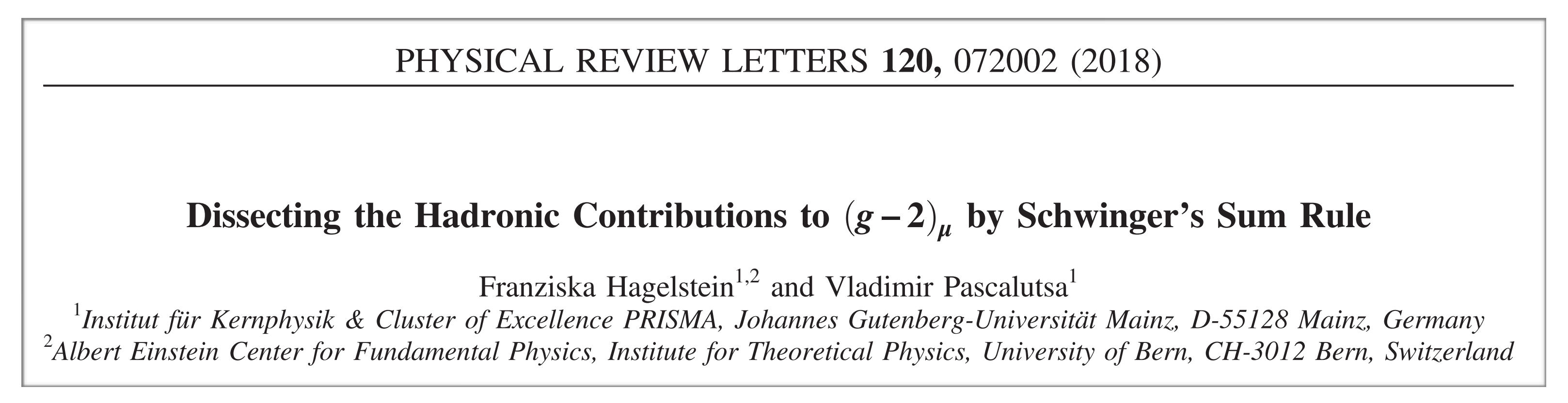
$$\mathrm{Im}\,\Pi^{\mathrm{had}}(s) = \frac{s}{4\pi\alpha}\,\sigma(\gamma^* \to \text{anything})$$

F. Jegerlehner, Springer Tracts Mod. Phys. 274 (2017).

M. Davier, Nucl. Part. Phys. Proc. 287-288, 70 (2017).

- HLbL: no analogue of the simple dispersive formula
- Is there an exact dispersive formula which treats HVP and HLbL (and everything else) in the same way?

OUTLINE



PHYSICAL REVIEW LETTERS **120,** 072002 (2018)

Dissecting the Hadronic Contributions to $(g-2)_\mu$ by Schwinger's Sum Rule

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- THE SCHWINGER SUM RULE $$a = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$
- THE SCHWINGER TERM $a^{(1)} = \alpha/2\pi$
- HADRONIC VACUUM POLARIZATION
- CONTRIBUTION OF THE PRIMAKOFF MECHANISM
- PSEUDOSCALAR-MESON CONTRIBUTIONS

THE SCHWINGER SUM RULE (1975)

J. S. Schwinger, Proc. Nat. Acad. Sci. 72, 1 (1975); ibid. 72, 1559 (1975) [Acta Phys. Austriaca Suppl. 14, 471 (1975)].

A. M. Harun ar-Rashid, Nuovo Cim. A 33, 447 (1976).

anomalous magnetic moment (a.m.m.) $a=\frac{1}{2}(g-2)_\mu$

muon mass m

photon lab-frame energy ν and virtuality $Q^2=-q^2$

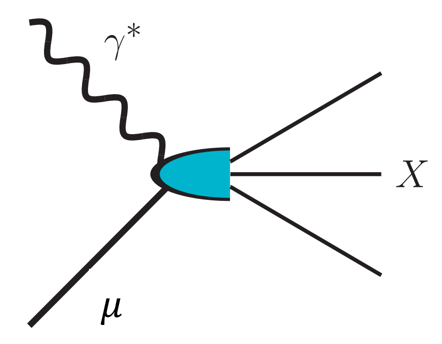
$$a=\frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty}\mathrm{d}\nu\left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$

fine-structure constant $\alpha\approx 1/137$

photo-absorption threshold ν_0

longitudinal-transverse photo-absorption cross section σ_{LT}

- Cross sections for photo-absorption on muon:



inelastic cross section

$X=\mu\gamma,\ \mu\gamma\gamma,\ \mu\pi^0,\ \mu\gamma\pi^0,\ \dots$

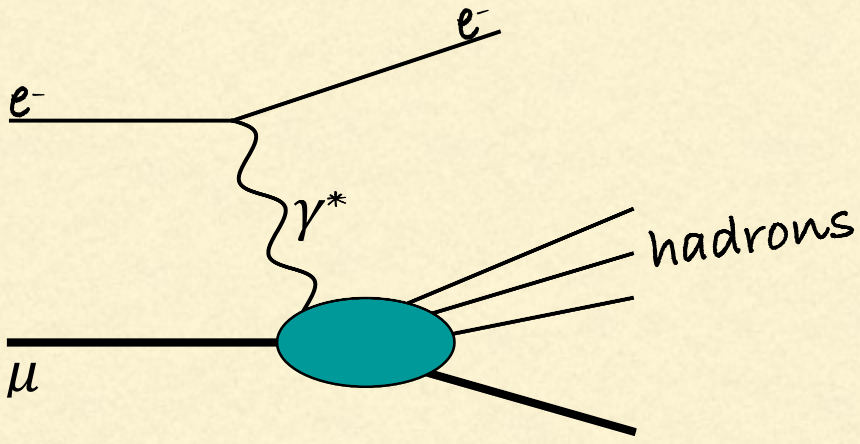
SPIN STRUCTURE FUNCTIONS

a.m.m.

$a = \frac{1}{2}(g-2)_\mu$

$$a = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$

$$= \lim_{Q^2\to 0} \frac{8m^2}{Q^2}\int_0^{x_0} \mathrm{d}x \, [\bar{g}_1 + \bar{g}_2](x,Q^2)$$

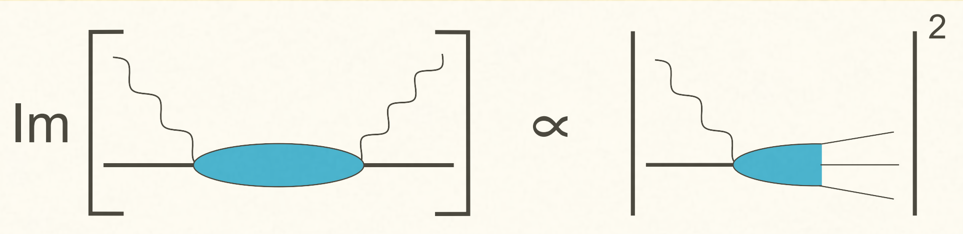


muon spin structure functions g_1 and g_2

Bjorken variable: $x = \frac{Q^2}{2m\nu}$

- Spin-dependent forward doubly-virtual Compton scattering:

$$T_A^{\mu\nu}(q,p) = -\frac{1}{M}\gamma^{\mu\nu\alpha}q_\alpha \, S_1(\nu,Q^2) + \frac{Q^2}{M^2}\gamma^{\mu\nu}S_2(\nu,Q^2)$$



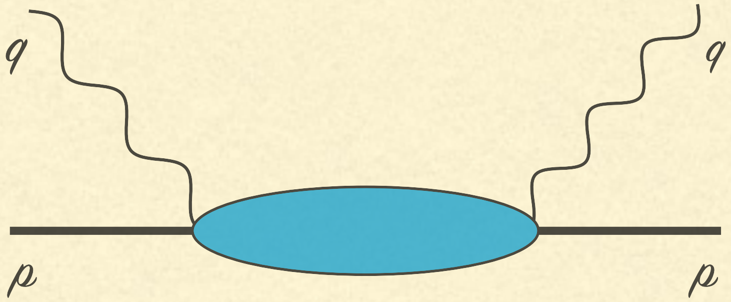
- Optical theorem:

$$\mathrm{Im}\, S_1(\nu,Q^2) = \frac{4\pi^2\alpha}{\nu}g_1(x,Q^2) = \frac{M\nu^2}{\nu^2+Q^2}\left[\frac{Q}{\nu}\sigma_{LT} + \sigma_{TT}\right](\nu,Q^2)$$

$$\mathrm{Im}\, S_2(\nu,Q^2) = \frac{4\pi^2\alpha M}{\nu^2}g_2(x,Q^2) = \frac{M^2\nu}{\nu^2+Q^2}\left[\frac{\nu}{Q}\sigma_{LT} - \sigma_{TT}\right](\nu,Q^2)$$

ORIGIN: THE GDH AND BC SUM RULES

- Sum rules are model-independent relations based on very general principles:
  - Analyticity/causality (dispersion relations)
  - Unitarity (optical theorem)
  - Crossing symmetry



- Some sum rules of Compton scattering off a spin-1/2 particle:

$$(1+a)a = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}}{Q} - \frac{\sigma_{TT}}{\nu}\right]_{Q^2=0}$$

Burkhardt—Cottingham sum rule (1970) $\int_0^1 \mathrm{d}x\, g_2(x,Q^2) = 0$

⊖ $$a^2 = -\frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu\, \frac{\sigma_{TT}(\nu)}{\nu}$$

Gerasimov—Drell—Hearn sum rule (1966)

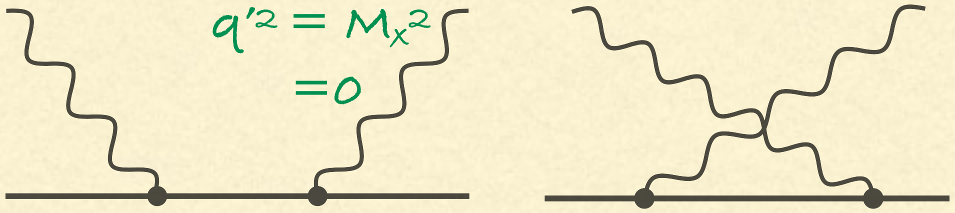
$$a = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$

Schwinger sum rule (1975)

REPRODUCING THE SCHWINGER TERM

- Input for Schwinger sum rule:
  longitudinal-transverse photo-absorption cross section

tree-level QED
Compton scattering

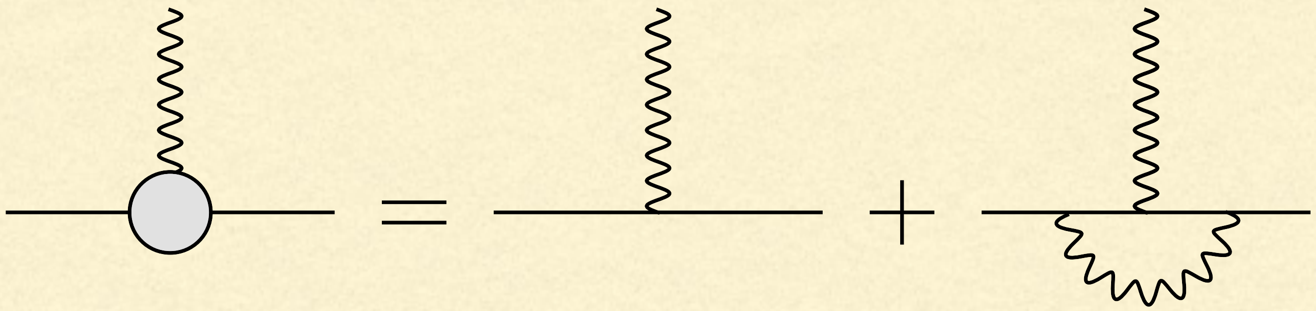


SR calculation is simpler,
has less of loops

$$\left[\frac{\sigma_{LT}^{\gamma\mu\to\gamma^*\mu}(\nu,Q^2)}{Q}\right]_{Q^2=0} = \frac{\pi\alpha^2}{2m^2\nu^3}\left[-(5s+m^2+M_X^2)\lambda+(s+2m^2-2M_X^2)\log\frac{\beta+\lambda}{\beta-\lambda}\right]$$

$$\lambda=(1/2s)\sqrt{[s-(m+M_X)^2]\,[s-(m-M_X)^2]},\quad \beta=(s+m^2-M_X^2)/2s,\quad s=m^2+2m\nu$$

$M_x^2=0$

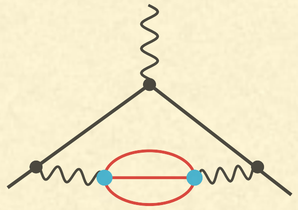


$F_2(0)=a$ $a^{(0)}=0$ $a^{(1)} = \alpha/2\pi$

$\frac{\alpha}{2\pi}$
JULIAN SCHWINGER
2·12·1918 — 7·16·1994
CLARICE CARROL SCHWINGER
9·23·1917 — 1·9·2011

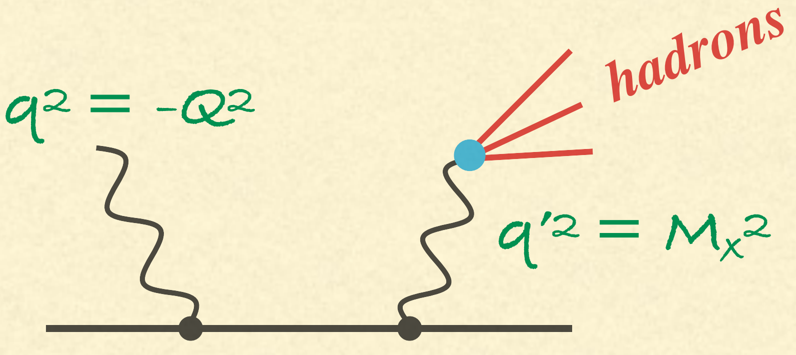
Schwinger term —
the leading QED result

HVP FROM SCHWINGER SUM RULE



- HVP from the Schwinger sum rule with the cross section of hadron production through timelike Compton scattering:

$$\boxed{a = \frac{m^2}{\pi^2\alpha}\int\limits_{4m_\pi^2}^{\infty} \mathrm{d}M_X^2 \int\limits_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{1}{Q}\frac{\mathrm{d}\sigma_{LT}^{\gamma\mu\to\mu X}(\nu,Q^2)}{\mathrm{d}M_X^2}\right]_{Q^2=0}}$$



+ crossed diagram

cross section factorizes:

$$= \frac{1}{\pi}\int\limits_{4m_\pi^2}^{\infty} \mathrm{d}M_X^2 \, \frac{\mathrm{Im}\,\Pi^{\mathrm{had}}(M_X^2)}{M_X^2} \, \underbrace{\frac{m^2}{\pi^2\alpha}\int\limits_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}^{\gamma\mu\to\gamma^*\mu}(\nu,Q^2)}{Q}\right]_{Q^2=0}}$$

virtual-photon decay into hadrons

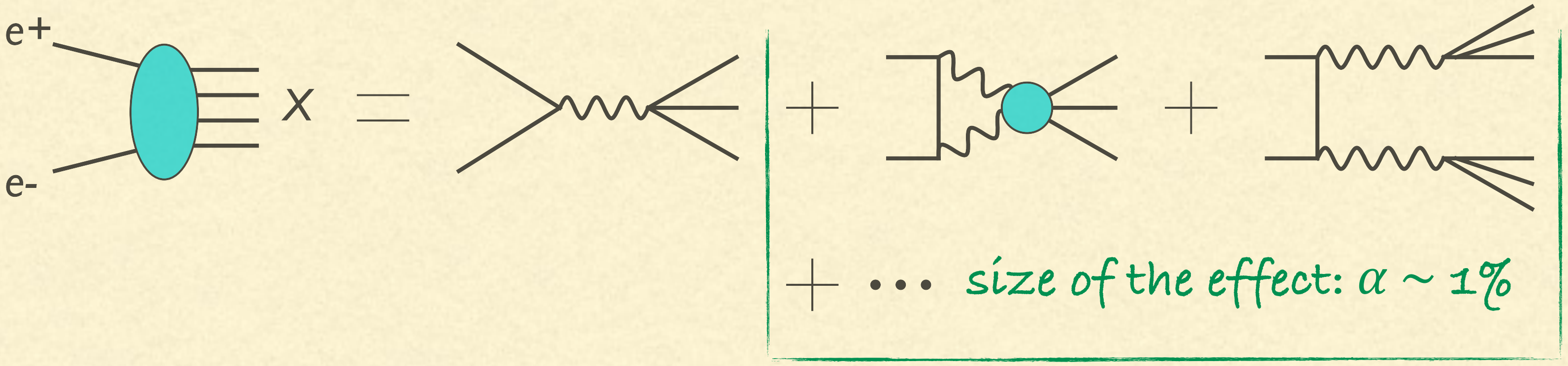
timelike Compton scattering

kernel function: $= \frac{\alpha}{\pi}K(M_X^2/m^2) \equiv \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x \, \frac{x^2(1-x)}{x^2+(1-x)(M_X^2/m^2)}$

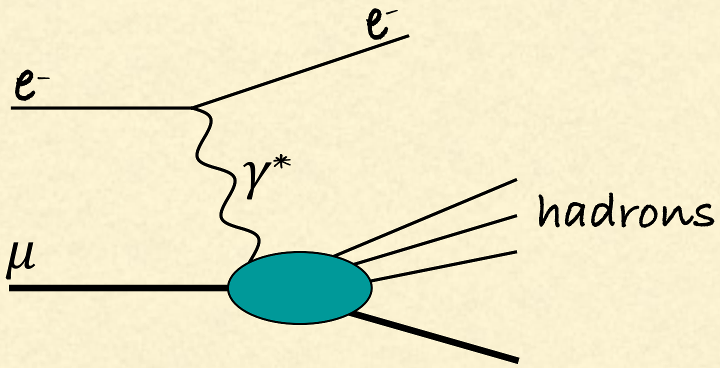
- Schwinger sum rule can reproduce the HVP standard formula

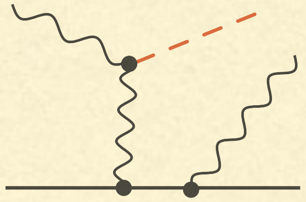
$$\boxed{a^{\mathrm{HVP}} = \frac{\alpha}{\pi^2}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s}\,\mathrm{Im}\,\Pi^{\mathrm{had}}(s)\int_0^1 \mathrm{d}x \, \frac{x^2(1-x)}{x^2+(1-x)(s/m^2)}}$$

LIMITATIONS & ADVANTAGES FOR HVP

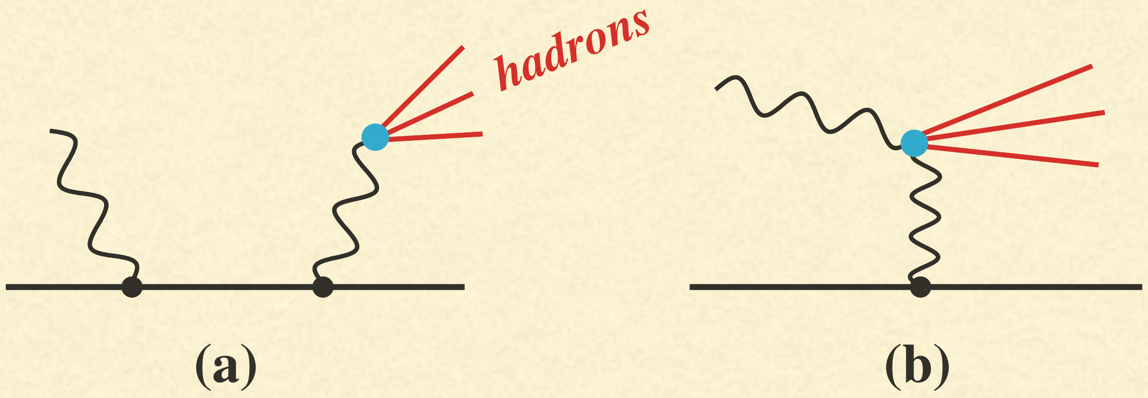


- Limitations of the standard approach:
  - Two-photon exchange corrections
- Advantages of the Schwinger sum rule:
  - Different mechanisms contribute and will be included in the measured cross section (… of course there is no data yet!)
  - No adjustment of the Schwinger sum rule needed for different mechanisms
  - No separation of final state radiation necessary — as long as there are hadrons in the final state





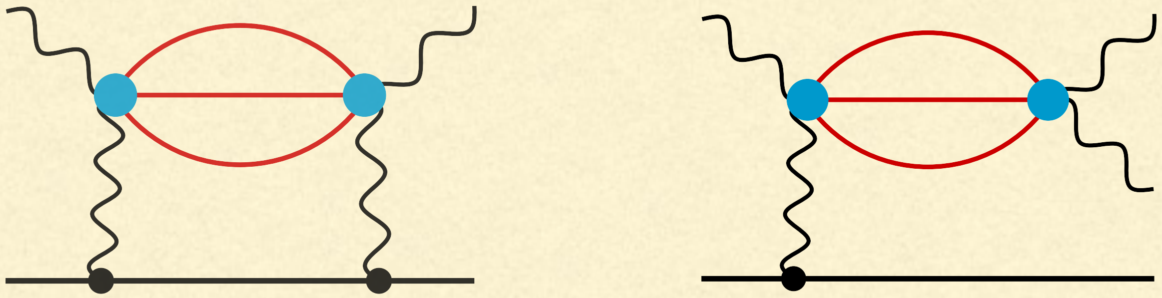
HADRONIC CONTRIBUTIONS



$\mu\gamma \to \mu + \text{hadrons}$

$\mu\gamma \to \mu\gamma + \text{hadrons}$

- Hadron photo-production off the muon:
  - (a) timelike Compton scattering
  - (b) Primakoff effect



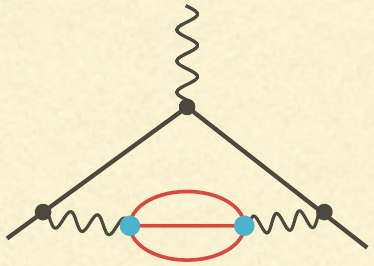
$\mu\gamma \to \mu\gamma$

$\mu\gamma \to \mu\gamma\gamma$

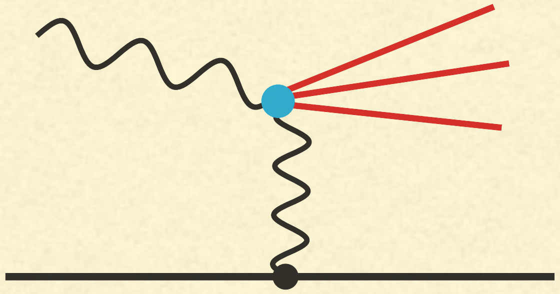
- HLbL contribution to Compton scattering

FH, Pascalutsa and Vanderhaeghen, in prep.

PROOF OF VANISHING PRIMAKOFF CONTR.



- HVP Contribution to (g−2) starts at $\mathcal{O}(\alpha^2)$
  - whereas HLbL starts at $\mathcal{O}(\alpha^3)$
- Contribution of the Primakoff mechanism:
  - Primakoff cross section is of $\mathcal{O}(\alpha^3)$, hence effect on (g−2) is of $\mathcal{O}(\alpha^2)$
  - It belongs to HLbL topology, hence must be absent
- To prove that the Primakoff contribution by itself is vanishing exactly:



can be expressed through the photon structure functions observed in LbL scattering

- **LbL sum rules:**

Pascalutsa et al.,
Phys. Rev. D 85 (2012) 116001.

see also talk
by I. Danilkin

$$\int_0^1 \mathrm{d}x\, g_1^\gamma(x,Q^2) = 0$$

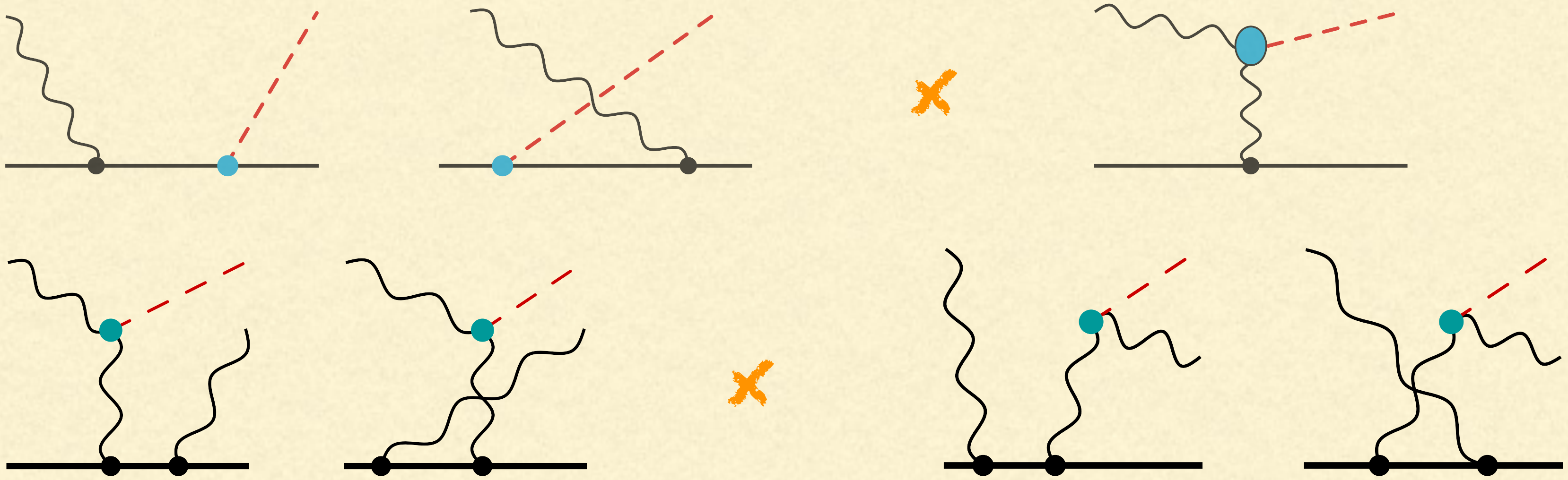
$$\int_0^1 \mathrm{d}x\, g_2^\gamma(x,Q^2) = 0$$

PSEUDOSCALAR-MESON CONTRIBUTION

$\mathcal{O}(\alpha^3)$

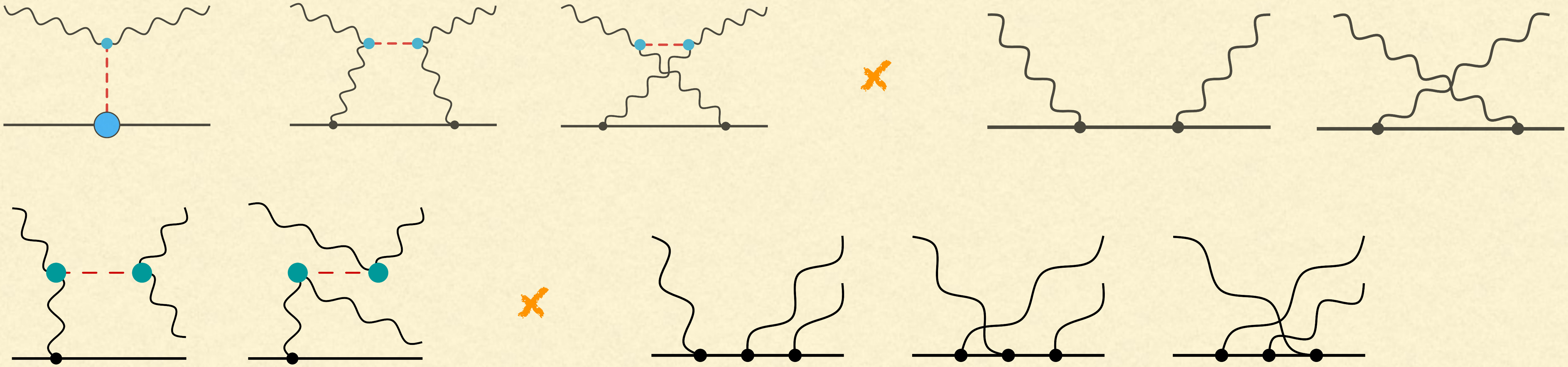
- 4 different channels:

I. Hadron photo-production channels



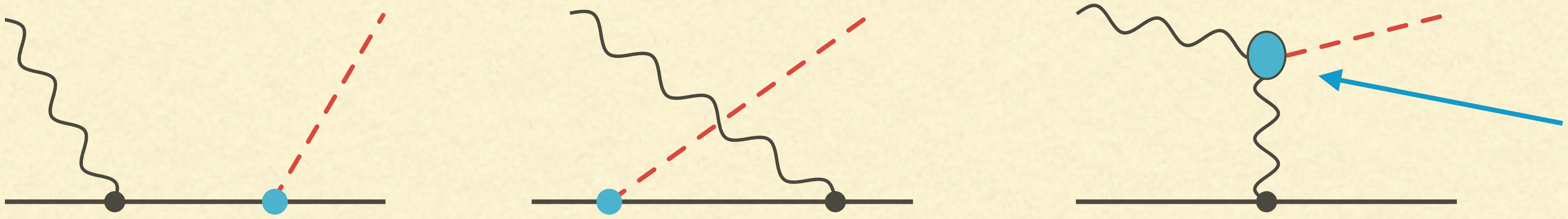
No doubly-virtual transition form factors needed, only $F_{\pi\gamma\gamma^*}$

II. Electromagnetic channels



NEUTRAL-PION CONTRIBUTION

1 out of 4 channels (to order α^3): $\mu + \gamma \rightarrow \mu + \pi^0$



using the LMD+V single-virtual transition form factor, $F_{\pi\gamma\gamma^*}$

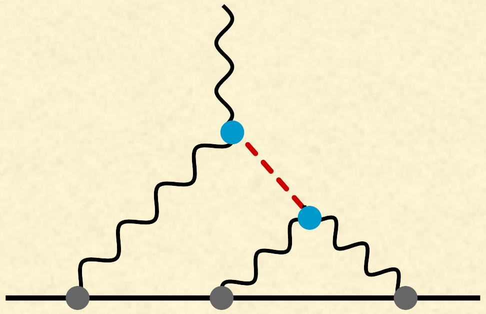
$$\boxed{a_\mu^{\pi^0\text{-prod.}} = 11.9(9) \times 10^{-10}}$$

- Compare with the full π^0-pole contribution:

$a_\mu^{\pi^0\text{-pole}} = 5.8(1.0) \times 10^{-10}$ Knecht & Nyffeler 2002

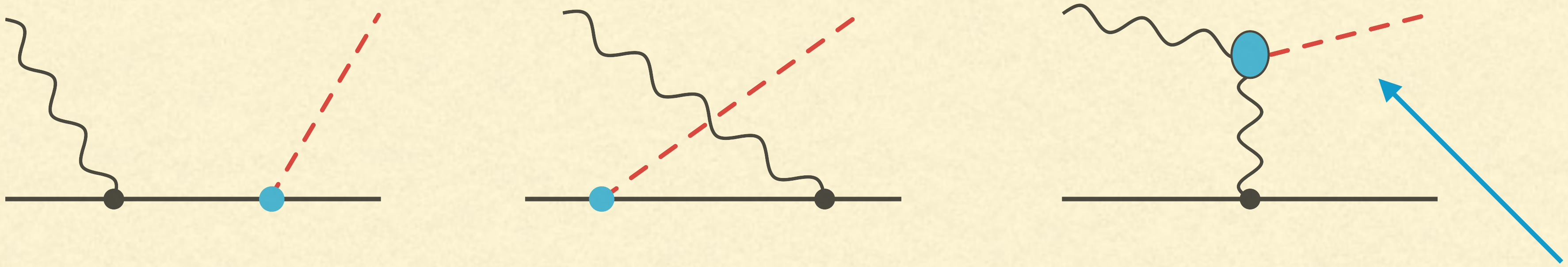
$a_\mu^{\pi^0\text{-pole}} = 7.65 \times 10^{-10}$ Melnikov & Vainshtein 2004

$a_\mu^{\pi^0\text{-pole}} = 6.26^{+0.30}_{-0.25} \times 10^{-10}$ Hoferichter et al. 2018



ETA- AND ETA'- CONTRIBUTIONS

1 out of 4 channels (to order α^3): $\mu + \gamma \rightarrow \mu + (\eta,\eta')$



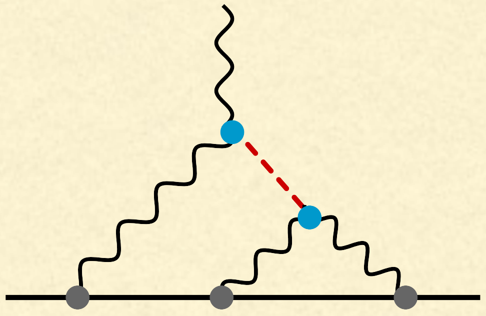
using the VMD transition form factor, $F_{(\eta,\eta')\gamma\gamma^*}$, from Nyffeler 2009

$$\boxed{\begin{aligned} a_\mu^{\eta\text{-prod.}} &= 7.4(6) \times 10^{-10} \\ a_\mu^{\eta'\text{-prod.}} &= 5.5(4) \times 10^{-10} \end{aligned}}$$

- Compare with the full η,η'-pole contributions:

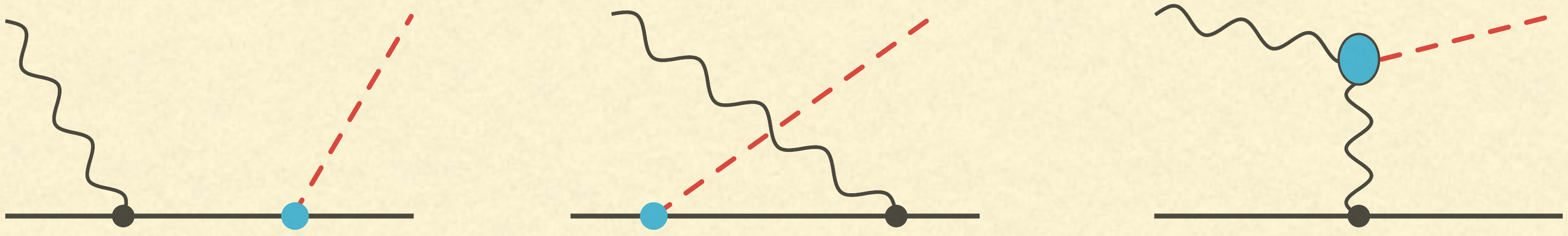
$$a_\mu^{\eta\text{-pole}} = 1.45 \times 10^{-10},\; a_\mu^{\eta'\text{-pole}} = 1.25 \times 10^{-10} \qquad \text{Nyffeler 2009}$$

$$a_\mu^{\eta\text{-pole}} = a_\mu^{\eta'\text{-pole}} = 1.8 \times 10^{-10} \qquad \text{Melnikov \& Vainshtein 2004}$$



PSEUDOSCALAR-MESON CONTRIBUTION

1 out of 4 channels (to order α^3): $\mu + \gamma \to \mu + (\pi^0,\eta,\eta')$

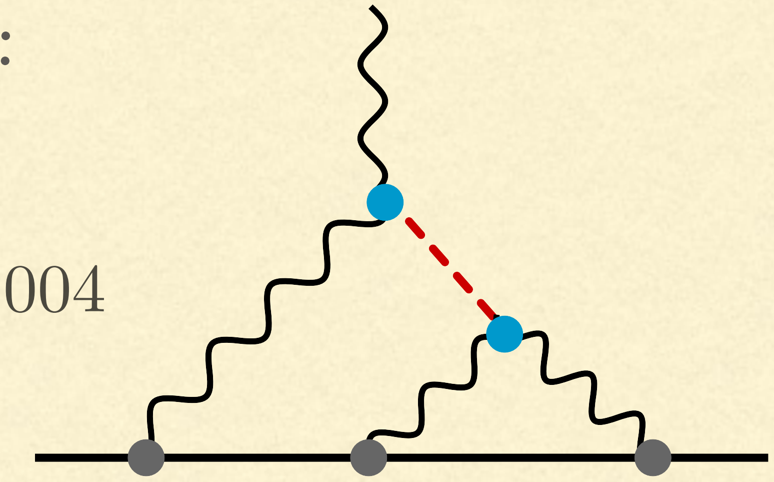


$$\boxed{a_\mu^{\pi^0,\eta,\eta'\text{-prod.}} = 24.8(1.2) \times 10^{-10}}$$

- Compare with the full pseudoscalar-meson contributions:

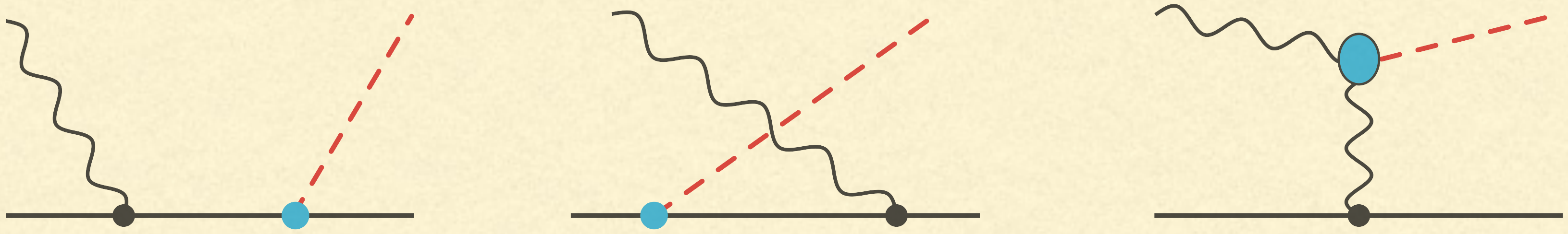
$a_\mu^{\text{PS-pole}} = 8.3(1.2) \times 10^{-10}$ Knecht & Nyffeler 2002

$a_\mu^{\text{PS-pole}} = 11.4(10) \times 10^{-10}$ Melnikov & Vainshtein 2004

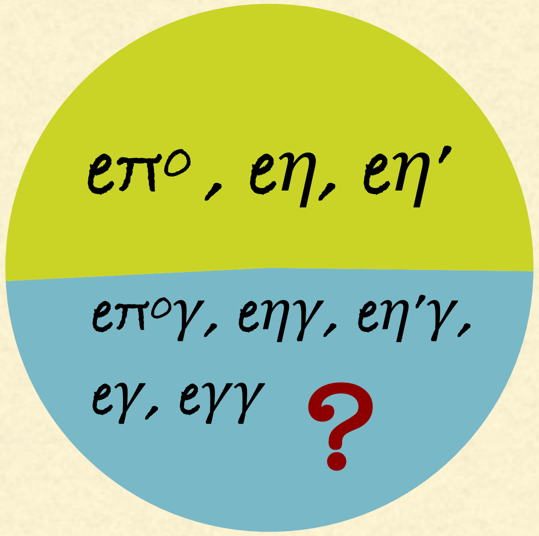


$(g\text{-}2)_e$

1 out of 4 channels (to order α^3): $e + \gamma \rightarrow e + (\pi^0, \eta, \eta')$



$$a_e^{\pi^0\text{-prod}} = 3.63(10) \times 10^{-13}$$
$$a_e^{\eta\text{-prod}} = 2.05(5) \times 10^{-13}$$
$$a_e^{\eta'\text{-prod}} = 1.42(4) \times 10^{-13}$$
$$a_e^{\pi^0,\eta,\eta'\text{-prod}} = 7.11(12) \times 10^{-13}$$



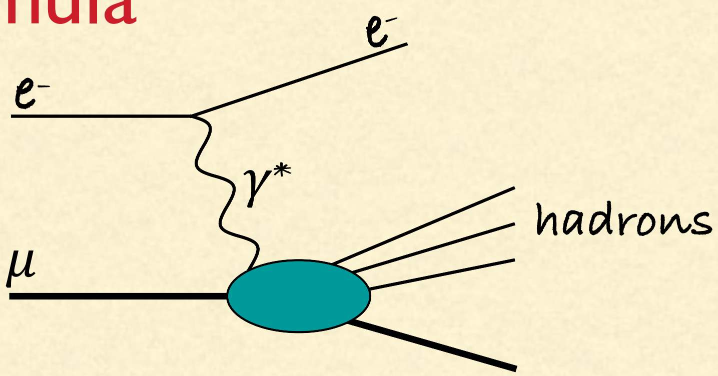
$$a_e^{\text{exp.}} - a_e^{\text{theo.}} = -1.14(82) \times 10^{-12} \quad \text{Jegerlehner 2016}$$

$\rightarrow -1.53(82) \times 10^{-12}$ neglecting $e\pi^0\gamma$, $e\gamma$, $e\gamma\gamma$, ... channels

$$a_e^{\text{had. LbL}} = 3.7(0.5) \times 10^{-14} \qquad \text{Jegerlehner 2016}$$

SUMMARY

- Schwinger sum rule is an exact dispersive formula for evaluation of hadronic contributions to (g−2) — both HVP and HLbL
- Reproduces $\alpha/2\pi$ and the standard HVP formula
- Splits contributions into

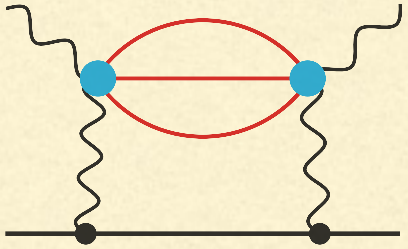


(a) hadron photo-production:

  - inelastic spin structure functions (directly measurable ?)

(b) e.m. (HLbL) channels:

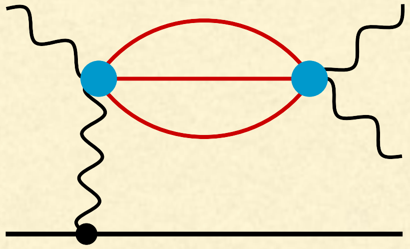
  - LQCD ?



see f.i. talk
by A. Gerardin

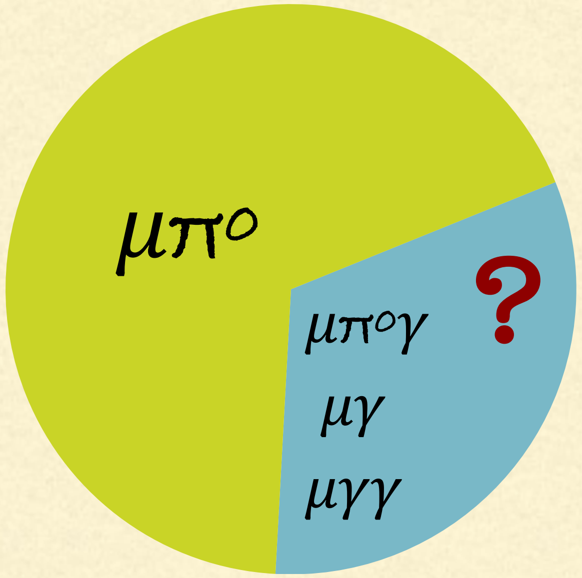
direct LbL scattering (exp.) ?

ATLAS Coll., Nature Physics 13, 852–858 (2017)



OUTLOOK

- Pseudoscalar-meson production, $\gamma\mu \to \mu(\pi^0,\eta,\eta')$, gives $(g-2)_\mu$ contribution which is a factor of 2 to 3 larger than the conventional PS-pole calculations

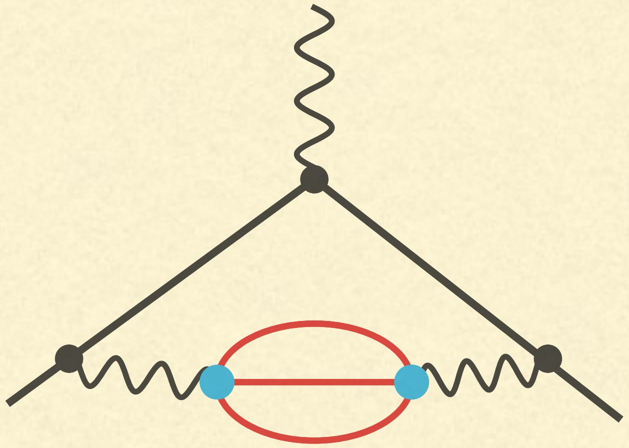


to be continued …

BACK-UP SLIDES

HVP: STANDARD FORMULA

- Hadronic vacuum polarization: 2 Data-driven approaches based on dispersion theory
  - A) **Standard Formula**
  - B) **Schwinger Sum Rule**



A

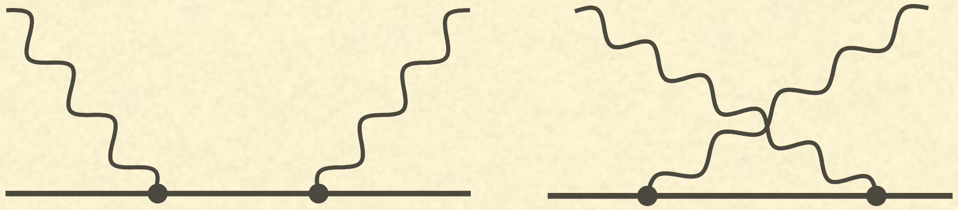
$$\boxed{\varkappa^{\mathrm{HVP}} = \frac{\alpha}{\pi^2}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{d}s}{s}\,\mathrm{Im}\,\Pi^{\mathrm{had}}(s)\int_0^1\mathrm{d}x\,\frac{x^2(1-x)}{x^2+(1-x)(s/m^2)}}$$

$$\underset{\substack{\uparrow\\ \text{photon selfenergy}}}{\mathrm{Im}\,\Pi^{\mathrm{had}}(s)} = \frac{s}{4\pi\alpha}\,\sigma(\gamma^* \underset{\substack{\uparrow\\ \text{decay rate of a virtual timelike}\\ \text{photon into hadrons}}}{\rightarrow} \text{anything})$$

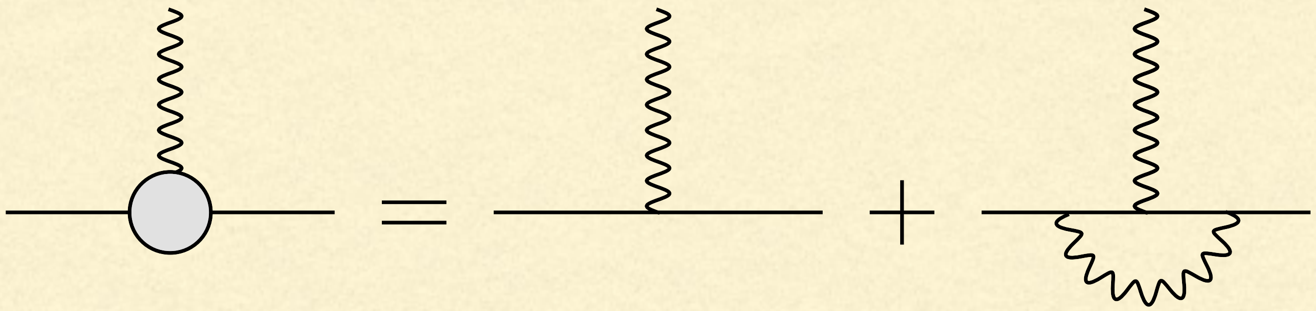
THE SCHWINGER TERM

- Schwinger sum rule: $\varkappa = \frac{m^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q} \right]_{Q^2=0}$
- Input: longitudinal-transverse photo-absorption cross section

tree-level QED
Compton scattering

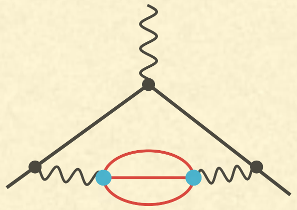


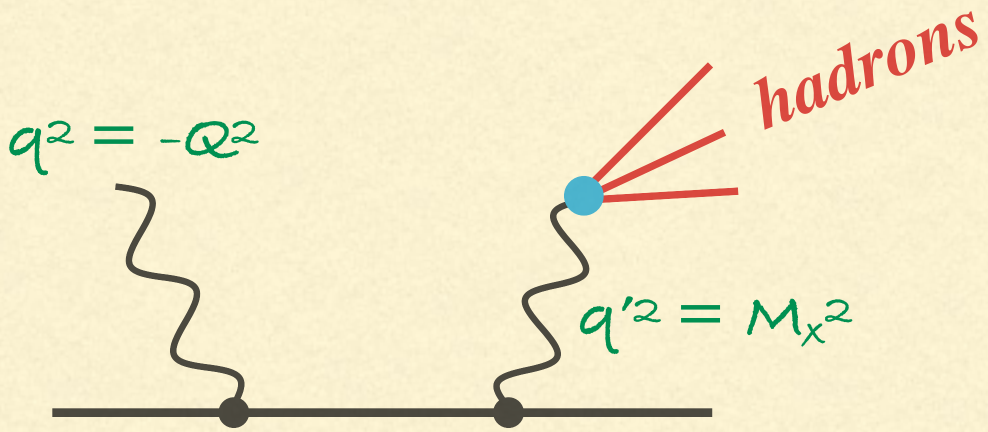
$$\sigma_{LT}^{\gamma^* \mu \to \gamma \mu}(\nu, Q^2) = \frac{\pi \alpha^2 Q \, (s - m^2)^2}{4 m^3 \nu^2 \, (\nu^2 + Q^2)} \left(-2 - \frac{m(m+\nu)}{s} + \frac{3m + 2\nu}{\sqrt{\nu^2 + Q^2}} \, \mathrm{arccoth} \frac{m + \nu}{\sqrt{\nu^2 + Q^2}} \right)$$



$F_2(0) = \varkappa$ $\quad$ $\varkappa^{(0)} = 0$ $\quad$ $\varkappa^{(1)} = \alpha / 2\pi$

HVP: SCHWINGER SUM RULE





+ crossed diagram

- Cross section of hadron production through timelike Compton scattering:

factories into: $$\sigma(\gamma\mu \to \mu X) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}M_X^2}{M_X^2}\, \underbrace{\sigma(\gamma\mu \to \gamma^*\mu)}_{\text{timelike Compton scattering}}\, \underbrace{\operatorname{Im}\Pi_X(M_X^2)}_{\text{virtual-photon decay into hadrons}}$$

- Timelike Compton scattering cross section:

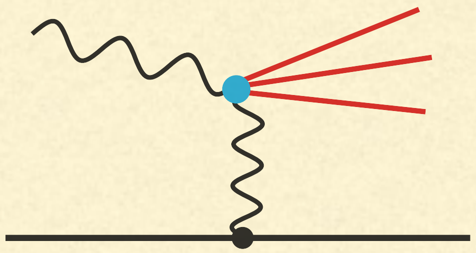
$$\boxed{\left[\frac{\sigma_{LT}^{\gamma\mu\to\gamma^*\mu}(\nu,Q^2)}{Q}\right]_{Q^2=0} = \frac{\pi\alpha^2}{2m^2\nu^3}\left[-(5s+m^2+M_X^2)\lambda + (s+2m^2-2M_X^2)\log\frac{\beta+\lambda}{\beta-\lambda}\right]}$$

$$\beta = (s+m^2-M_X^2)/2s \qquad s = m^2+2m\nu$$
$$\lambda = (1/2s)\sqrt{[s-(m+M_X)^2]\,[s-(m-M_X)^2]}$$

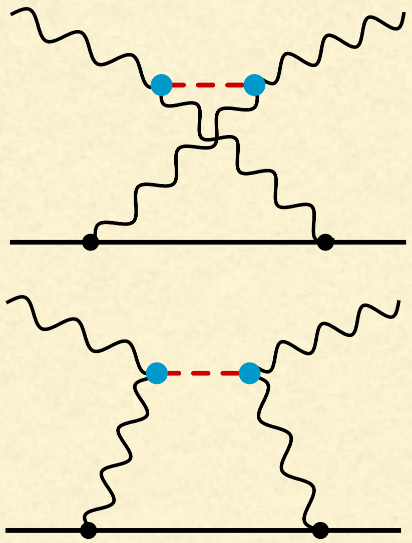
Hagelstein et al., in prep.

PROOF OF VANISHING PRIMAKOFF CONTR.

- Contribution of the Primakoff cross section is vanishing by itself [would be of order $\mathcal{O}(\alpha^2)$]
- Calculate the real part of the Compton scattering box diagram:



$$\boxed{\varkappa = -\frac{m^2}{2\pi\alpha} \lim_{Q^2\to 0} \lim_{\nu \to 0} \frac{T_{TL}(\nu, Q^2)}{Q}}$$



$$\begin{aligned} &\lim_{Q^2\to 0} \frac{T^{\text{Box}}_{LT}}{Q} \\ &= \lim_{Q^2\to 0} \frac{2}{3\pi^4} \int_0^\infty \mathrm{d}K \int_0^\pi \mathrm{d}\chi \, \sin^2\chi \int_{\nu_0}^\infty \mathrm{d}\tilde{\nu}' \\ &\times \left\{ S_1(iK\cos\chi, K^2) \left[\frac{\tau^a_{TL}}{Q} + \frac{K\cos^2\chi}{2} \frac{\tau^a_{TT}}{\tilde{\nu}'} \right] - \frac{iK\cos\chi}{m} S_2(iK\cos\chi, K^2) \left[\frac{\tau^a_{TL}}{Q} + \frac{K}{2} \frac{\tau^a_{TT}}{\tilde{\nu}'} \right] \right\} \\ &= 0 \end{aligned}$$

- LbL sum rules:

$$\lim_{Q^2\to 0} \int_{\nu_0}^\infty \mathrm{d}\tilde{\nu}' \, \frac{1}{\tilde{\nu}'} \, \tau^a_{TT}(\tilde{\nu}', K^2, Q^2) = 0$$

$$\lim_{Q^2\to 0} \int_{\nu_0}^\infty \mathrm{d}\tilde{\nu}' \, \frac{1}{Q} \, \tau^a_{TL}(\tilde{\nu}', K^2, Q^2) = 0$$

Pascalutsa et al.,
Phys. Rev. D 85 (2012) 116001.

see also talk
by I. Danilkin

THE A.M.M. OF THE PROTON

Gerasimov—Drell—Hearn sum rule

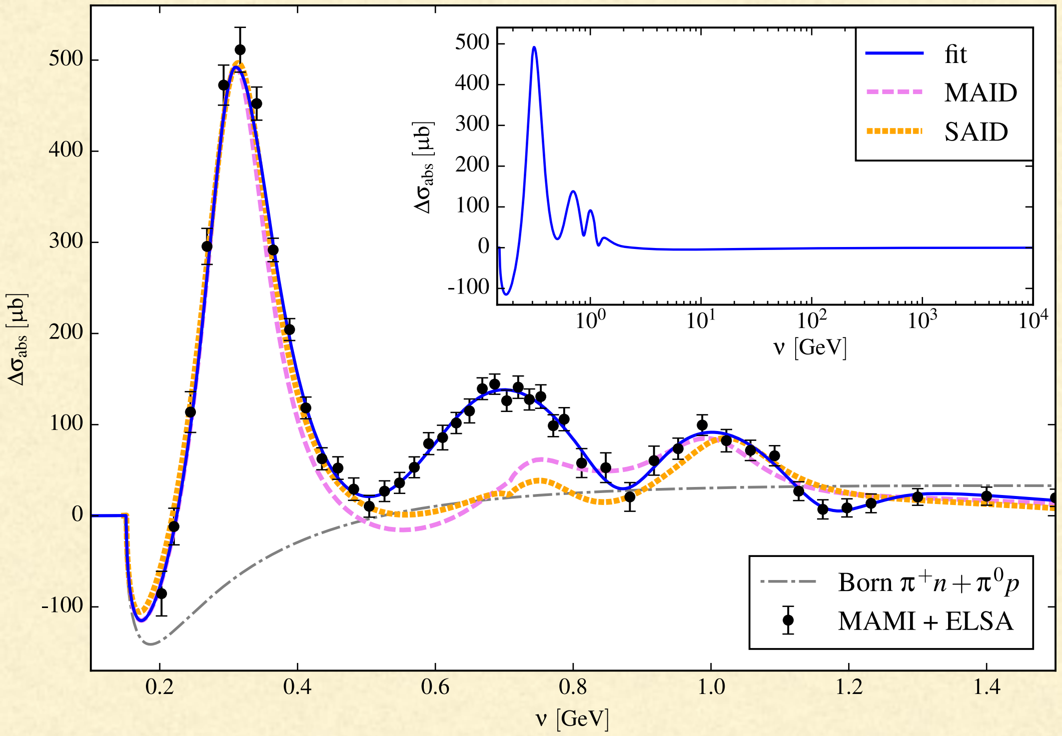
$$\boxed{I_{\mathrm{GDH}} = \frac{2\pi^2\alpha}{m^2}\varkappa^2 = -2\int_{\nu_0}^{\infty} \mathrm{d}\nu\, \frac{\sigma_{\mathrm{TT}}(\nu)}{\nu}}$$

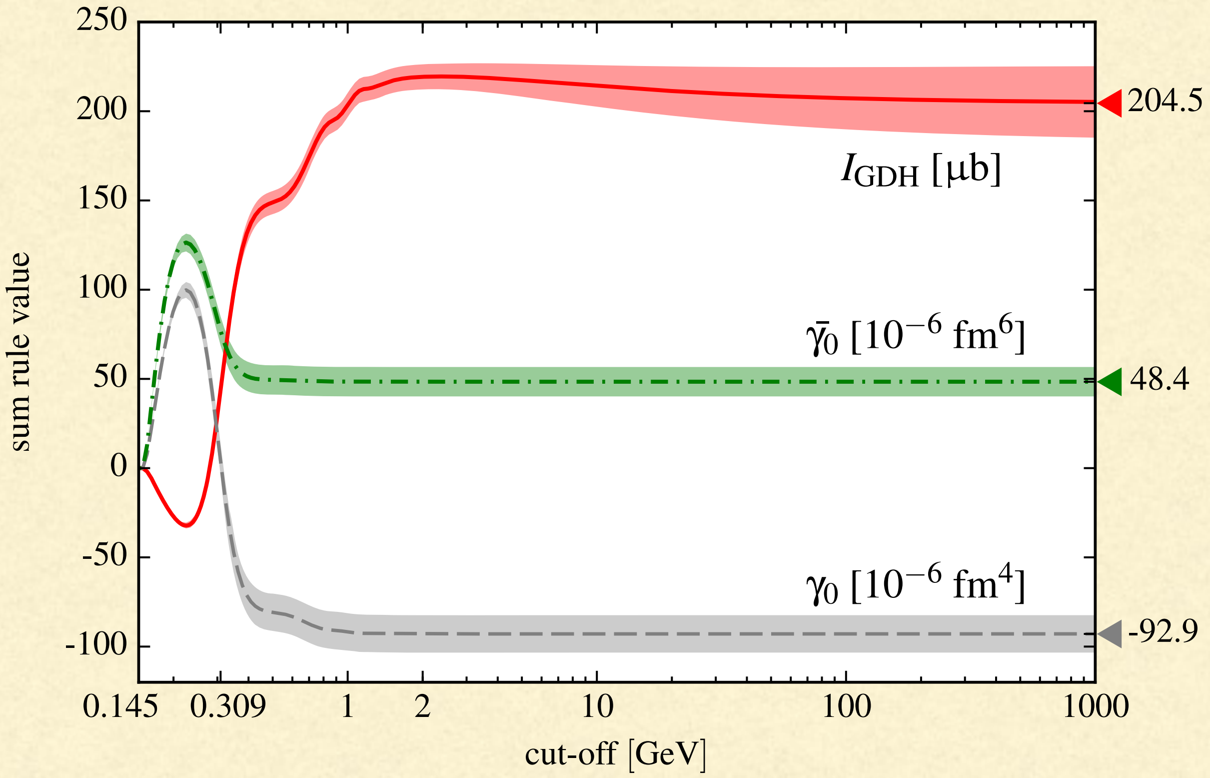
$\varkappa_p \approx 1.7929$ and

$I_{\mathrm{GDH}} = 204.784481\ \mu b$ [CODATA]

$\varkappa_\mu \approx 0.0011659209(6)$ [BNL]

- GDH sum rule for the muon:
  - huge cancelation requires measurements with incredible accuracy
    - r.h.s.: HVP starts at $\mathcal{O}(\alpha^2)$, I_{GDH} starts at $\mathcal{O}(\alpha^5)$
    - l.h.s.: hadronic photo-production cross section starts at $\mathcal{O}(\alpha^3)$



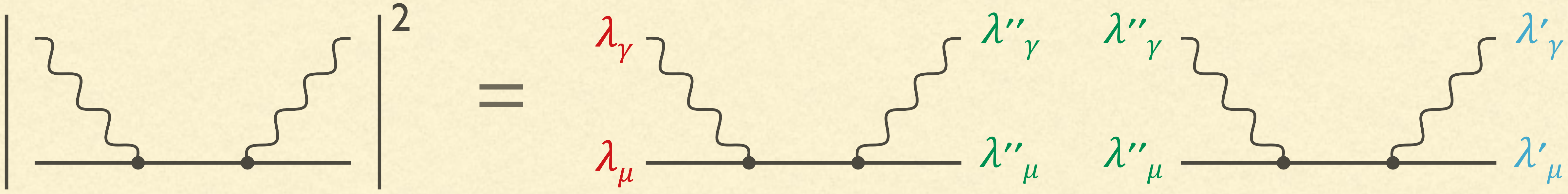


THE CROSS SECTION σ_{LT}

- Example: tree-level QED Compton scattering cross section

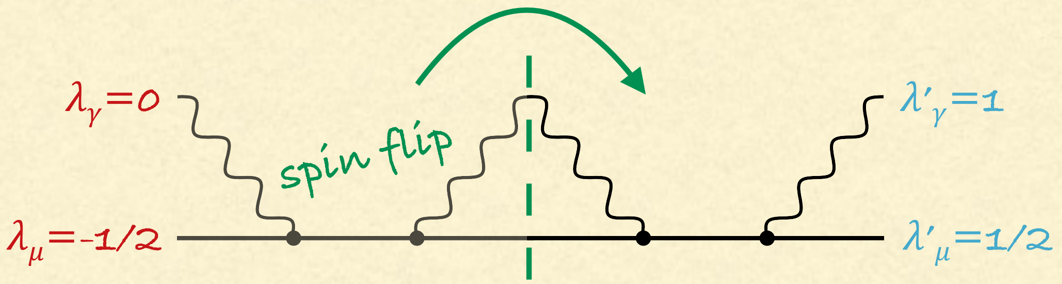
$$\mathrm{d}\sigma_{\lambda'_\gamma\lambda'_\mu\lambda_\gamma\lambda_\mu} = (2\pi)^4\delta^{(4)}(p_f - p_i)\sum_{\lambda''_\gamma,\lambda''_\mu}\frac{\mathcal{M}^\dagger_{\lambda'_\gamma\lambda'_\mu\lambda''_\gamma\lambda''_\mu}\mathcal{M}_{\lambda''_\gamma\lambda''_\mu\lambda_\gamma\lambda_\mu}}{4I}\prod_a\frac{\mathrm{d}^3p'_a}{(2\pi)^3 2E'_a},$$

with conserved helicity: $H=\lambda'_\gamma-\lambda'_\mu=\lambda_\gamma-\lambda_\mu$

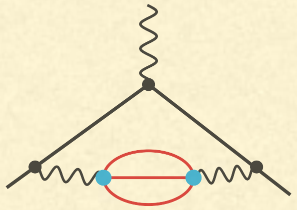


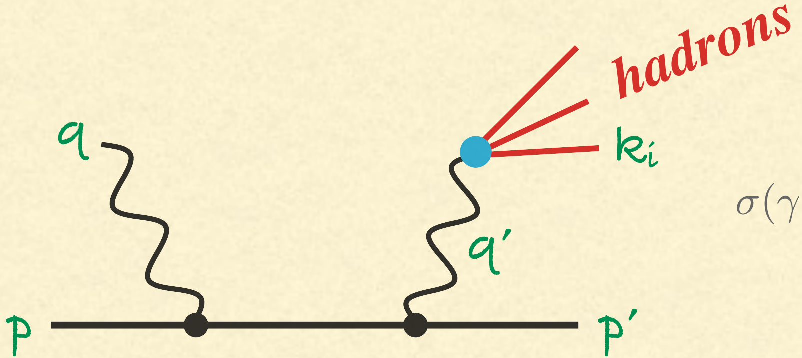
- helicity difference photo-absorption cross section: $\boxed{\sigma_{TT} = 1/2\,(\sigma_{1/2} - \sigma_{3/2})}$
- longitudinal-transverse photo-absorption cross section:

$\gamma^*(\lambda_\gamma=0) + \mu(\lambda_\mu=-1/2) \rightarrow \gamma(\lambda'_\gamma=1) + \mu(\lambda'_\mu=1/2)$



TIME-LIKE CS & PHOTON DECAY





$q^2 = -Q^2$
$q'^2 = M_x^2$

+ crossed diagram

Λ^ν: virtual-photon decay vertex

$$\sigma(\gamma\mu \to \mu X) = \frac{(2\pi)^4}{4I} \int \mathrm{d}^4 q' \underbrace{\int \prod_i \frac{\mathrm{d}^3 \boldsymbol{k}_i}{2E_{k_i}(2\pi)^3} \int \frac{\mathrm{d}^3 \boldsymbol{p'}}{2E_{p'}(2\pi)^3}}_{\text{phase space of the final state}} \left[\frac{\Lambda^{\dagger\mu}\Lambda^\nu \rho_{\mu\nu}}{(-q'^2)^2} \right] \delta^4(q' - \sum_i k_i)\, \delta^4(p + q - p' - q')$$

initial flux factor
$I^2 = (p \cdot q)^2 - p^2 q^2$

phase space of the final state

$\rho_{\mu\nu}$: squared matrix element of timelike CS

- Virtual-photon decay width into hadronic state X:

$$[\Gamma(\gamma^* \to X)]^{\mu\nu} = \int \prod_i \frac{\mathrm{d}^3 \boldsymbol{k_i}}{2E_{k_i}(2\pi)^3} \frac{\Lambda^{\dagger\mu}\Lambda^\nu}{2E_{q'}} (2\pi)^4 \delta^4(q' - \sum_i k_i)$$
$$= -\frac{1}{\sqrt{q'^2}} (q'^2 g^{\mu\nu} - q'^\mu q'^\nu) \mathrm{Im}\, \Pi_X(q'^2)$$

Im Π_X: contribution of state X to the VP

- Combine into: $\sigma(\gamma\mu \to \mu X) = -\frac{1}{2I} \int \mathrm{d}^4 q' \int \frac{\mathrm{d}^3 \boldsymbol{p'}}{2E_{p'}(2\pi)^3} \rho^\mu_\mu \frac{\mathrm{Im}\, \Pi_X(q'^2)}{q'^2} \delta^4(p + q - p' - q')$

- Final factorized cross section: $\boxed{\sigma(\gamma\mu \to \mu X) = \frac{1}{\pi} \int_{4m_\pi^2}^\infty \frac{\mathrm{d}M_X^2}{M_X^2} \sigma(\gamma\mu \to \gamma^* \mu)\, \mathrm{Im}\, \Pi_X(M_X^2)}$

COMPTON SCATTERING SUM RULES

- optical theorem:

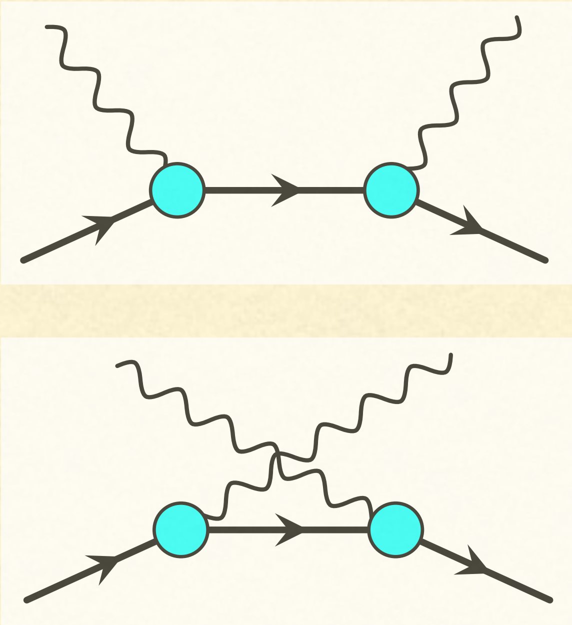
$$\boxed{\mathrm{Im}\, S_1(\nu,Q^2)} = \frac{4\pi^2\alpha}{\nu}\, g_1(x,Q^2)$$

$$\mathrm{Im}\, S_2(\nu,Q^2) = \frac{4\pi^2\alpha m}{\nu^2}\, g_2(x,Q^2)$$

- dispersion relations:

$$S_1(\nu,Q^2) = \frac{2}{\pi}\int_{\nu_{\mathrm{el}}}^{\infty} \mathrm{d}\nu' \, \frac{\nu' \boxed{\mathrm{Im}\, S_1(\nu',Q^2)}}{\nu'^{\,2}-\nu^2-i0^+}$$

$$\nu S_2(\nu,Q^2) = \frac{2}{\pi}\int_{\nu_{\mathrm{el}}}^{\infty} \mathrm{d}\nu' \, \frac{\nu'^{\,2}\, \mathrm{Im}\, S_2(\nu',Q^2)}{\nu'^{\,2}-\nu^2-i0^+}$$



$$\longrightarrow \boxed{\begin{aligned} S_1(\nu,Q^2) &= \frac{16\pi\alpha m}{Q^2}\int_0^1 \mathrm{d}x\, \frac{g_1(x,Q^2)}{1-x^2(\nu/\nu_{\mathrm{el}})^2-i0^+} \\ \nu S_2(\nu,Q^2) &= \frac{16\pi\alpha m^2}{Q^2}\int_0^1 \mathrm{d}x\, \frac{g_2(x,Q^2)}{1-x^2(\nu/\nu_{\mathrm{el}})^2-i0^+} \end{aligned}}$$

with $\nu_{\mathrm{el}} = Q^2/2m$

- low-energy expansion of CS amplitudes:

$$\frac{1}{4\pi}\left[S_1 - S_1^{\mathrm{pole}}\right](\nu,0) = -\frac{\alpha\varkappa^2}{2m} + \mathcal{O}(\nu^2)$$

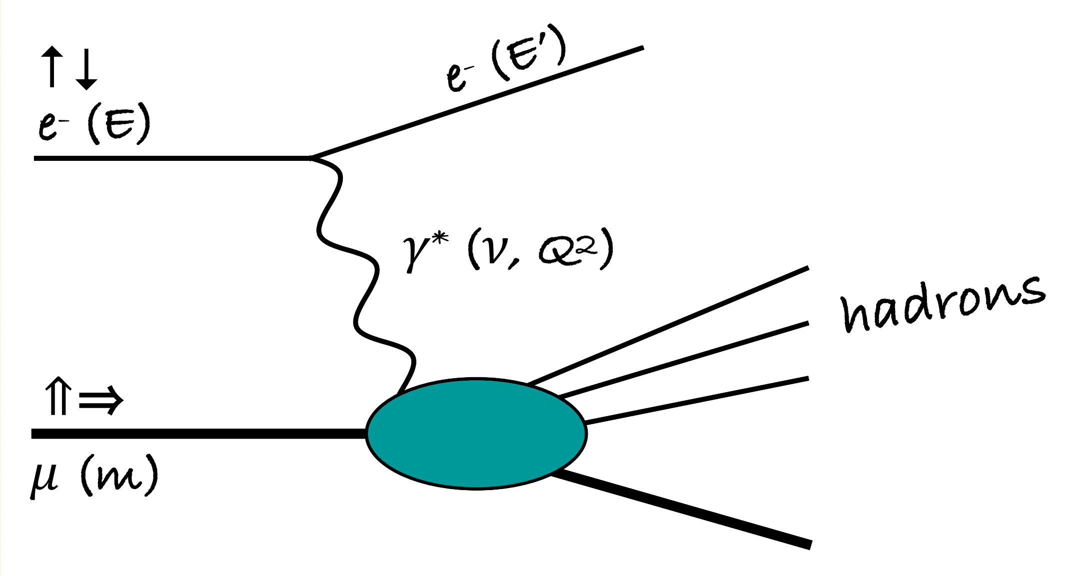
$$\frac{\nu}{4\pi}\left[S_2 - S_2^{\mathrm{pole}}\right](\nu,0) = \frac{\alpha\varkappa(1+\varkappa)}{2} + \mathcal{O}(\nu^2)$$

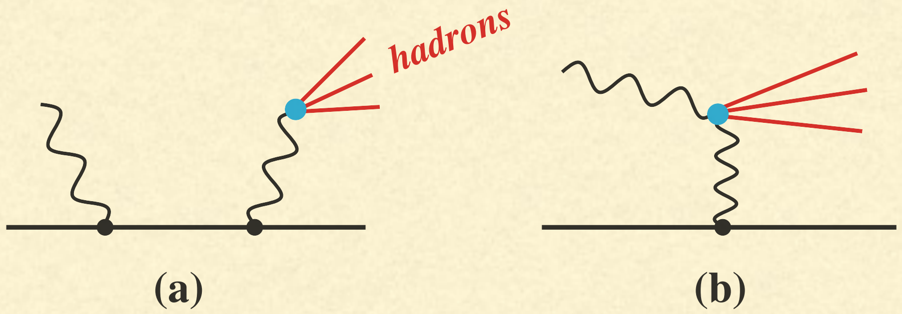
MUON STRUCTURE FUNCTIONS

- Muon spin structure functions measured in inelastic electron-muon scattering:
  - $\underset{\Uparrow}{\mu} e \to \mu \underset{\uparrow}{e}$ + hadrons

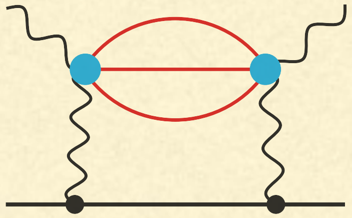
$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega}(\downarrow\Uparrow - \uparrow\Uparrow) = \frac{4\alpha^2}{mQ^2}\frac{E'}{\nu E}\left[(E + E'\cos\theta)\, g_1(x,Q^2) - \frac{Q^2}{\nu}\, g_2(x,Q^2)\right]$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega}(\downarrow\Rightarrow - \uparrow\Rightarrow) = \frac{4\alpha^2\sin\theta}{mQ^2}\frac{E'^2}{\nu^2 E}\left[\nu\, g_1(x,Q^2) - 2E\, g_2(x,Q^2)\right]$$





- Hadron photo-production off the muon:
  - (a) timelike Compton scattering
  - (b) Primakoff effect

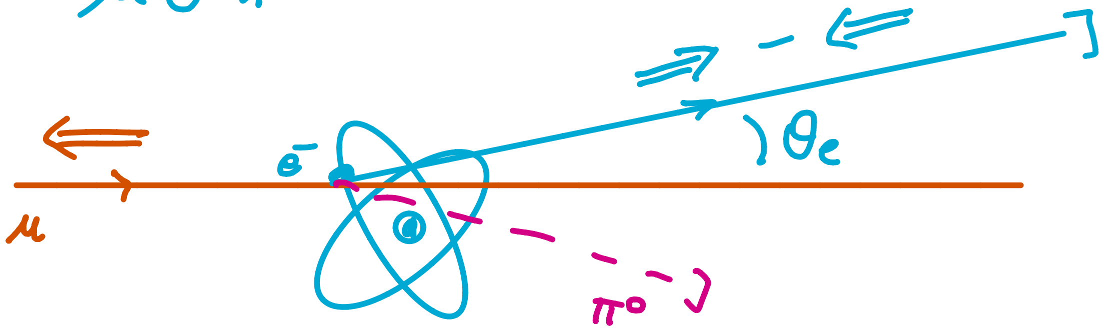


- HLbL contribution to Compton scattering

Feasibility of measurement at COMPASS as part of MUonE ?

cf. The Workshop on
Evaluation of the Leading Hadronic Contribution
to the Muon Anomalous Magnetic Moment
Mainz (Germany), 2 - 5 April 2017

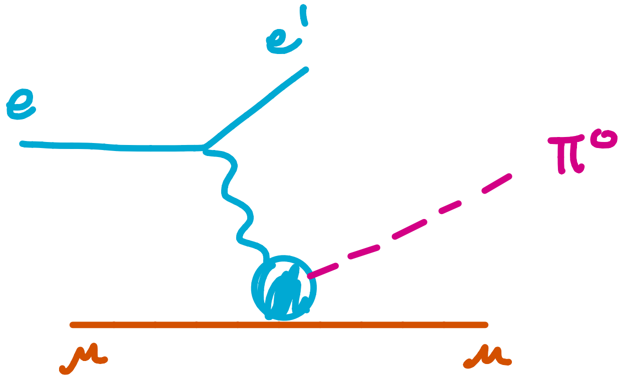
$\mu e \to \mu e \pi^0$



$E_\mu = 150, \ 200 \text{ GeV}$

$E_e' \simeq 1 \text{ GeV}$

$\theta_e \simeq 10 \text{ mrad}$



$$Q^2 \simeq 2 m_e E_e' \simeq 10^{-3} \text{ GeV}^2$$

$$\nu \simeq \frac{m_e E_\mu}{m_\mu} \left(1 - 2 \frac{E_e'}{m_e} \sin^2 \frac{\theta}{2} \right) = \left(\nu_{\pi^0}, \ 1 \text{ GeV} \right)$$

$$\nu_{\pi^0} = \frac{m_{\pi^0}}{m_\mu} \left(\tfrac{1}{2} m_{\pi^0} + m_\mu \right) \simeq 230 \text{ MeV}$$

slide: V. Pascalutsa @CIPANP'18