# The pion transition form factor from lattice QCD

Antoine Gérardin

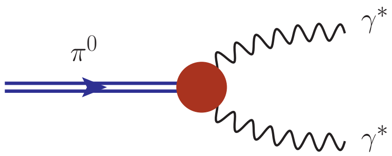
Institut für Kernphysik, Johannes Gutenberg-Universität Mainz

In collaboration with Harvey Meyer and Andreas Nyffeler

June 18, 2018 - $g-2$ Workshop - Mainz

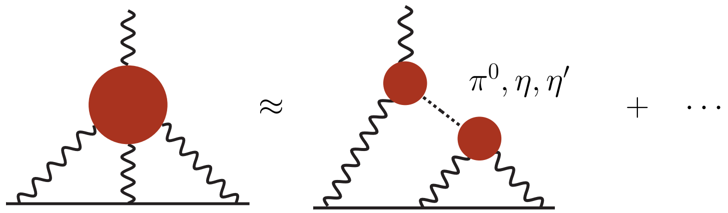
## Motivations

- Interaction between a neutral pion and two off-shell photons




- Tests of the dynamics of QCD
  - → Chiral anomaly
  - → Singly-virtual : Brodsky-Lepage behaviour, pion distribution amplitude
  - → Doubly-virtual : test of the operator product expansion (OPE)

- Pion-pole contribution to the hadronic light-by-light scattering contribution to the $(g-2)_\mu$
  - → dominant contribution in model calculations
  - → input to the dispersive approach [Colangelo et al. '14, '15] [Pauk, Vanderhaeghen, '15, '16]




Frequent estimates :

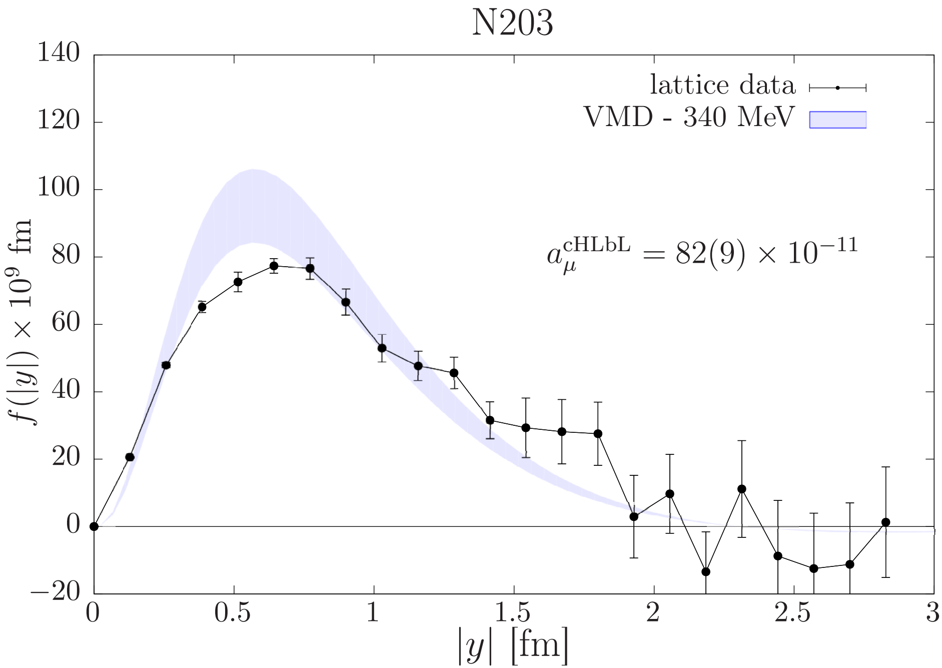
$$a_\mu^{\text{HLbL}}(\pi^0) \approx 63.0 \times 10^{-11}$$
$$a_\mu^{\text{HLbL}}(\eta) \approx 14.5 \times 10^{-11}$$
$$a_\mu^{\text{HLbL}}(\eta') \approx 12.5 \times 10^{-11}$$

✦ most results were based on model calculations

✦ new results based on a dispersive analysis [Talk by Bai Long Hoid tomorrow]

# Motivations : HLbL calculation on the lattice

- Integrand for the fully-connected contribution to $a_\mu^{\text{HLbL}}$ ($m_\pi = 340$ MeV)




- The signal deteriorates at large values of $|y|$ where the pion-pole is expected to be dominant
- The pion-pole contribution could lead to large finite-size effects in the lattice calculation

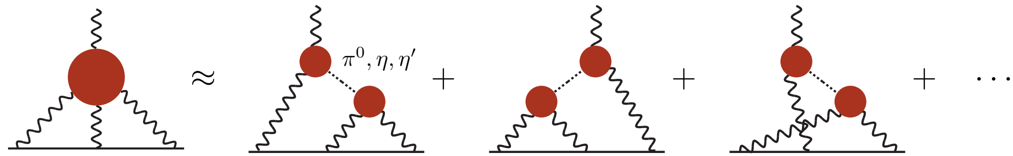
[See H. Meyer Talk]

→ Use the lattice results for the TFF to estimate the tail of the integrand

# Motivations : the pion-pole contribution

[Jegerlehner & Nyffeler '09]

$\tau = \cos(\theta)$
$Q_1 \cdot Q_2 = Q_1 Q_2 \cos(\theta)$

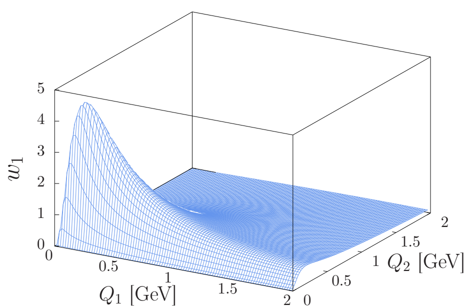


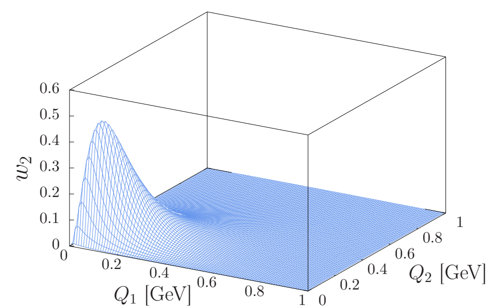

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \; + $$
$$w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

→ Product of one single-virtual and one double-virtual transition form factors (spacelike virtualities)

→ $w_{1,2}(Q_1, Q_2, \tau)$ are model-independent weight functions

→ The weight functions are concentrated at small momenta below 1 GeV (here for $\tau = -0.5$)




↪ Need the pion TFF for arbitrary spacelike virtualities in the momentum range $[0-3]$ GeV$^2$

## The pion transition form factor : experimental status

- Adler-Bell-Jackiw (ABJ) anomaly in the chiral limit : $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \dfrac{1}{4\pi^2 F_\pi}$

  ↪ Compatible with the PrimEx results [PrimEx '10] (normalization measured at 1.4% level)

  ↪ New results should increase the precision by a factor of two [Talk by PrimEx-II tomorrow]

- The single-virtual transition form factor has been measured (CELLO, CLEO, BaBar, Belle)

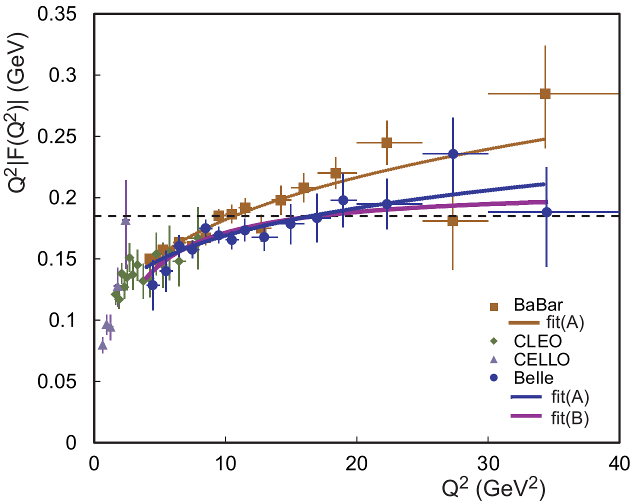
  → Belle data agree with Brodsky-Lepage ($\sim 1/Q^2$)

  → Belle and Babar results are quite different

  → No measurement for $Q < 0.8~\mathrm{GeV}$ yet (dominant contribution)

  → But new results soon by BESIII [Talk by C. Redmer]

[Belle '12]




- No result yet for the double-virtual transition form factor

  ↪ very challenging (small cross section)

- Short-distance constraints

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,0) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{Q^2}$$

Brodsky-Lepage behavior

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,-Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{3Q^2}$$

OPE prediction

# Lattice calculation

## Lattice calculation

In Minkowski space-time :

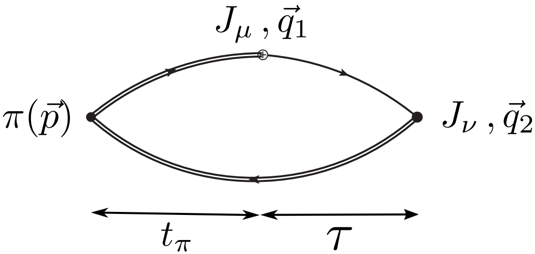
$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\; \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) = i \int \mathrm{d}^4x\, e^{iq_1 x}\, \langle 0|T\{J_\mu(x) J_\nu(0)\}|\pi^0(p)\rangle = M_{\mu\nu}(q_1^2, q_2^2)$$

- $J_\mu(x)$ hadronic component of the electromagnetic current : $J_\mu(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) + \dots$

In Euclidean space-time : [Ji & Jung '01] [Cohen et al. '08] [Feng et al. '12]

$$M^E_{\mu\nu}(q_1^2, q_2^2) = -\int \mathrm{d}\tau\, e^{\omega_1 \tau} \int \mathrm{d}^3z\, e^{-i\vec{q}_1\vec{z}}\, \langle 0|T\left\{J_\mu(\vec{z},\tau) J_\nu(\vec{0},0)\right\}|\pi(p)\rangle$$

- Analytical continuation : $q_1 = (\omega_1, \vec{q}_1)$
- We must kept $q_{1,2}^2 < M_V^2 = \min(M_\rho^2, 4m_\pi^2)$ to avoid poles




The main object to compute is the Euclidean three-point correlation function :

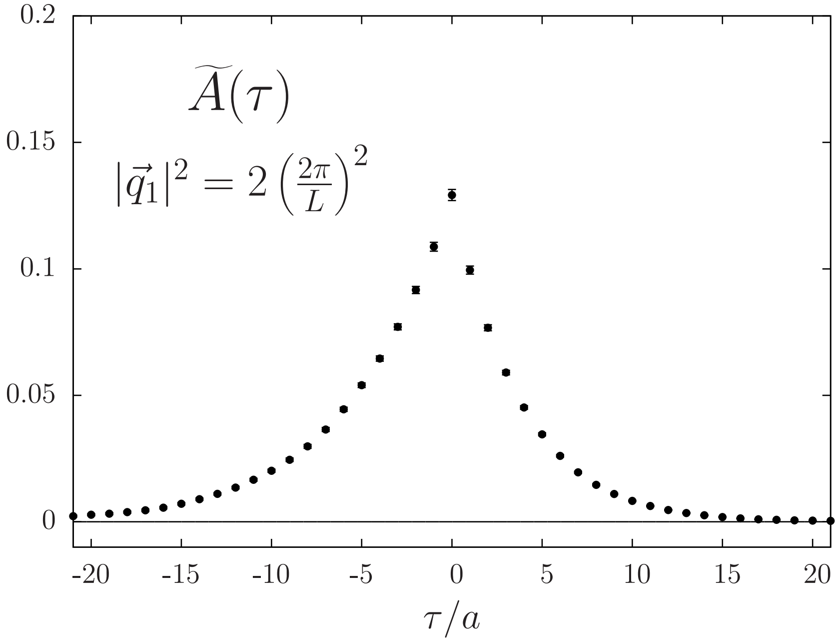
$$C^{(3)}_{\mu\nu}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) = \sum_{\vec{x},\vec{z}} \langle T\left\{J_\nu(\vec{0}, t_f) J_\mu(\vec{z}, t_i) P(\vec{x}, t_0)\right\}\rangle\, e^{i\vec{p}\vec{x}}\, e^{-i\vec{q}_1\vec{z}}$$

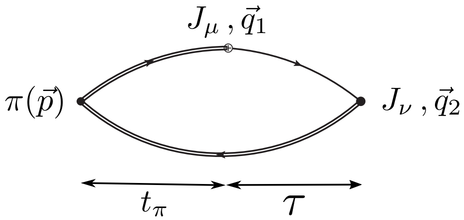
## Shape of the integrand $(a = 0.064 \text{ fm and } m_\pi = 270 \text{ MeV})$

$$\mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) \propto \frac{2E_\pi}{Z_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \widetilde{A}_{\mu\nu}(\tau) \, e^{\omega_1 \tau}$$

$$A_{\mu\nu}(\tau) = \lim_{t_\pi \to \infty} C_{\mu\nu}(\tau, t_\pi) e^{E_\pi t_\pi}$$

$$\widetilde{A}_{\mu\nu}(\tau) = \begin{cases} A_{\mu\nu}(\tau) & \tau > 0 \\ A_{\mu\nu}(\tau) \, e^{-E_\pi \tau} & \tau < 0 \end{cases}$$




On the lattice :

- Discrete sum over lattice points :

$$\int \mathrm{d}\tau \to a \sum_\tau$$

- Finite size of the box :

$$|\tau| \leq \tau_{\max} \neq \infty$$

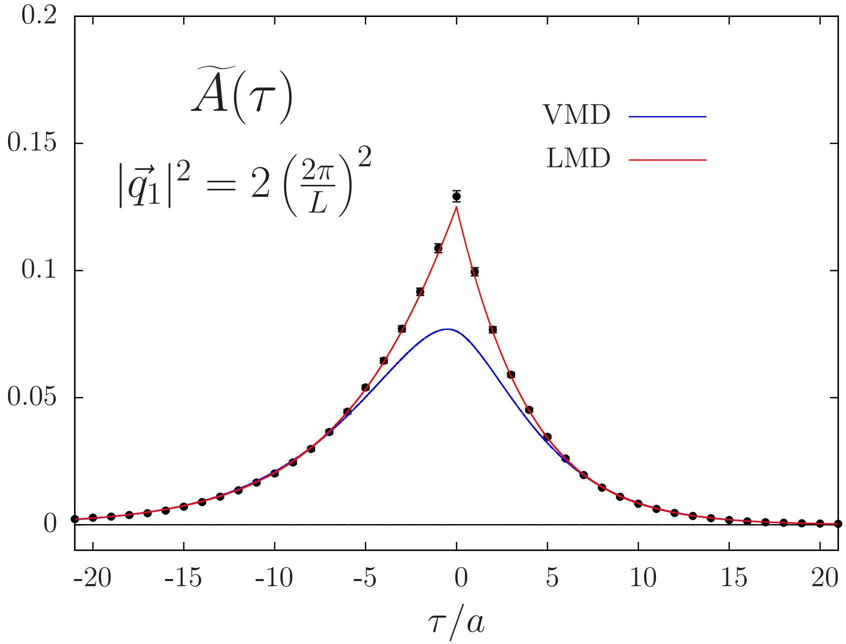
- Signal deteriorates at large $|\tau|$ : $e^{\omega_1 \tau}$

## Shape of the integrand ($a = 0.064$ fm and $m_\pi = 270$ MeV)

- The vector meson dominance (VMD) model is expected to give a good description of the data at large $\tau$

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)} \quad \rightarrow \quad \widetilde{A}(\tau) = \ \ldots \quad \text{(known analytical expression)}$$

- Fit the data at large $\tau$ and use the result of the fit for $\tau > \tau_c \gtrsim 1.3$ fm



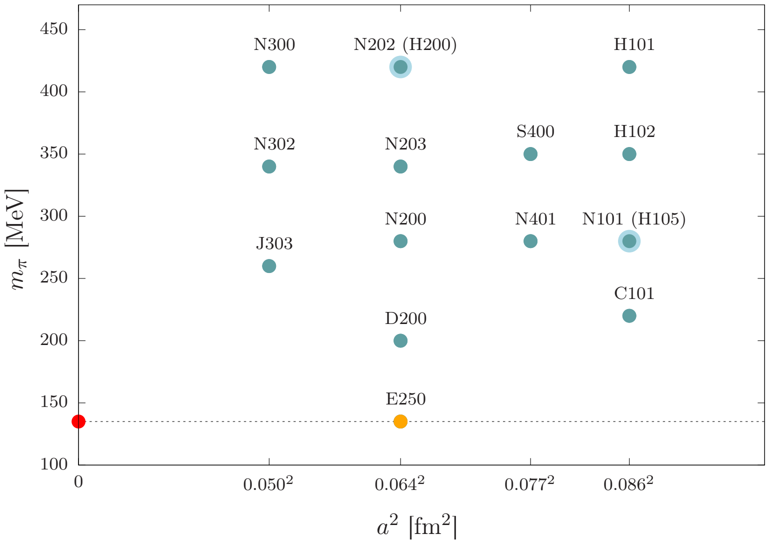

- Check the dependance on the model using LMD rather than VMD :

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- The difference between the two models is included in the systematic error.

# Results

# Improvements compared to our previous study with $N_f = 2$




CLS ensembles

- $N_f = 2+1$ : dynamical strange quark
- Ensembles with different volumes
  $\hookrightarrow$ dedicated study of finite-size effects
- Chiral & continuum extrapolations under control
  $\hookrightarrow$ we plan to include a physical pion mass lattice

- Add full $\mathcal{O}(a)$-improvement of the vector current :
  $\hookrightarrow$ Continuum extrapolation $\propto a^2$
  $\hookrightarrow$ Two discretizations of the vector current are used : combined continuum extrapolation

$$J_\mu^l(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$$
$$J_\mu^c(x) = \frac{1}{2a}\left(\overline{\psi}(x+a\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger(x)\psi(x) - \overline{\psi}(x)(1-\gamma_\mu)U_\mu(x)\psi(x+a\hat{\mu})\right)$$

## Kinematical reach : add a new moving frame

- Finite volume : discrete spatial momenta $\vec{q} = 2\pi/L\vec{n}$ ($L$ = size of the box, $q_1 = (\omega_1, \vec{q}_1)$)

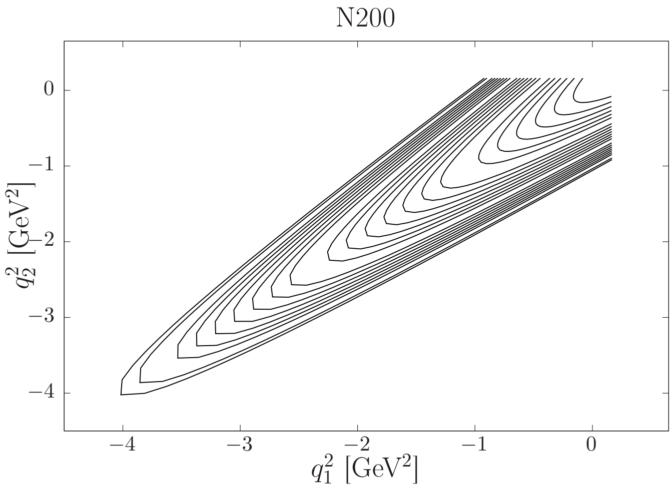
$$q_1^2 = \omega_1^2 - |\vec{q}_1|^2$$
$$q_2^2 = (E_\pi - \omega_1)^2 - |\vec{q}_2|^2$$

## Kinematical reach : add a new moving frame

- **Finite volume** : discrete spatial momenta $\vec{q} = 2\pi/L\vec{n}$ ($L =$ size of the box, $q_1 = (\omega_1, \vec{q}_1)$)

$$q_1^2 = \omega_1^2 - |\vec{q}_1|^2$$
$$q_2^2 = (E_\pi - \omega_1)^2 - |\vec{q}_2|^2$$

Pion rest frame ($\vec{p} = \vec{0}$)




$\hookrightarrow$ difficult to reach large $Q^2$ for $\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2, 0)$

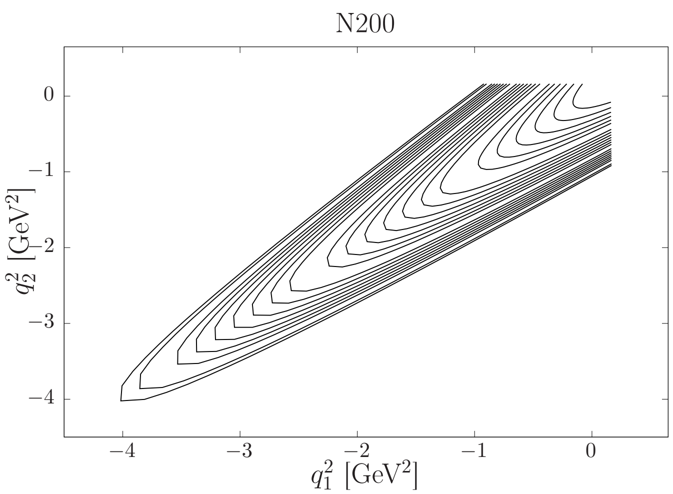
$\hookrightarrow$ add a new frame with $\vec{p} \neq \vec{0}$ (pion momentum)

# Kinematical reach : add a new moving frame

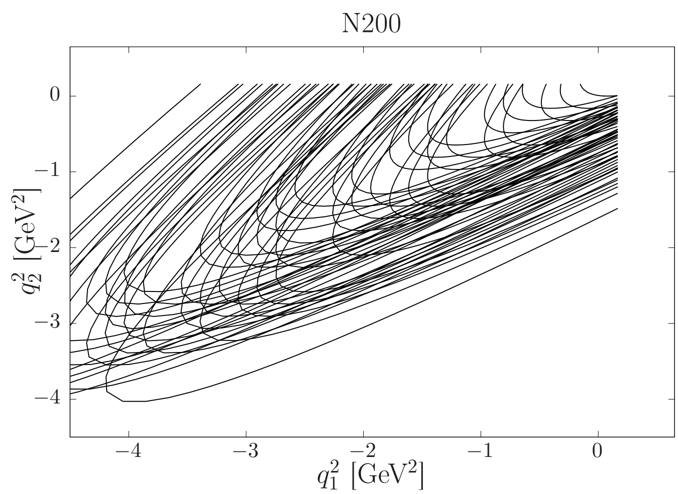
- Finite volume : discrete spatial momenta $\vec{q} = 2\pi/L\vec{n}$ $(L = \text{size of the box}, q_1 = (\omega_1, \vec{q}_1))$

$$q_1^2 = \omega_1^2 - |\vec{q}_1|^2$$
$$q_2^2 = (E_\pi - \omega_1)^2 - |\vec{q}_2|^2$$

Pion rest frame $(\vec{p} = \vec{0})$




Moving frame $(\vec{p} = 2\pi/L\vec{z})$



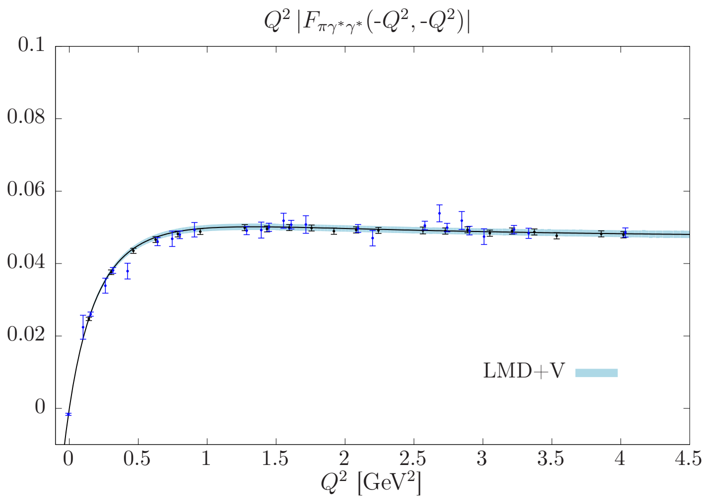

- We have computed all spatial momenta $\vec{q}_1 = (q_{1,x},\ q_{1,y},\ q_{1,z})$ to cover the plane $0 < Q_1^2, Q_2^2 < 3\ \text{GeV}^2$

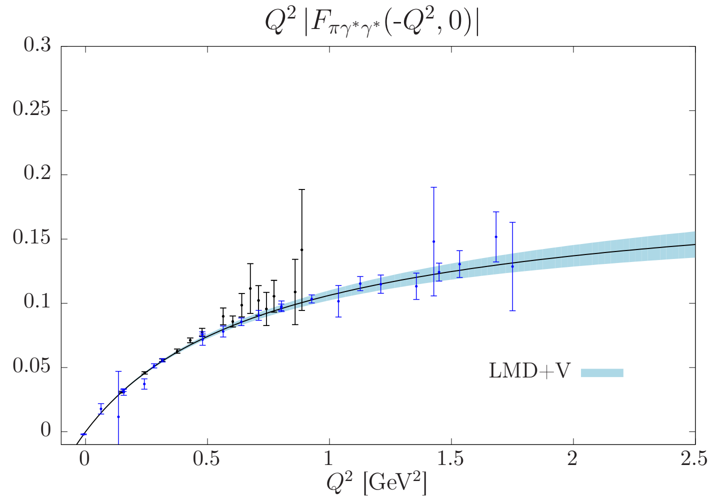
  $\hookrightarrow$ important to reduce statistical noise (multiplicity)

## Results : $m_\pi \approx 280$ MeV at $a \approx 0.065$ fm

- Legend : black points : pion rest frame
  blue points : moving frame

(Preliminary)

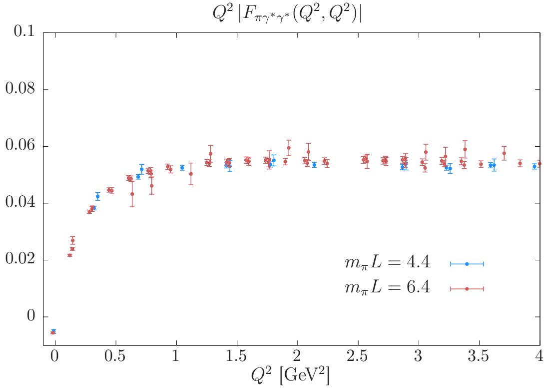


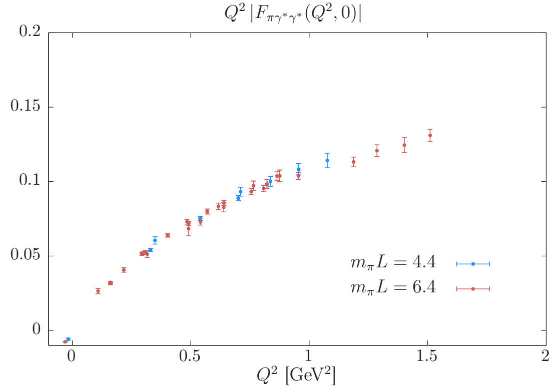




- improved statistical precision compared to our published results with $N_f = 2$
- new data points with $\vec{p} \neq \vec{0}$ are valuable
- data with $\vec{p} = \vec{0}$ and $\vec{p} \neq \vec{0}$ are in very good agreement

## Systematic uncertainties

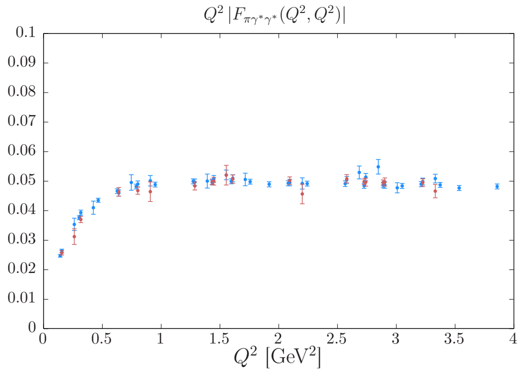
- Finite-volume corrections ($a = 0.065$ fm and $m_\pi = 340$ MeV)




$\rightarrow$ corresponds to our smaller volume : FSE negligible at our level of precision

- Hypercubic artifacts :




➠ rotational symmetry is broken on the lattice

➠ e.g. : $\vec{q}_1 = (3,0,0)$ and $\vec{q}_1 = (2,2,1)$ : same $|q_1|^2$

➠ might be affected by different discretization effects

$\hookrightarrow$ compatible results within error bars

$\hookrightarrow$ average over all points with same $(|q_1|^2, |q_2|^2)$

## Extrapolation to the physical point

1) Comparison with phenomenological models :

- VMD and LMD models (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2+q_2^2)}{(M_V^2-q_1^2)(M_V^2-q_2^2)}$$

- LMD+V model [Knecht & Nyffeler '01]

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\,(q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2-q_1^2)(M_{V_2}^2-q_1^2)(M_{V_1}^2-q_2^2)(M_{V_2}^2-q_2^2)}$$

→ Ansatz for the fit parameters : $\tilde{p}(a,m_\pi) = p + \gamma_1\, a^2 + \gamma_2\, m_\pi^2$

## Extrapolation to the physical point

1) Comparison with phenomenological models :

- VMD and LMD models (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- LMD+V model [Knecht & Nyffeler '01]

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2 + q_2^2) + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2 + q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_1^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}$$

→ Ansatz for the fit parameters : $\tilde{p}(a, m_\pi) = p + \gamma_1\, a^2 + \gamma_2\, m_\pi^2$

2) Canterbury approximants (generalization of Pade approximants) [see P. Sanchez-Puertas talk]

→ disadvantage : many fit parameters

$$C_2^1(Q_1^2, Q_2^2) = \frac{a_{00} + a_{01}(Q_1^2 + Q_2^2) + a_{11} Q_1^2 Q_2^2}{1 + b_{01}(Q_1^2 + Q_2^2) + b_{11} Q_1^2 Q_2^2 + b_{20}(Q_1^4 + Q_2^4) + b_{21} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2) + b_{22} Q_1^4 Q_2^4}$$

# Extrapolation to the physical point

1) Comparison with phenomenological models :

- VMD and LMD models (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- LMD+V model [Knecht & Nyffeler '01]

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2(q_1^2 + q_2^2) + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2 + q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_1^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}$$

→ Ansatz for the fit parameters : $\tilde{p}(a, m_\pi) = p + \gamma_1\, a^2 + \gamma_2\, m_\pi^2$

2) Canterbury approximants (generalization of Pade approximants) [see P. Sanchez-Puertas talk]

→ disadvantage : many fit parameters

$$C_2^1(Q_1^2, Q_2^2) = \frac{a_{00} + a_{01}(Q_1^2 + Q_2^2) + a_{11} Q_1^2 Q_2^2}{1 + b_{01}(Q_1^2 + Q_2^2) + b_{11} Q_1^2 Q_2^2 + b_{20}(Q_1^4 + Q_2^4) + b_{21} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2) + \cancel{b_{22} Q_1^4 Q_2^4}}$$

# Extrapolation to the physical point

1) Comparison with phenomenological models :

- VMD and LMD models (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\rm LMD}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2+q_2^2)}{(M_V^2-q_1^2)(M_V^2-q_2^2)}$$

- LMD+V model [Knecht & Nyffeler '01]

$$\mathcal{F}^{\rm LMD+V}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2-q_1^2)(M_{V_2}^2-q_1^2)(M_{V_1}^2-q_2^2)(M_{V_2}^2-q_2^2)}$$

→ Ansatz for the fit parameters : $\tilde{p}(a, m_\pi) = p + \gamma_1\, a^2 + \gamma_2\, m_\pi^2$

2) Canterbury approximants

3) Assume the following double $z$-expansion for space-like momenta : $F(Q_1^2, Q_2^2) = \sum_{n,m=0}^{N} c_{nm}\, z_1^n\, z_2^n$

where $c_{nm} = c_{mn}$ and with $z_i = \dfrac{\sqrt{t_c + Q_i^2} - \sqrt{t_c - t_0}}{\sqrt{t_c + Q_i^2} + \sqrt{t_c - t_0}}$ , $t_0 = t_c\left(1 - \sqrt{1 + Q_{\rm max}^2/t_c}\right)$

→ $t_c = 4m_\pi^2$

→ $t_0$ : reduces the maximum value of $|z_i|$ in the range $[0, Q_{\rm max}^2]$

→ VMD or LMD models : coefficients are known

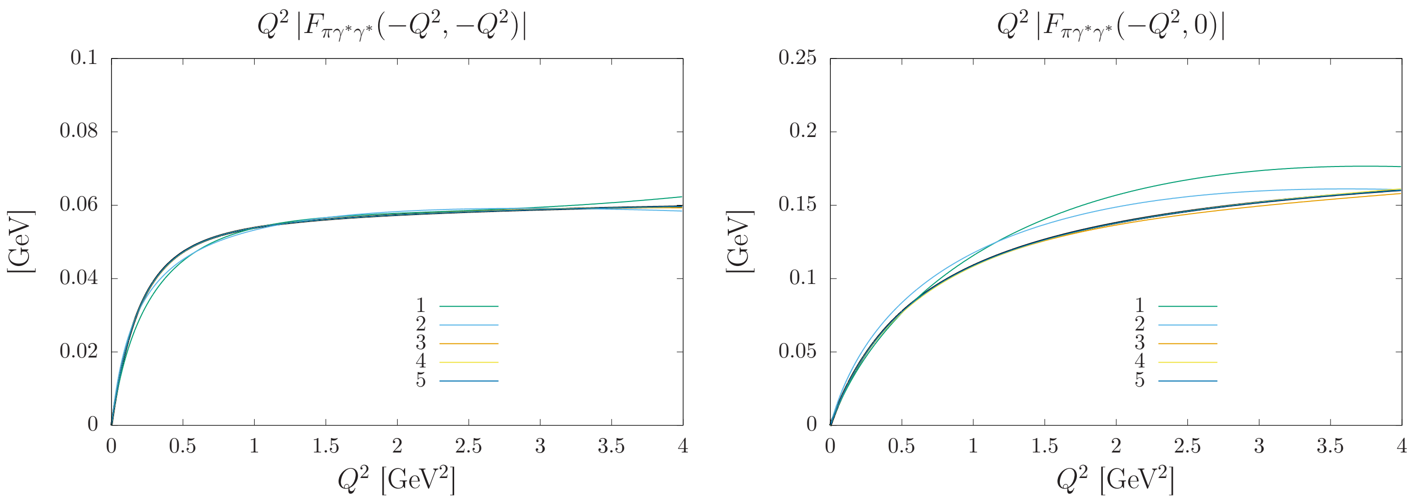
→ Fit using : $\tilde{c}_{mn}(a, m_\pi) = c_{mn} + \gamma_{mn}^{(1)}\, a^2 + \gamma_{mn}^{(2)}\, m_\pi^2$

# $z$-expansion inspired fits : test of the method

- Fit the LMD or LMD+V model using a double $z$-expansion

  $\hookrightarrow$ Estimate the systematic error from the truncation of the sum (finite $N$)

- Results for the LMD+V model with $Q^2_{\max} = 4\ \mathrm{GeV}^2$ :



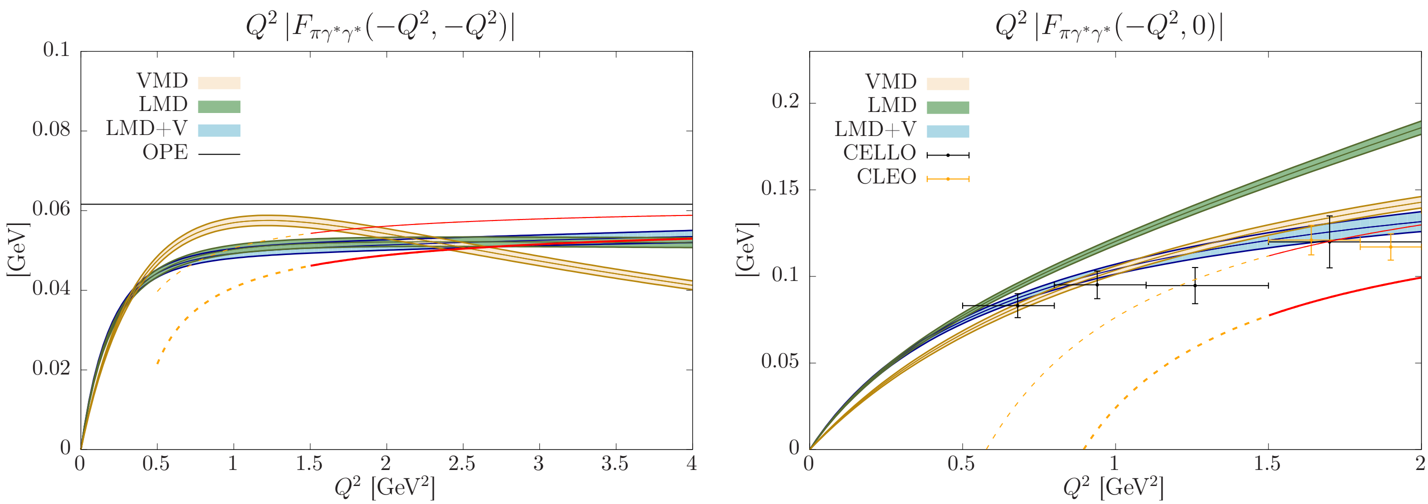

$\hookrightarrow$ black curve = exact results

$\hookrightarrow$ With $Q^2_{\max} = 4\ \mathrm{GeV}^2$, $N = 3$ is already sufficient to get a precision bellow 1 % for the TFF

$\hookrightarrow$ The anomaly is recovered with a precision better than 2 %

# Results at the physical point

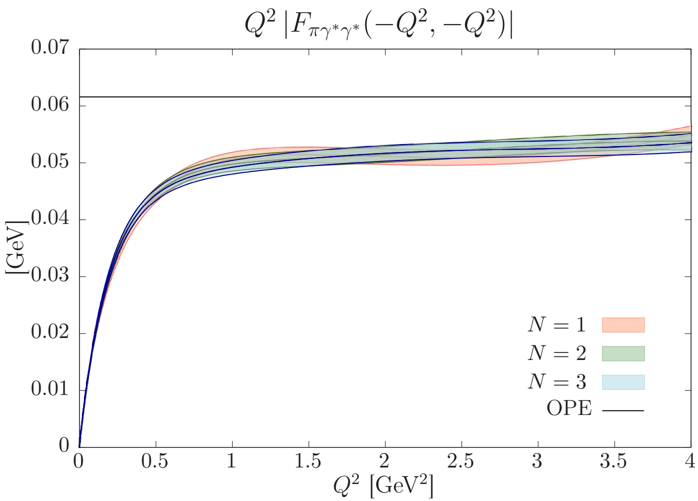
## Phenomenological models (Preliminary)

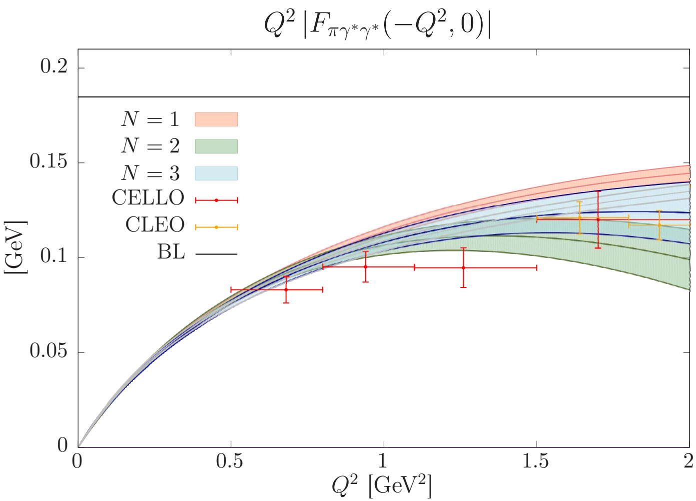



- VMD model : bad $\chi^2$ $\rightarrow$ wrong asymptotic behavior in the double-virtual case
- LMD+V : good $\chi^2$. Results also in good agreement with experimental data
- $\alpha^{\mathrm{LMD+V}} = 268(7)\ \mathrm{GeV}^{-1}$. Other fit parameters also in good agreement with phenomenology.

# Results at the physical point

Double $z-$expansion (Preliminary)



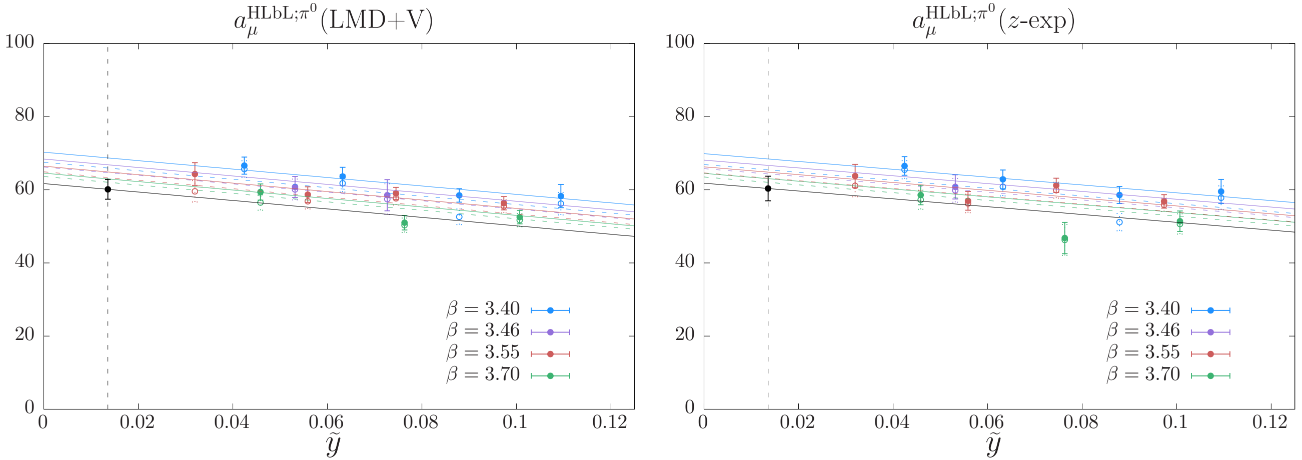




- $\alpha = 0.278(14)\ \mathrm{GeV}^{-1}$ : compatible with the PRIMEX experiment (precision $\approx\ 5\%$)
- Results in good agreement with experimental data

# The pion-pole contribution (preliminary results)

[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \; + $$
$$w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$




LMD+V model : $a_\mu^{\mathrm{HLbL};\pi^0} = (60.1 \pm 2.7) \times 10^{-11}$

Double $z-$expansion : $a_\mu^{\mathrm{HLbL};\pi^0} = (60.4 \pm 3.4) \times 10^{-11}$

$\rightarrow$ Compatible with the $N_f = 2$ results but with smaller errors [Gérardin et al. '16]

$\rightarrow$ Compatible with the dispersive result $a_\mu^{\mathrm{HLbL};\pi^0} = 62.6^{+3.0}_{-2.5} \times 10^{-11}$ [Hoferichter et al. '18]

## Conclusion

- We can compute the TFF for arbitrary spacelike virtualities at low energy

- Several major improvements compared to our previous study

  → Full $\mathcal{O}(a)$-improvement to reduce discretisation effects + dynamical strange quark

  → New kinematics included, higher statistics

- We reproduce the anomaly constraint with a precision of $5\ \%$

- Dedicated study of systematic effects

  → Finite-volume effects are small

  → Hypercubic artifacts seem to be small

  → Disconnected contributions were shown to be small (still to be done with $N_f = 2+1$)

- Result for the pion-pole contribution (Preliminary !)

$$a_\mu^{\mathrm{HLbL};\pi^0} = (60.4 \pm 3.4) \times 10^{-11}$$

- Future :

  → use our results to constraint the tail of the integrand in the HLbL calculation

  → estimation of finite-size effects

  → add physical pion-mass ensemble