Dispersive approach to hadronic light-by-light: An overview

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FOR FUNDAMENTAL PHYSICS

Muon $g-2$ Theory Initiative Workshop, Mainz, 19.6.2018

Based on:

JHEP09(14)091, JHEP09(15)074, JHEP04(17)161, PRL(17)
in collab. with M. Hoferichter, M. Procura and P. Stoffer and
PLB738(2014)6 .... +B. Kubis

Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light (HLbL)

Setting up the stage:
- Gauge invariance and crossing symmetry
- Master Formula

A dispersion relation for HLbL

Individual contributions
- - Pion-pole contribution
- - Pion-box contribution
- - Pion rescattering contribution
- - Missing contributions

Outlook and Conclusions

Different analytic evaluations of HLbL

Jegerlehner-Nyffeler 2009

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
" " + subl. in N_c	–	–	–	0 ± 10	–	–	–
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3	21 ± 3
total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (*Ks are subdominant*)
- heavier single-particle poles decreasingly important

Advantages of the dispersive approach

- model independent
- unambiguous definition of the various contributions
- makes a data-driven evaluation possible
  (in principle)
- if data not available: use theoretical calculations of subamplitudes, short-distance constraints etc.

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- why hasn't this been adopted before?

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- First attempts: GC, Hoferichter, Procura, Stoffer (14)

  Pauk, Vanderhaeghen (14)
- why hasn't this been adopted before?
- similar philosophy, with a different implementation: Schwinger sum rule Hagelstein, Pascalutsa (17) → talk by F. Hagelstein

Dispersive approach for hadronic vacuum polarization

$$\Pi_{\mu\nu}(q) = i\int d^4x e^{iqx}\langle 0|Tj_\mu(x)j_\nu(0)|0\rangle = \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi(q^2)$$

where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x)\gamma^\mu q_i(x)$, $i = u, d, s$ is the em current

- Lorentz invariance: 2 structures
- gauge invariance: reduction to 1 structure
- Lorentz-tensor defined in such a way that the function $\Pi(q^2)$ does not have kinematic singularities or zeros
- $\bar{\Pi}(q^2) := \Pi(q^2) - \Pi(0)$ satisfies

$$\bar{\Pi}(q^2) = \frac{q^2}{\pi}\int_{4M_\pi^2}^{\infty} dt \frac{\mathrm{Im}\bar{\Pi}(t)}{t(t-q^2)}$$

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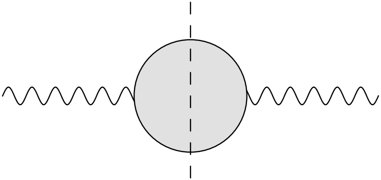
Easy!

Unitarity for HVP

For HVP the unitarity relation is simple and looks the same for all possible intermediate states



$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

which implies ($H_i(t) = i$-th hadronic state)

$$\bar{\Pi}(q^2) = \sum_i \frac{q^2}{\pi} \int_{4M_\pi^2}^{\infty} dt \frac{\sigma(e^+e^- \to H_i(t))}{4\pi\alpha(t)(t-q^2)}$$

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The HLbL tensor (much less easy...)

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz \, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

with Mandelstam variables

$$s = (q_1 + q_2)^2 \quad t = (q_1 + q_3)^2 \quad u = (q_2 + q_3)^2$$

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General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \ldots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \ldots\}$, but in $d = 4$ only 136 are linearly independent

Eichmann *et al.* (14)

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Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

- 43 basis tensors (BT) — in $d = 4$: 41=no. of helicity amplitudes
- 11 additional ones (T) — to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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$$
\begin{aligned}
T_1^{\mu\nu\lambda\sigma} &= \epsilon^{\mu\nu\alpha\beta}\epsilon^{\lambda\sigma\gamma\delta} q_{1\,\alpha} q_{2\,\beta} q_{3\,\gamma} q_{4\,\delta}, \\
T_4^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_4^\lambda q_3^\sigma - q_3\cdot q_4 g^{\lambda\sigma}\right), \\
T_7^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_1\cdot q_4 \left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_1^\sigma q_1\cdot q_3 - q_1^\lambda q_1^\sigma q_3\cdot q_4\right), \\
T_{19}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2\cdot q_4 \left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_2^\sigma q_1\cdot q_3 - q_1^\lambda q_2^\sigma q_3\cdot q_4\right), \\
T_{31}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2^\lambda q_1\cdot q_3 - q_1^\lambda q_2\cdot q_3\right)\left(q_2^\sigma q_1\cdot q_4 - q_1^\sigma q_2\cdot q_4\right), \\
T_{37}^{\mu\nu\lambda\sigma} &= \left(q_3^\mu q_1\cdot q_4 - q_4^\mu q_1\cdot q_3\right)\Big(q_3^\nu q_4^\lambda q_2^\sigma - q_4^\nu q_2^\lambda q_3^\sigma + g^{\lambda\sigma}\left(q_4^\nu q_2\cdot q_3 - q_3^\nu q_2\cdot q_4\right) \\
&\qquad + g^{\nu\sigma}\left(q_2^\lambda q_3\cdot q_4 - q_4^\lambda q_2\cdot q_3\right) + g^{\lambda\nu}\left(q_3^\sigma q_2\cdot q_4 - q_2^\sigma q_3\cdot q_4\right)\Big), \\
T_{49}^{\mu\nu\lambda\sigma} &= q_3^\sigma \Big(q_1\cdot q_3 q_2\cdot q_4 q_4^\mu g^{\lambda\nu} - q_2\cdot q_3 q_1\cdot q_4 q_4^\nu g^{\lambda\mu} + q_4^\mu q_4^\nu\left(q_1^\lambda q_2\cdot q_3 - q_2^\lambda q_1\cdot q_3\right) \\
&\qquad + q_1\cdot q_4 q_3^\mu q_4^\nu q_2^\lambda - q_2\cdot q_4 q_4^\mu q_3^\nu q_1^\lambda + q_1\cdot q_4 q_2\cdot q_4\left(q_3^\nu g^{\lambda\mu} - q_3^\mu g^{\lambda\nu}\right)\Big) \\
&\quad - q_4^\lambda \Big(q_1\cdot q_4 q_2\cdot q_3 q_3^\mu g^{\nu\sigma} - q_2\cdot q_4 q_1\cdot q_3 q_3^\nu g^{\mu\sigma} + q_3^\mu q_3^\nu\left(q_1^\sigma q_2\cdot q_4 - q_2^\sigma q_1\cdot q_4\right) \\
&\qquad + q_1\cdot q_3 q_4^\mu q_3^\nu q_2^\sigma - q_2\cdot q_3 q_3^\mu q_4^\nu q_1^\sigma + q_1\cdot q_3 q_2\cdot q_3\left(q_4^\nu g^{\mu\sigma} - q_4^\mu g^{\nu\sigma}\right)\Big) \\
&\quad + q_3\cdot q_4\Big(\left(q_1^\lambda q_4^\mu - q_1\cdot q_4 g^{\lambda\mu}\right)\left(q_3^\nu q_2^\sigma - q_2\cdot q_3 g^{\nu\sigma}\right) - \left(q_2^\lambda q_4^\nu - q_2\cdot q_4 g^{\lambda\nu}\right)\left(q_3^\mu q_1^\sigma - q_1\cdot q_3 g^{\mu\sigma}\right)\Big).
\end{aligned}
$$

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

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- 43 basis tensors (BT) — in $d = 4$: 41=no. of helicity amplitudes
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- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

The 54 scalar functions Π_i are free of kinematic singularities and zeros and as such are amenable to a dispersive treatment

HLbL contribution to a_μ

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) = -k^\rho \frac{\partial}{\partial k^\sigma} \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2).$$

Contribution to a_μ: $m := m_\mu$

$$a_\mu = \frac{-1}{48m} \mathrm{Tr}\left\{ (\not{p} + m)[\gamma^\rho, \gamma^\sigma](\not{p} + m)\Gamma^{\mathrm{HLbL}}_{\rho\sigma}(p) \right\}$$

$$\Gamma_{\rho\sigma} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \frac{\gamma^\mu (\not{p} + \not{q}_1 + m)\gamma^\lambda (\not{p} - \not{q}_2 + m)\gamma^\nu}{((p+q_1)^2 - m^2)((p-q_2)^2 - m^2)} \times \\ \times \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)\Big|_{k=0}$$

BTT basis (no kin. singularities!) ⇒ limit $k_\mu \to 0$ unproblematic

Master Formula

$$a_\mu^{\text{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the Π_i
- 5 integrals can be performed with Gegenbauer polynomial techniques

GC, Hoferichter, Procura, Stoffer (2015)

Master Formula

After performing the 5 integrations:

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{48\pi^2}\int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\sum_{i=1}^{12} T_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau)$$

where Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

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Setting up the dispersive calculation

We split the HLbL tensor as follows:

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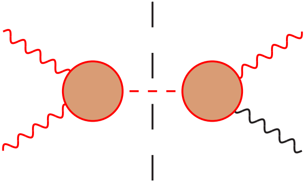

Pion pole: imaginary parts = δ-functions

Projection on the BTT basis: easy ✓

Our master formula=explicit expressions in the literature ✓

Input: pion transition form factor → talks by Hoid, Fischer, Roig, Sanchez-Puertas

First results of direct lattice calculations → talks by L. Jin and by Gerardin

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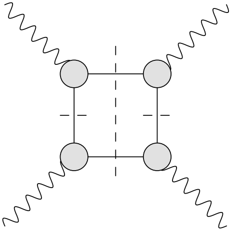
π-box with the BTT set:



– we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole
– we have explicitly checked that this is identical to sQED multiplied by $F_V^\pi(s)$ (FsQED)

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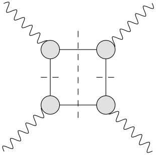
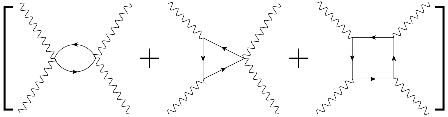


$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2)$$

$\times$



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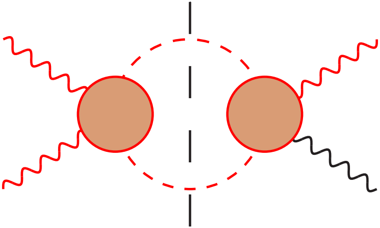

The "rest" with 2π intermediate states has cuts only in one channel and will be calculated dispersively after partial-wave expansion

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Contributions of cuts with anything else other than one and two pions in intermediate states are neglected in first approximation

of course, the η, η' and other pseudoscalars pole contribution, or the kaon-pole contribution can be calculated within the same formalism

Partial wave expansion for 2π contributions

To complete the program of writing down a dispersion relation for two-pion contributions **is not easy**:

- unitarity relations are diagonal in a helicity amplitude basis;
- the helicity basis relevant for $(g-2)_\mu$ is the one with one on-shell photon, which has 27 elements;
- in the limit $q_4^2, q_4^\sigma \to 0$ of the HLbL tensor the number of independent elements of the BTT set drops from 41 to 27;
- there is freedom in the choice of this subset (singly-on-shell basis);
- the arbitrariness in the choice of the 27 elements of the singly-on-shell basis does not influence the final result because of sum rules
- these sum rules follow from the assumption that the HLbL tensor has a uniform behaviour at short distances
- Pascalutsa, Pauk, Vanderhaeghen (12) forward-kinematics sum-rules are a special case of our general sum rules

S-wave 2π contributions

$$\hat{\Pi}_4^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(4s'\,\mathrm{Im}h^0_{++,++}(s') - (s'+q_1^2-q_2^2)(s'-q_1^2+q_2^2)\,\mathrm{Im}h^0_{00,++}(s')\Big)$$

$$\hat{\Pi}_5^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{-2}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(4t'\,\mathrm{Im}h^0_{++,++}(t') - (t'+q_1^2-q_3^2)(t'-q_1^2+q_3^2)\,\mathrm{Im}h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}_6^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{-2}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(4u'\,\mathrm{Im}h^0_{++,++}(u') - (u'+q_2^2-q_3^2)(u'-q_2^2+q_3^2)\,\mathrm{Im}h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}_{11}^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{4}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(2\,\mathrm{Im}h^0_{++,++}(u') - (u'-q_2^2-q_3^2)\,\mathrm{Im}h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}_{16}^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{4}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(2\,\mathrm{Im}h^0_{++,++}(t') - (t'-q_1^2-q_3^2)\,\mathrm{Im}h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}_{17}^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(2\,\mathrm{Im}h^0_{++,++}(s') - (s'-q_1^2-q_2^2)\,\mathrm{Im}h^0_{00,++}(s')\Big)$$

Analogous expressions for the D, G and all higher waves have been derived but are too long to be shown

Calculation of D-wave contributions: $\rightarrow$ talk by P. Stoffer

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Pion-pole contribution

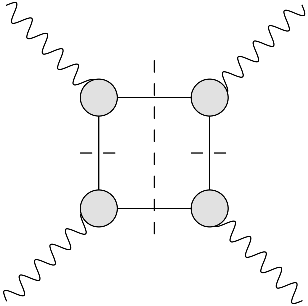
- Expression of this contribution in terms of the pion transition form factor already known — Knecht-Nyffeler (01)
- Both transition form factors must be included:

$$\bar{\Pi}_1 = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2, 0)}{q_3^2 - M_{\pi^0}^2}$$

  dropping one to satisfy short-distance constraints can only be seen as a model — Melnikov-Vainshtein (04)
- data on singly-virtual form factor available — CELLO, CLEO, BaBar, Belle
- several calculations of the transition form factors in the literature, recent developments — → talks by Fischer, Roig, Sanchez-Puertas
- dispersive approach works here too — → talks by Hoid
- quantity where lattice calculations can have a significant impact — → talks by L. Jin and A. Gerardin

Pion-box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion-box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy\, I_i(x,y),$$

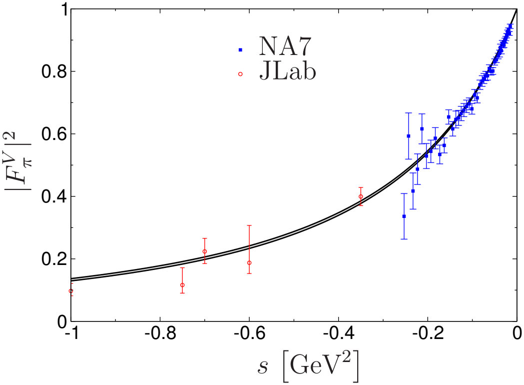
where

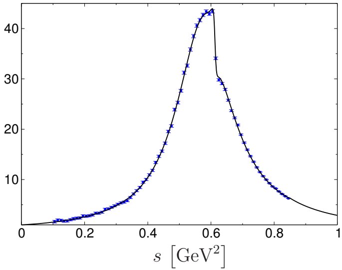
$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

Pion-box contribution





Uncertainties are negligibly small:

$$a_\mu^{\text{FsQED}} = -15.9(2) \cdot 10^{-11}$$

Pion-box contribution

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
" " + subl. in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

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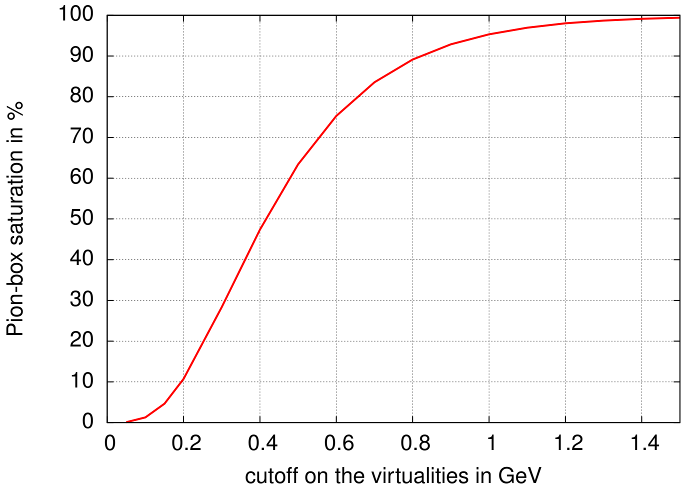
Pion-box saturation with photon virtualities



Check of the partial-wave formalism

Comparison partial-wave expansion of the pion-box vs. full result

J_{max}	$\delta_{J_{\text{max}}}$	$\Delta_{J_{\text{max}}}$
0	29.2%	55.4%
2	10.4%	20.9%
4	4.3%	11.0%
6	2.4%	6.2%
8	1.5%	3.7%
10	1.0%	2.4%
12	0.7%	1.6%
14	0.6%	1.1%

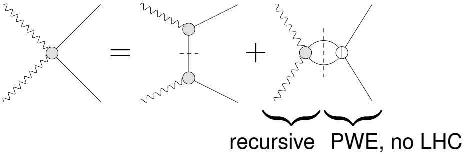
where

$$\delta_{J_{\text{max}}} := 1 - \frac{a_{\mu,J_{\text{max}}}^{\pi\text{-box, PW}}}{a_\mu^{\pi\text{-box}}} \qquad \Delta_{J_{\text{max}}} := \frac{\left|\mathbf{a}_{\mu,J_{\text{max}}}^{\pi\text{-box, PW}} - \mathbf{a}_\mu^{\pi\text{-box}}\right|}{\left|\mathbf{a}_\mu^{\pi\text{-box}}\right|}$$

Convergence for real helicity amplitudes should be much better

First evaluation of *S*- wave 2π-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:



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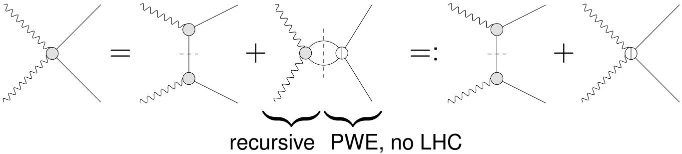

First evaluation of S- wave 2π-rescattering

Based on:

- taking the pion pole as the only left-hand singularity
- $\Rightarrow$ pion vector FF to describe the off-shell behaviour
- $\pi\pi$ phases obtained with the inverse amplitude method

  [realistic only below 1 Gev: accounts for the $f_0(500)$ + unique and well defined extrapolation to ∞]
- numerical solution of the $\gamma^*\gamma^* \to \pi\pi$ dispersion relation

S-wave contributions: $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

$a_\mu^{\rm HLbL}$ in 10^{-11} units

cutoff	1 GeV	1.5 GeV	2 GeV	∞
$I = 0$	-9.2	-9.5	-9.3	-8.8
$I = 2$	2.0	1.3	1.1	0.9
sum	-7.3	-8.3	-8.3	-7.9

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Recall π-Box: $a_\mu^{\pi-\text{box}} = -15.9(2) \cdot 10^{-11}$

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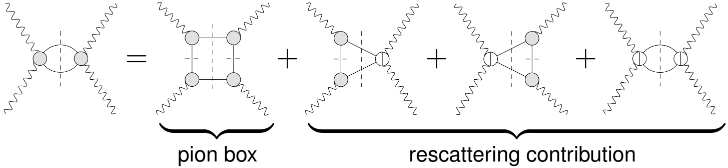
Our first numerical result

Two-pion contributions to HLbL:



Our first numerical result

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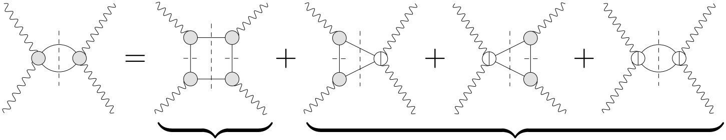


pion box rescattering contribution

$$a_\mu^{\pi-\text{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1) \cdot 10^{-11}$$

$\gamma^*\gamma^* \to \pi\pi$ contribution from other partial waves

- formulae get significantly more involved with several subtleties in the calculation → talk by P. Stoffer
- in particular sum rules which link different partial waves must be satisfied by different resonances in the narrow width approximation Danilkin, Pascalutsa, Pauk, Vanderhaeghen (12,14,17)

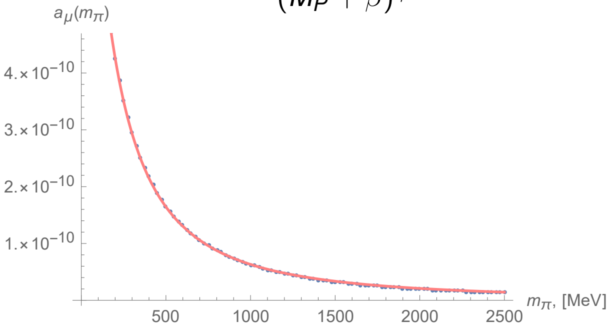
  → talk by Danilkin
- data and dispersive treatments available for on-shell photons
- dispersive treatment for the singly-virtual case and check with forthcoming data is very important → talks by Danilkin & Redmer

Mass scaling of known contributions

Simple parametrization of the mass scaling of various contributions:

J. Monnard

$$a_\mu^P = \frac{\alpha}{(M_P + \beta)^\gamma}$$



Pion-pole (LMD+V form factor):

$$\alpha = 4.46 \cdot 10^{-5} \quad \beta = 257\ \mathrm{MeV} \quad \gamma = 1.89$$

Mass scaling of known contributions

Simple parametrization of the mass scaling of various contributions:

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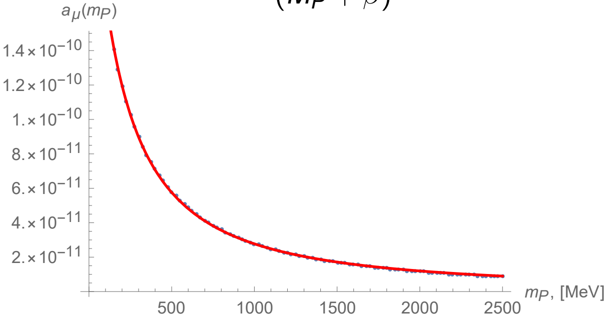


Pseudoscalar resonance (RχT couplings):

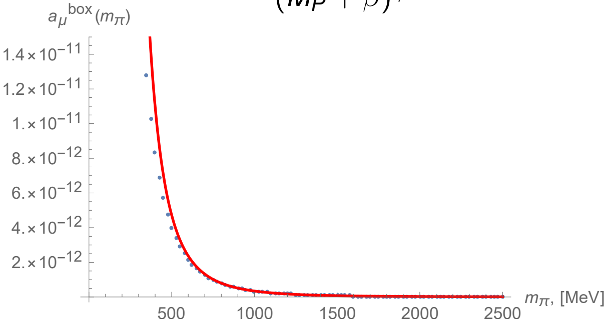
$$\alpha = 7.1 \cdot 10^{-7} \quad \beta = 246\ \mathrm{MeV} \quad \gamma = 1.42$$

Mass scaling of known contributions

Simple parametrization of the mass scaling of various contributions:

J. Monnard

$$a_\mu^P = \frac{\alpha}{(M_P + \beta)^\gamma}$$



Pion-box (VMD form factor):

$$\alpha = 7.5 \cdot 10^{-2} \quad \beta = 0\ \mathrm{MeV} \quad \gamma = 3.78$$

Short-distance contraints

- short-distance constraints on *n*-point functions in QCD is a well known issue
- low- and intermediate-energy representation in terms of hadronic states doesn't typically extrapolate to the right high-energy limit
- requiring that the latter be satisfied is often essential to obtain a description of spectral functions which leads to correct integrals over them vast literature [de Rafael, Goltermann, Peris,...]
- implementing such an approach for HLbL not very simple

  GC, Hagelstein, Laub, work in progress
- alternative strategy: describe the high-energy region by the quark loop. The transition from the hadronic to the quark regime must be done properly $\rightarrow$ talks by Hoferichter & Hoid

Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light (HLbL)

Setting up the stage:
Gauge invariance and crossing symmetry
Master Formula

A dispersion relation for HLbL

Individual contributions
- Pion-pole contribution
- Pion-box contribution
- Pion rescattering contribution
- Missing contributions

Outlook and Conclusions

Conclusions

- The HLbL contribution to $(g-2)_\mu$ can be expressed in terms of measurable quantities in a dispersive approach
- master formula: HLbL contribution to a_μ as triple-integral over scalar functions which satisfy dispersion relations
- the relevant measurable quantity entering the dispersion relation depends on the intermediate state:
  - single-pion contribution: pion transition form factor
  - pion-box contribution: pion vector form factor
  - 2-pion rescattering: $\gamma^*\gamma^{(*)} \to \pi\pi$ helicity amplitudes

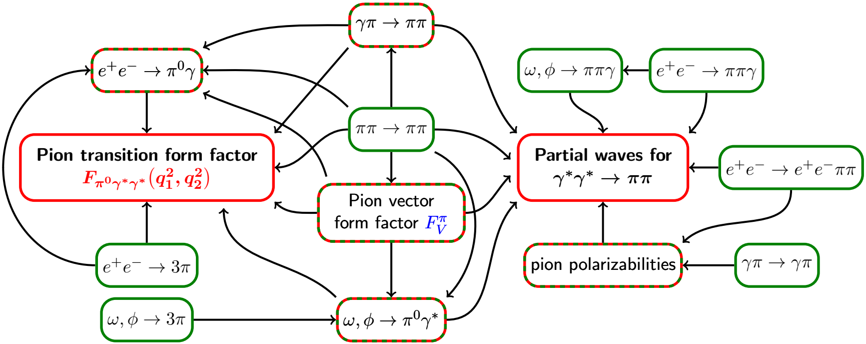
  these three contributions (S-wave for the latter) have been calculated with remarkably small uncertainties
- work on calculating other contributions and estimating missing pieces is in progress

Outlook

- More work is needed to complete the evaluation of contributions of 2π intermediate states esp. for $\ell \geq 2$
  - take into account experimental constraints on $\gamma^{(*)}\gamma \to \pi\pi$
  - estimate the dependence on the q^2 of the second photon (theoretically, there are no data on $\gamma^*\gamma^* \to \pi\pi$ – Lattice?)
  - $\Rightarrow$ solve the dispersion relation for the helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$, including a full treatment of the LHC
- same formulae apply to heavier $n \leq 2$ intermediate states ($\eta^{(\prime)}$ or $\bar{K}K$); for $n > 2$ the formalism must be extended;
- a satisfactory implementation of short-distance constraints is in progress

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer arXiv:1408.2517 (PLB '14)



Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among (lattice) theorists and experimentalists