Hadronic light-by-light dispersion relations: short-distance constraints

Martin Hoferichter

Institute for Nuclear Theory

University of Washington

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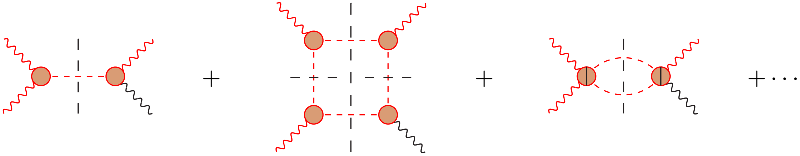
G. Colangelo, MH, M. Procura, P. Stoffer, work in progress

Dispersive representation: overview



$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

- Organized in terms of **on-shell intermediate states**
- Numerics for $a_\mu^{\pi^0\text{-pole}}$ talk by B.-L. Hoid and $a_\mu^{\pi\text{-box}}$, $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}}$ talks by G. Colangelo and P. Stoffer
- Other pseudoscalar (η, η') and two-meson states ($K\bar{K}$, $\pi\eta$) to be included along the same lines
- Here: attacking the ellipsis with **short-distance constraints**

BTT decomposition: reminder

$$a_\mu^{\text{HLbL}} = \frac{\alpha^3}{432\pi^2}\int_0^\infty d\Sigma\,\Sigma^3\int_0^1 dr\,r\sqrt{1-r^2}\int_0^{2\pi}d\phi\sum_{i=1}^{12}T_i(Q_1,Q_2,Q_3)\bar{\Pi}_i(Q_1,Q_2,Q_3)$$

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\Pi_i = \sum_{i=1}^{54}\hat{T}_i^{\mu\nu\lambda\sigma}\hat{\Pi}_i \qquad \bar{\Pi}_i \text{ subset of } \hat{\Pi}_i \qquad Q_i = Q_i(\Sigma, r, \phi)$$

- **Bardeen–Tung–Tarrach (BTT) decomposition**
- Π_i free of kinematic singularities and zeros

  $\hookrightarrow$ dispersive treatment
- A lot of the complexity separated into kernel functions T_i
- Dispersion relations for the Π_i at small virtualities, but need to account for
  - **Asymptotic region**: all Q_i^2 large
  - **Mixed regions**: $Q_3^2 \ll Q_1^2 \sim Q_2^2$ etc.

$\hookrightarrow$ **short-distance constraints**

Familiar contributions in BTT form

- **Pion pole**

$$\hat{\Pi}_1(q_1^2,q_2^2,q_3^2) = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi^0\gamma^*\gamma^*}(q_3^2,0)}{q_3^2-M_{\pi^0}^2} \qquad \hat{\Pi}_2(q_1^2,q_2^2,q_3^2) = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_3^2)F_{\pi^0\gamma^*\gamma^*}(q_2^2,0)}{q_2^2-M_{\pi^0}^2}$$

- **Pion loop**

$$\hat{\Pi}_i^{\pi\text{-box}}(q_1^2,q_2^2,q_3^2) = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy\, I_i(x,y)$$

$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}} \qquad I_7(x,y) = -\frac{8xy(1-x-y)(1-2x)^2(1-2y)}{\Delta_{123}^3} \qquad \ldots$$

$$\Delta_{ijk} = M_\pi^2 - xyq_i^2 - x(1-x-y)q_j^2 - y(1-x-y)q_k^2 \qquad \Delta_{ij} = M_\pi^2 - x(1-x)q_i^2 - y(1-y)q_j^2$$

- BTT decomposition **isolates the dynamical content**, separates the kinematics

  $\hookrightarrow$ do the same for the fermion loop

Fermion loop in BTT decomposition

- **Fermion loop**

$$\hat{\Pi}_i^{f\text{-loop}}(q_1^2, q_2^2, q_3^2) = N_c Q_f^4 \frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy\, I_i(x,y)$$

$$I_1(x,y) = -\frac{16x(1-x-y)}{\Delta_{132}^2} - \frac{16xy(1-2x)(1-2y)}{\Delta_{132}\Delta_{32}} \qquad I_7(x,y) = -\frac{64xy^2(1-x-y)(1-2x)(1-y)}{\Delta_{132}^3}$$

$$\Delta_{ijk} = m_f^2 - xyq_i^2 - x(1-x-y)q_j^2 - y(1-x-y)q_k^2 \qquad \Delta_{ij} = m_f^2 - x(1-x)q_i^2 - y(1-y)q_j^2$$

- Numerical cross checks

f	e	μ	τ	c	b
$a_\mu^{f\text{-loop}}$ $[10^{-11}]$	26257(3)	464.97(5)	2.686(3)	3.038(3)	0.018(3)
Jegerlehner, Nyffeler 2009	26253.5102(2)	464.971652	2.68556(86)		
Asymptotic expansion, Kühn et al. 2003				3.04	0.0182

Asymptotic region and pQCD quark loop

- Four point function

$$\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = -i\int d^4x\, d^4y\, d^4z\, e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)}\langle 0|T\{j^\mu(x)j^\nu(y)j^\lambda(z)j^\sigma(0)\}|0\rangle$$

$$j^\mu(x) = \bar\psi(x)Q\gamma^\mu\psi(x) \qquad \psi=(u,d,s)^T \qquad Q=\frac{1}{3}\text{diag}\big(2,-1,-1\big)$$

- All q_i^2 large: free propagators give the most singular configuration in position space

$\hookrightarrow$ **pQCD quark loop should be adequate for the asymptotic region**

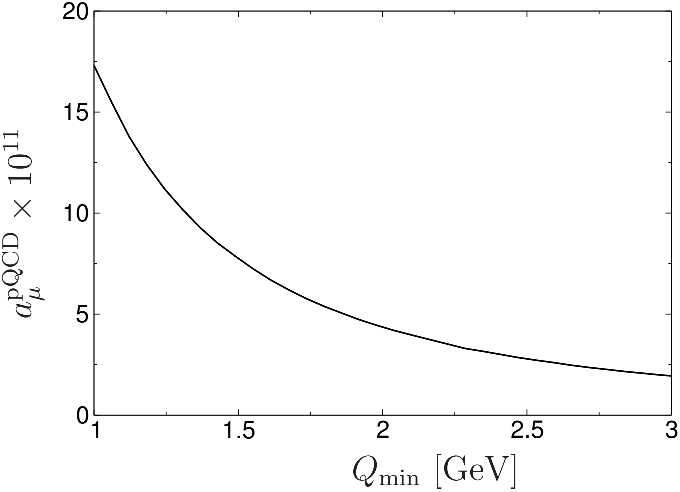
- For $q_1^2=q_2^2=q_3^2\equiv q^2$ simple analytic results

$$\hat\Pi_1^{\text{pQCD}} = -\frac{4}{9\pi^2q^4} \qquad \hat\Pi_4^{\text{pQCD}} = -\frac{8}{243\pi^2q^4}\bigg[33-16\sqrt{3}\,\text{Cl}_2\Big(\frac{\pi}{3}\Big)\bigg] \qquad \cdots$$

- For rough estimate, implement step function

$$\theta(Q_1-Q_{\text{min}})\theta(Q_2-Q_{\text{min}})\theta(Q_3-Q_{\text{min}})$$

Asymptotic region and pQCD quark loop



- For $Q_{\text{min}} \sim 2\,\text{GeV}$ asymptotic region $\lesssim 5 \times 10^{-11}$, but quite sensitive to matching scale

Mixed regions: OPE and triangle amplitude

- What to do for mixed regions $q_1^2 \sim q_2^2 \gg q_3^2$? OPE! Melnikov, Vainshtein 2004
- **Non-renormalization theorems** for VVA triangle (in chiral limit), c.f. a_μ^{EW}
  Czarnecki, Marciano, Vainshtein 2003, Knecht, Peris, Perrottet, de Rafael 2002, 2004, Mondejar, Melnikov 2013
- Proposed interpolation between ABJ anomaly and asymptotic behavior

$$\hat{\Pi}_1^{\text{MV}}(q_1^2, q_2^2, q_3^2) = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\boldsymbol{F_{\pi^0\gamma^*\gamma^*}(0,0)}}{q_3^2 - M_{\pi^0}^2} \qquad \hat{\Pi}_2^{\text{MV}}(q_1^2, q_2^2, q_3^2) = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_3^2)\boldsymbol{F_{\pi^0\gamma^*\gamma^*}(0,0)}}{q_2^2 - M_{\pi^0}^2}$$

- **Ad-hoc model** that disturbs the low-energy properties

$$a_\mu^{\pi^0\text{-pole, VMD}} = 57.1 \times 10^{-11} \to 69.8 \times 10^{-11}$$
$$a_\mu^{\pi^0\text{-pole, disp}} = 62.6 \times 10^{-11} \to 79.9 \times 10^{-11}$$

$\hookrightarrow$ sizable effect, (13–17) $\times\ 10^{-11}$ for pion pole alone!

- Here: revisit OPE in BTT formalism

Mixed regions: OPE and triangle amplitude

- Starting point: **OPE of two vector currents** for $(q_1+q_2)^2 \ll (q_1-q_2)^2$

$$i\int d^4x\, d^4y\, e^{-i(q_1\cdot x+q_2\cdot y)}\,T\{j_\mu(x)j_\nu(y)\} = -\int d^4z\, e^{-i(q_1+q_2)\cdot z}\,\frac{2i}{\hat{q}^2}\epsilon_{\mu\nu\lambda\sigma}\hat{q}^\lambda j_5^\sigma(z)+\cdots$$

$$j^\mu=\bar{\psi}Q\gamma^\mu\psi \qquad j_5^\mu=\bar{\psi}Q^2\gamma^\mu\gamma_5\psi \qquad \hat{q}=\frac{q_1-q_2}{2} \qquad Q=\frac{e}{3}\text{diag}(2,-1,-1)$$

- HLbL tensor in terms of VVA correlator $W_{\mu\nu\lambda}$, valid for $q_1^2\sim q_2^2\gg q_3^2, q_4^2$

$$\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,q_3)=\frac{8}{\hat{q}^2}\epsilon_{\mu\nu\alpha\beta}\hat{q}^\alpha W^{\beta}_{\lambda\sigma}(-q_3,q_4)\sum_{a=3,8,0}C_a^2$$

$$C_a=\frac{1}{2}\text{Tr}(Q^2\lambda_a) \qquad C_3=\frac{1}{6} \qquad C_8=\frac{1}{6\sqrt{3}} \qquad C_0=\frac{2}{3\sqrt{6}}$$

- For BTT projection, need $W_{\mu\nu\lambda}(q_1,q_2)$ for general kinematics

  Knecht, Peris, Perrottet, de Rafael 2004

  $\hookrightarrow$ **one longitudinal** and **three transversal** structures

$$w_L(q_1^2,q_2^2,(q_1+q_2)^2) \qquad w_T^{\pm}(q_1^2,q_2^2,(q_1+q_2)^2) \qquad \tilde{w}_T^{-}(q_1^2,q_2^2,(q_1+q_2)^2)$$

VVA non-renormalization theorems

- Axial anomaly

$$w_L(q_1^2, q_2^2, (q_1+q_2)^2) = \frac{2N_c}{(q_1+q_2)^2}$$

- Transversal structures

$$\begin{aligned}
0 &= (w_T^+ + w_T^-)(q_1^2, q_2^2, (q_1+q_2)^2) - (w_T^+ + w_T^-)((q_1+q_2)^2, q_2^2, q_1^2),\\
0 &= (\tilde{w}_T^- + w_T^-)(q_1^2, q_2^2, (q_1+q_2)^2) + (\tilde{w}_T^- + w_T^-)((q_1+q_2)^2, q_2^2, q_1^2),\\
w_L((q_1+q_2)^2, q_2^2, q_1^2) &= (w_T^+ + \tilde{w}_T^-)(q_1^2, q_2^2, (q_1+q_2)^2) + (w_T^+ + \tilde{w}_T^-)((q_1+q_2)^2, q_2^2, q_1^2)\\
&\quad + \frac{2q_2\cdot(q_1+q_2)}{q_1^2} w_T^+((q_1+q_2)^2, q_2^2, q_1^2) - \frac{2q_1\cdot q_2}{q_1^2} w_T^-((q_1+q_2)^2, q_2^2, q_1^2)
\end{aligned}$$

- Validity:
  - All theorems apply **in the chiral limit**

    $\hookrightarrow$ application requires further assumptions such as pion dominance Vainshtein 2003
  - w_L is renormalized neither perturbatively nor non-perturbatively
  - The transversal theorems only hold perturbatively

Mapping onto BTT

- For $q_1^2 \sim q_2^2 \equiv q^2 \gg q_3^2$ (other combinations from crossing)

$$\hat{\Pi}_1 = 2\xi(q^2) w_L(q_3^2, 0, q_3^2) \qquad \hat{\Pi}_{\{2,3,4,7,8,9,11,13,16,54\}} = 0$$

$$\hat{\Pi}_{\{5,6\}} = \xi(q^2)\big(w_T^+ + \tilde{w}_T^-\big)(q_3^2, 0, q_3^2) = \frac{\xi(q^2)}{2} w_L(q_3^2, 0, q_3^2)$$

$$\hat{\Pi}_{\{10,14\}} = -\hat{\Pi}_{\{17,39,50,51\}} = \frac{\xi(q^2)}{q_1 \cdot q_2}\big(w_T^+ + \tilde{w}_T^-\big)(q_3^2, 0, q_3^2) = \frac{\xi(q^2)}{2 q_1 \cdot q_2} w_L(q_3^2, 0, q_3^2)$$

$$\xi(q^2) = -\frac{1}{2\pi^2 q^2} \sum_{a=3,8,0} C_a^2 = -\frac{1}{18\pi^2 q^2}$$

Mapping onto BTT

- For $q_1^2 \sim q_2^2 \equiv q^2 \gg q_3^2$ (other combinations from crossing)

$$\hat{\Pi}_1 = 2\xi(q^2) w_L(q_3^2, 0, q_3^2) \qquad \hat{\Pi}_{\{2,3,4,7,8,9,11,13,16,54\}} = 0$$

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$$\hat{\Pi}_{\{10,14\}} = -\hat{\Pi}_{\{17,39,50,51\}} = \frac{\xi(q^2)}{q_1 \cdot q_2}\big(w_T^+ + \tilde{w}_T^-\big)(q_3^2, 0, q_3^2) = \frac{\xi(q^2)}{2q_1 \cdot q_2} w_L(q_3^2, 0, q_3^2)$$

$$\xi(q^2) = -\frac{1}{2\pi^2 q^2} \sum_{a=3,8,0} C_a^2 = -\frac{1}{18\pi^2 q^2}$$

- In $\hat{\Pi}_1^{a=3}$ recover for the pion channel (similarly for η, η')

$$\hat{\Pi}_1^{\pi^0\text{-pole}} = \frac{F_{\pi^0\gamma^*\gamma^*}(q^2, q^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2, 0)}{q_3^2 - M_{\pi^0}^2} \to -\frac{2F_\pi}{3q^2 q_3^2} \frac{1}{4\pi^2 F_\pi} \frac{F_{\pi^0\gamma^*\gamma^*}(q_3^2, 0)}{F_{\pi\gamma\gamma}}$$

$$\hat{\Pi}_1^{a=3,\,\text{OPE}} = -\frac{1}{6\pi^2 q^2 q_3^2}$$

- Applicability:
  - Need $q_3^2 \ll q^2$, but $q_3^2 \ll \Lambda_{\text{QCD}}^2$ not a requirement
  - At “small” q_3^2 chiral corrections important, where to match?
  - MV essentially assume pion dominance everywhere

Comparison to pQCD

- Formally evaluating the pQCD quark loop in the OPE limit we find

$$\hat{\Pi}_1^{\text{pQCD}} = -\frac{2}{3\pi^2 q^2 q_3^2} \qquad \hat{\Pi}_{\{5,6\}}^{\text{pQCD}} = -\frac{2}{9\pi^2 q^2 q_3^2}$$

$$\hat{\Pi}_{\{10,14\}}^{\text{pQCD}} = -\hat{\Pi}_{\{17,30\}}^{\text{pQCD}} = -2\hat{\Pi}_{\{50,51\}}^{\text{pQCD}} = \frac{2}{9\pi^2 q^4 q_3^2}$$

  to be compared to

$$\hat{\Pi}_1^{\text{OPE}} = -\frac{2}{3\pi^2 q^2 q_3^2} \qquad \hat{\Pi}_{\{5,6\}}^{\text{OPE}} = -\frac{1}{6\pi^2 q^2 q_3^2}$$

$$\hat{\Pi}_{\{10,14\}}^{\text{OPE}} = -\hat{\Pi}_{\{17,30,50,51\}}^{\text{OPE}} = \frac{1}{6\pi^2 q^4 q_3^2}$$

- Longitudinal amplitudes exactly right
- Transversal ones seem to be off by factors $4/3$ or $2/3$, respectively
- Not even $\hat{\Pi}_1^{\text{OPE}}$ has the correct asymptotic behavior for $q_3^2 \to q^2$

  $\hookrightarrow$ MV model does not map correctly onto pQCD (not even in $\hat{\Pi}_1$)

Asymptotic and mixed regions from pQCD quark loop

- Quark loop gets $\hat{\Pi}^{\text{OPE}}_{1,2,3}$ right and the others "nearly"

  $\hookrightarrow$ obtain a rough estimate by extending the integration region

- **Chiral corrections** important below Λ_{QCD}, thus OPE constraint not applicable

  $\hookrightarrow$ where to **match between pseudoscalar poles and OPE**?

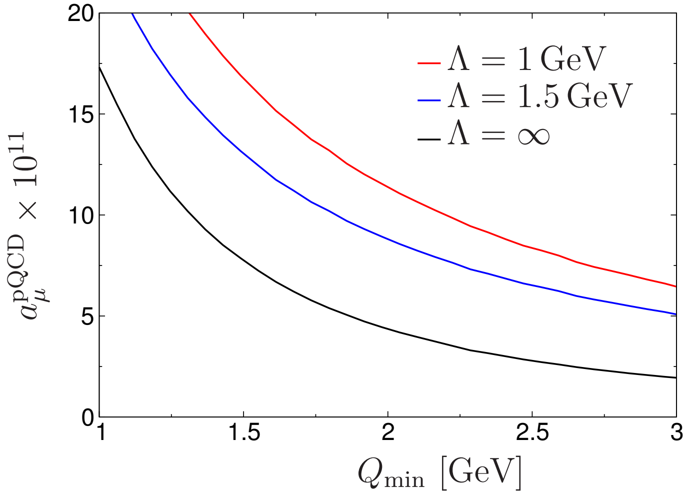
- Take that matching scale $\Lambda = 1$–$1.5\,\text{GeV}$ as a benchmark
- New integration region

$$\theta(Q_1 - Q_{\text{min}})\theta(Q_2 - Q_{\text{min}})\theta(Q_3 - Q_{\text{min}})$$
$$+\theta(Q_1 - Q_{\text{min}})\theta(Q_2 - Q_{\text{min}})\theta(Q_{\text{min}} - Q_3)\frac{Q_3^2}{Q_3^2 + \Lambda^2} + \text{crossed}$$

- **Main caveat**: a proper matching requires the consideration of other states between 1 and 2 GeV

  $\hookrightarrow$ depends on which part of the ellipsis can be captured dispersively!

Asymptotic and mixed regions from pQCD quark loop



- For $Q_{\mathrm{min}} \sim 2\,\mathrm{GeV}$, a contribution $\mathcal{O}(10 \times 10^{-11})$ seems plausible

Towards an OPE implementation based on hadronic states

- MV remark that the difference

$$\frac{F_{\pi^0\gamma^*\gamma^*}(q_3^2, 0) - F_{\pi^0\gamma^*\gamma^*}(0, 0)}{q_3^2 - M_{\pi^0}^2}$$

  could be generated by **excited states** in the same channel

- There is some amount of information about excited π, η, η' Klempt, Zaitsev 2007

  $\hookrightarrow$ can one make this idea work in practice? Colangelo, Hagelstein, Laub

- Ideally, this should
  - turn $1/q_3^4 \to 1/q_3^2$ by summing an infinite series
  - shift the weight of the OPE correction in $g-2$ integral to higher momenta
  - allow one to better understand the matching scales Λ, Q_{min}
  - yield a complementary estimate of the numerical impact of the mixed regions

- Large-N_c Regge models for higher resonances Ruiz Arriola, Broniowski 2006, 2010
- Key observation:

$$\frac{1}{4\pi^2 F_\pi}\sum_{n=0}^{\infty}\frac{M_\rho^2 M_\omega^2-8\pi^2F_\pi^2 q_3^2}{(q_3^2-M_\pi^2-n\sigma_\pi^2)(q_3^2-M_\rho^2-n\sigma_\rho^2)(q_3^2-M_\omega^2-n\sigma_\omega^2)}$$
$$=\frac{2F_\pi}{q_3^2}\frac{\sigma_\pi^2\sigma_\rho^2\log\frac{\sigma_\pi^2}{\sigma_\rho^2}+\sigma_\pi^2\sigma_\omega^2\log\frac{\sigma_\omega^2}{\sigma_\pi^2}+\sigma_\rho^2\sigma_\omega^2\log\frac{\sigma_\rho^2}{\sigma_\omega^2}}{(\sigma_\pi^2-\sigma_\rho^2)(\sigma_\pi^2-\sigma_\omega^2)(\sigma_\rho^2-\sigma_\omega^2)}+\mathcal{O}\left(q_3^{-4}\right)$$

- Phenomenological analysis in progress Colangelo, Hagelstein, Laub

Conclusions

- **Short-distance constraints:**
  - Asymptotic region: pQCD quark loop
  - Mixed regions: OPE and VVA non-renormalization theorems
- Towards a practical implementation:
  - Details of the matching important (e.g., sensitivity to Q_{min} and Λ)
  - Will depend on how well other states between 1 and 2 GeV can be accounted for
  - Our estimates indicate $\mathcal{O}(10 \times 10^{-11})$ from asymptotics, **less than MV model** implies
- Outlook
  - OPE constraints for $\hat{\Pi}_{i>3}$
  - Phenomenology of excited pseudoscalars
  - Improved matching from interplay with other states in the ellipsis

OPE and Brodsky–Lepage limit

- Separation into **hard scattering kernel** and **meson distribution amplitudes**
  Brodsky, Lepage 1979, 1980, 1981
- Simplest case: pion transition form factor

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)=-\frac{2e^2F_\pi}{3}\int_0^1\mathrm{d}x\,\frac{\phi_\pi(x)}{xq_1^2+(1-x)q_2^2}$$

- Relation to OPE Manohar 1990: only strictly justified for $\omega=2\frac{q_1^2-q_2^2}{q_1^2+q_2^2}<1$
- **Brodsky–Lepage limit**

$$F_{\pi^0\gamma^*\gamma^*}(-Q^2,0)=\frac{2e^2F_\pi}{Q^2}$$

  amounts to resummation of OPE
- Constraints on $\gamma\gamma\to\pi\pi$ Brodsky, Lepage 1981 useful for asymptotic behavior?
- OPE for $\gamma^*\gamma^*\to\pi\pi$ Bijnens, Relefors 2016

$$\lim_{Q^2\to\infty}A\big(\gamma^*(q_1=Q+k)\gamma^*(q_2=-Q+k)\to\pi(p_1)\pi(p_2)\big)\sim\frac{1}{Q^2}$$