Scattering and resonances from finite-volume calculations

Maxwell T. Hansen

Institut für Kernphysik and Helmholtz-Institut Mainz
Johannes Gutenberg Universität, Mainz, Germany

In collaboration with Raùl Briceño *and* Steve Sharpe

Matter and Universe Day

December 12th, 2016

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Helmholtz-Institut Mainz

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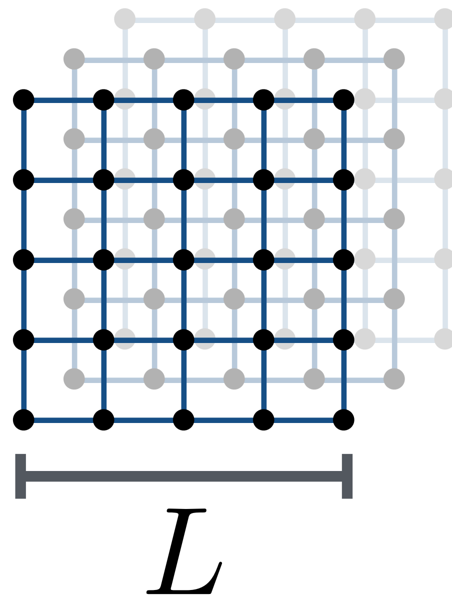
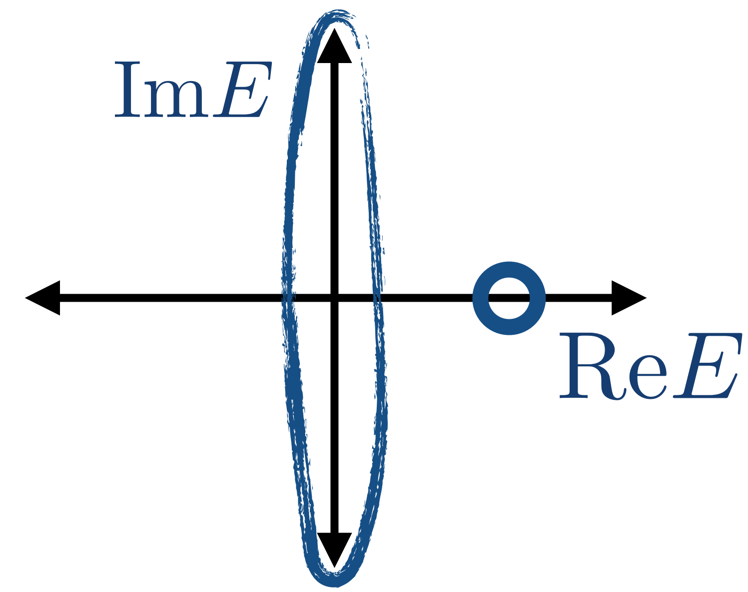
In LQCD it is *not* possible to directly calculate scattering amplitudes

finite volume



Euclidean momenta



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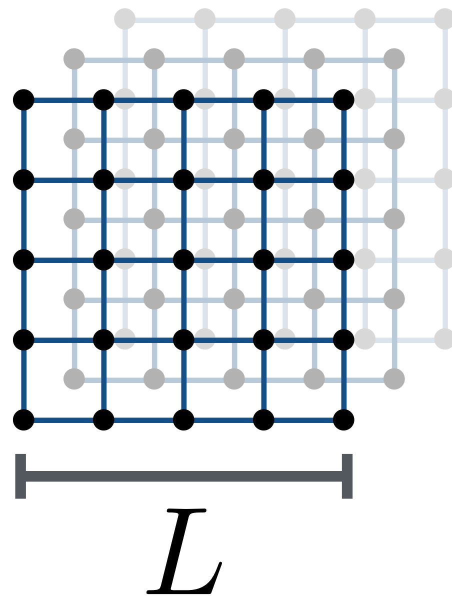

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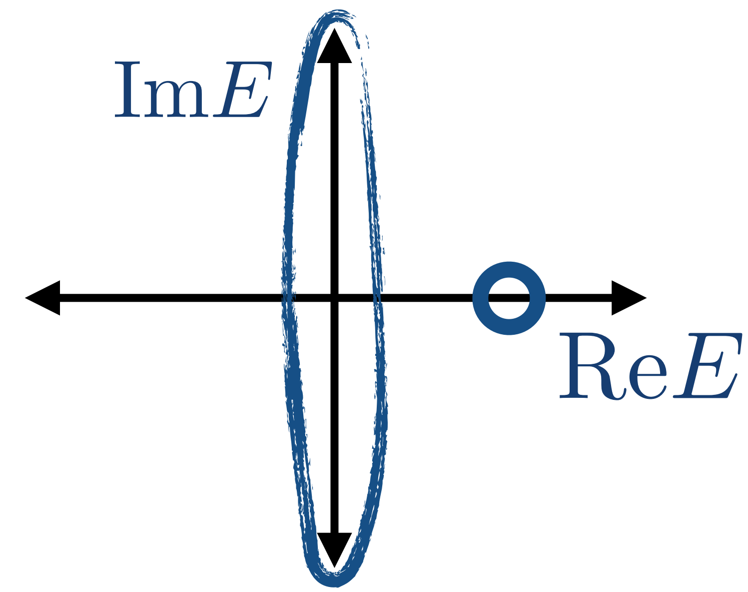
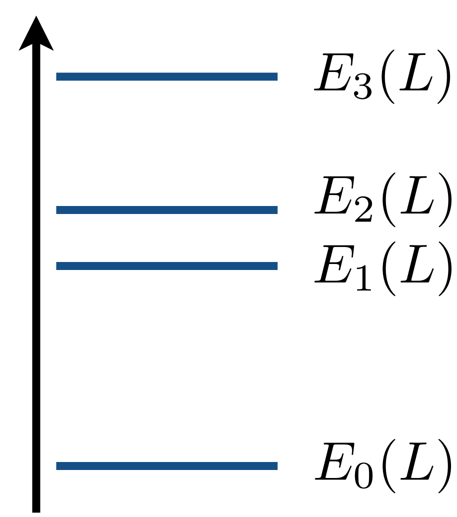
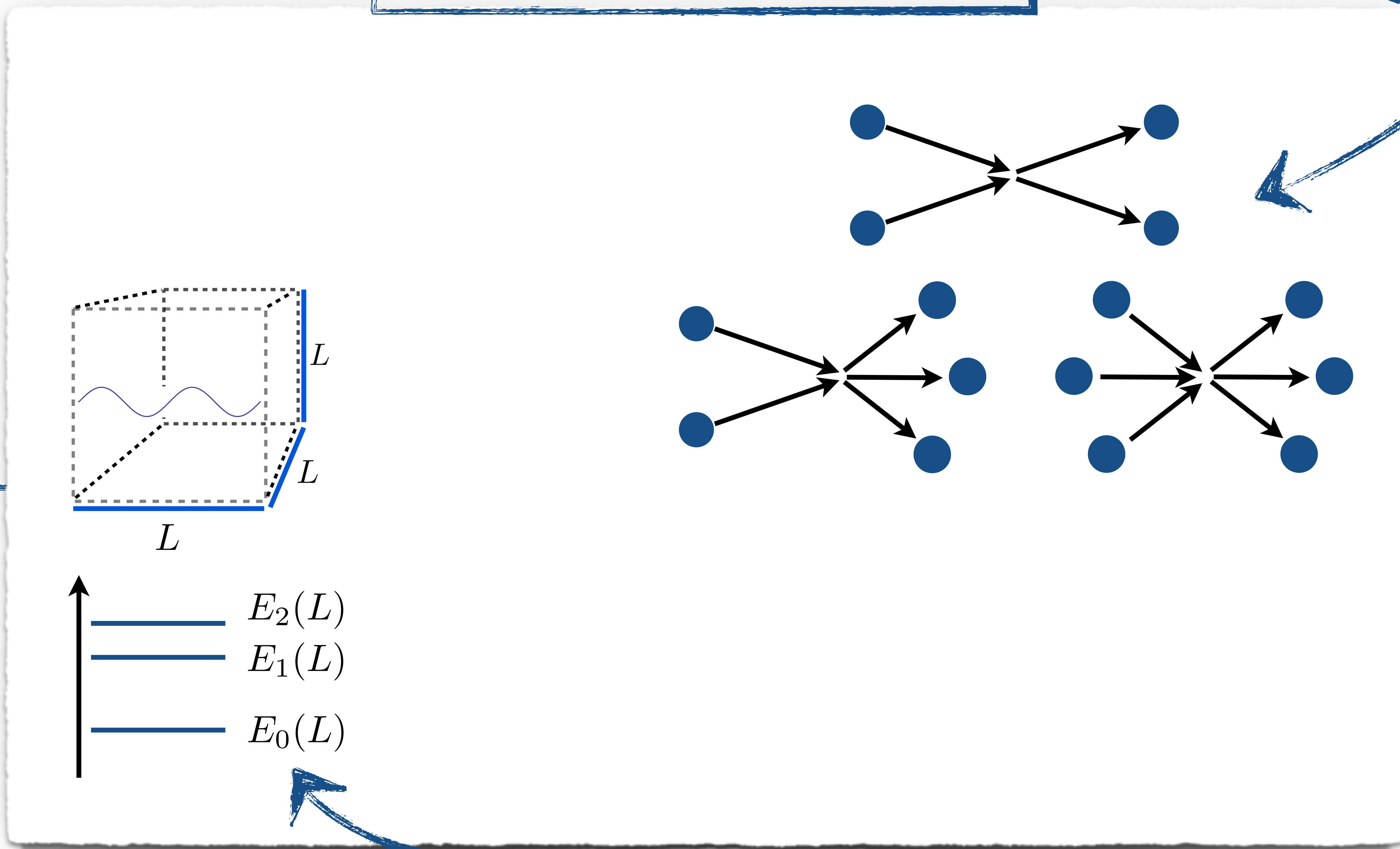


$$C(\tau) = \langle \mathcal{O}(\tau)\mathcal{O}^\dagger(0)\rangle$$
$$= \sum_n c_n \exp[-E_n(L)\tau]$$



But it is possible to calculate finite-volume energies of multi-particle systems

In LQCD it is *not* possible to directly calculate **scattering amplitudes**



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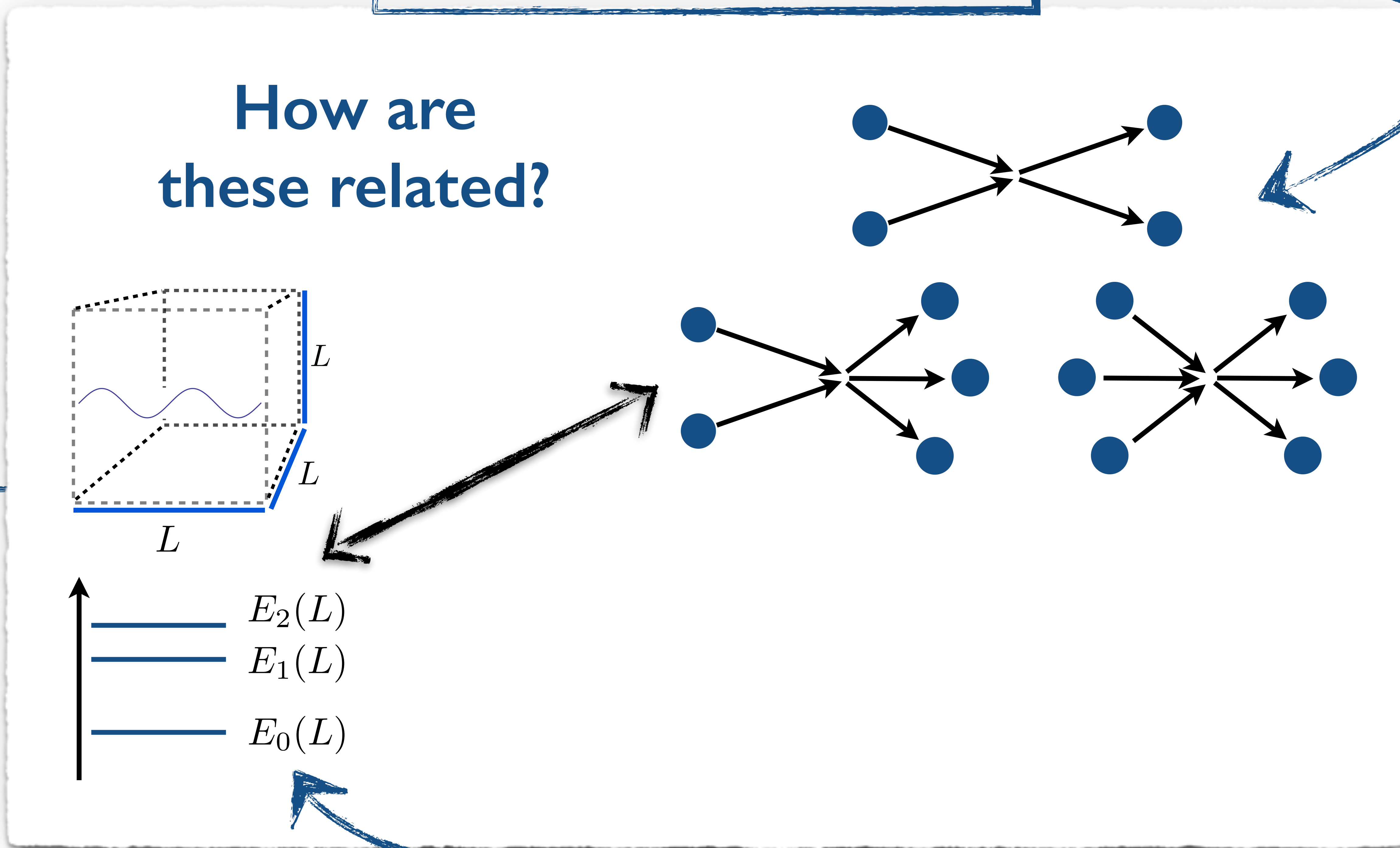

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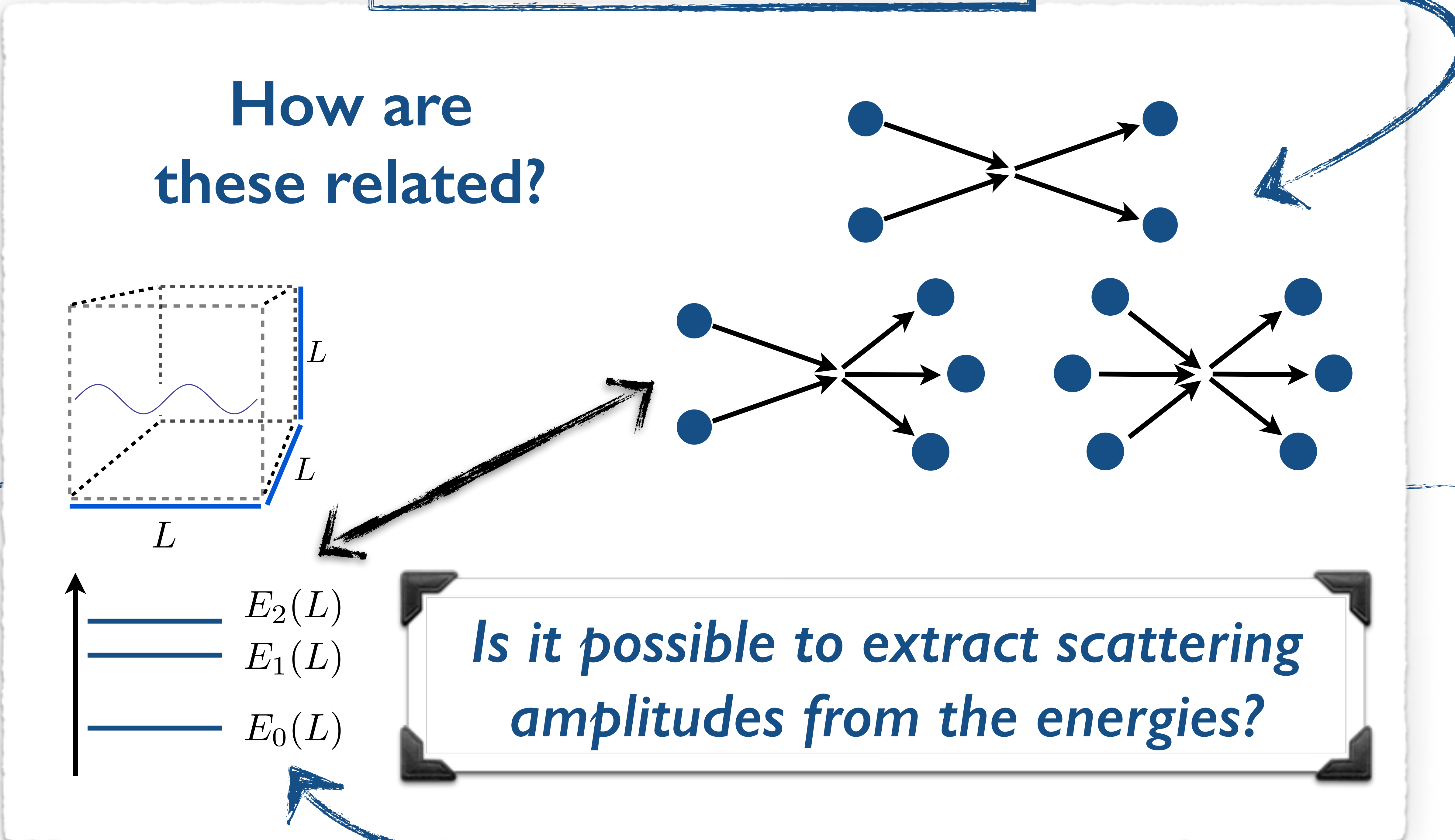
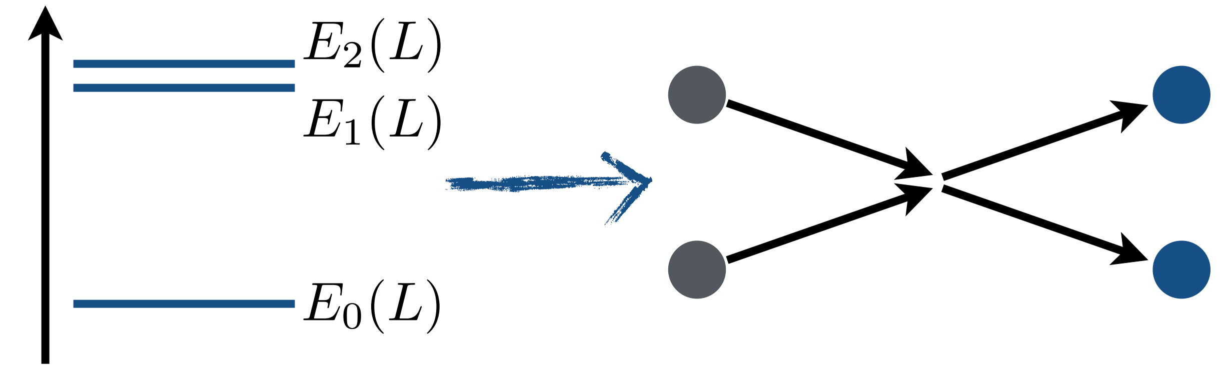


But it is possible to calculate finite-volume energies of multi-particle systems

In the case of two-to-two scattering
Lüscher's formalism + extensions have answered these questions

Lüscher (1991), Rummukainen and Gottlieb (1995), Beane et. al. (2005), Kim, Sachrajda, and Sharpe (2005), Christ, Kim, Yamazaki (2005), Bernard et. al. (2011), MTH and Sharpe (2012), Briceño and Davoudi (2013), Li and Liu (2013), Briceño (2014)



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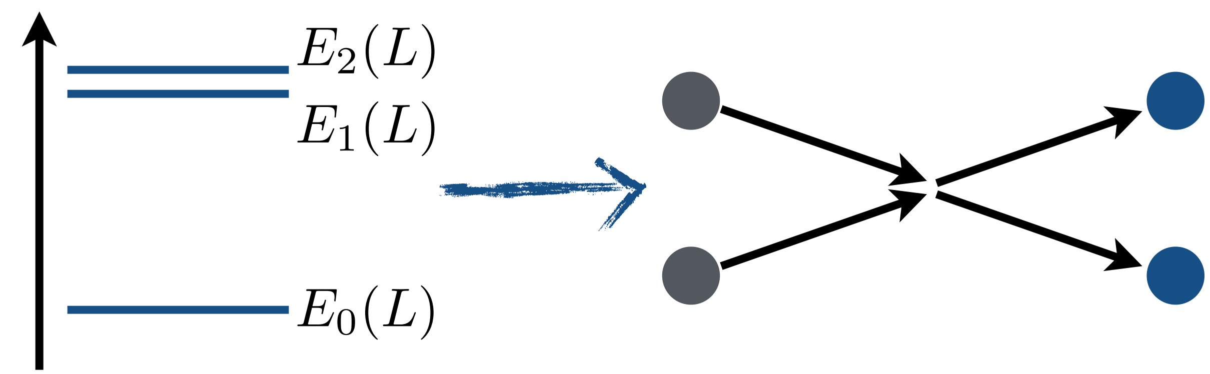

All results can be summarized by a generalized *quantization condition*

$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude — known geometric function

Both are matrices in angular momentum, spin and channel space

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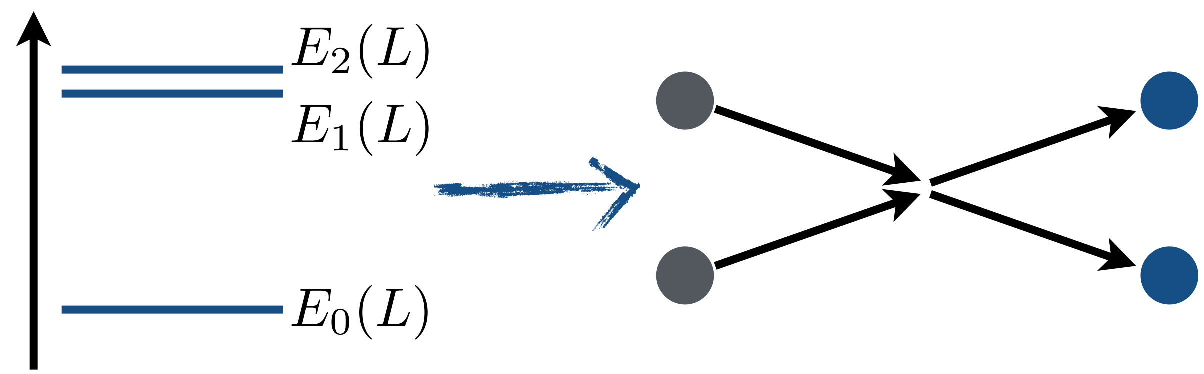

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Requires total energy to be below three-particle production threshold

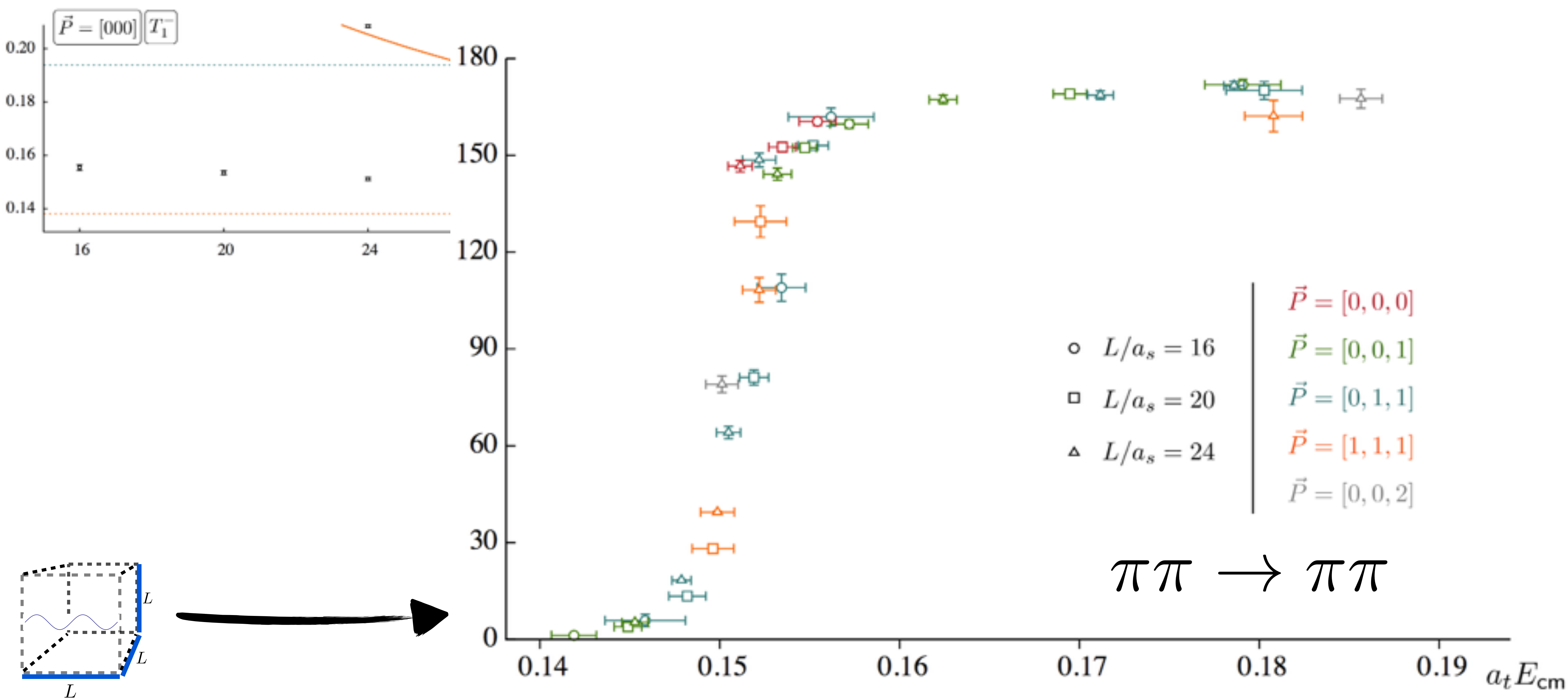
Ignores (drops) exponentially suppressed corrections $e^{-M_\pi L}$

Only useful if one truncates angular momentum space

Using the result: Simplest case is a single-channel

e.g. for pions in a p-wave the relation reduces to

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



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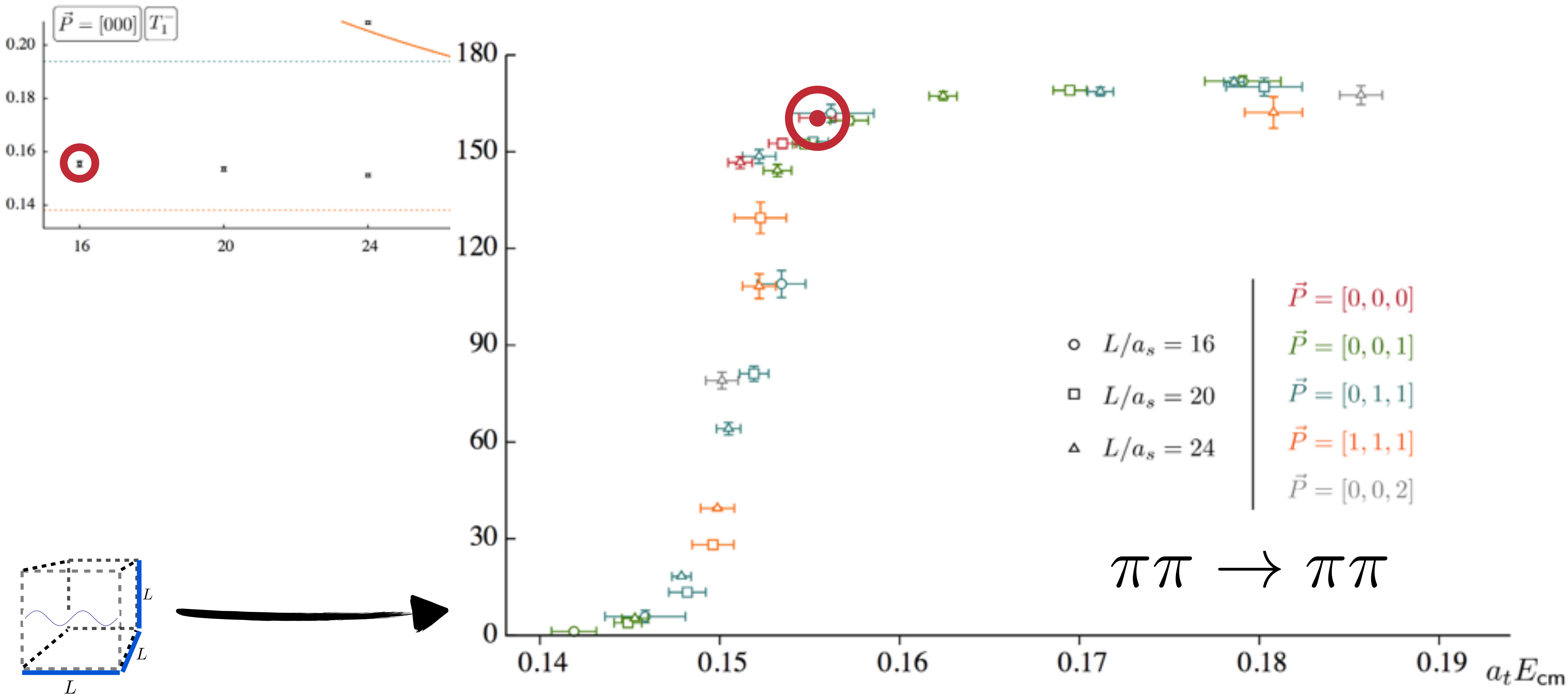

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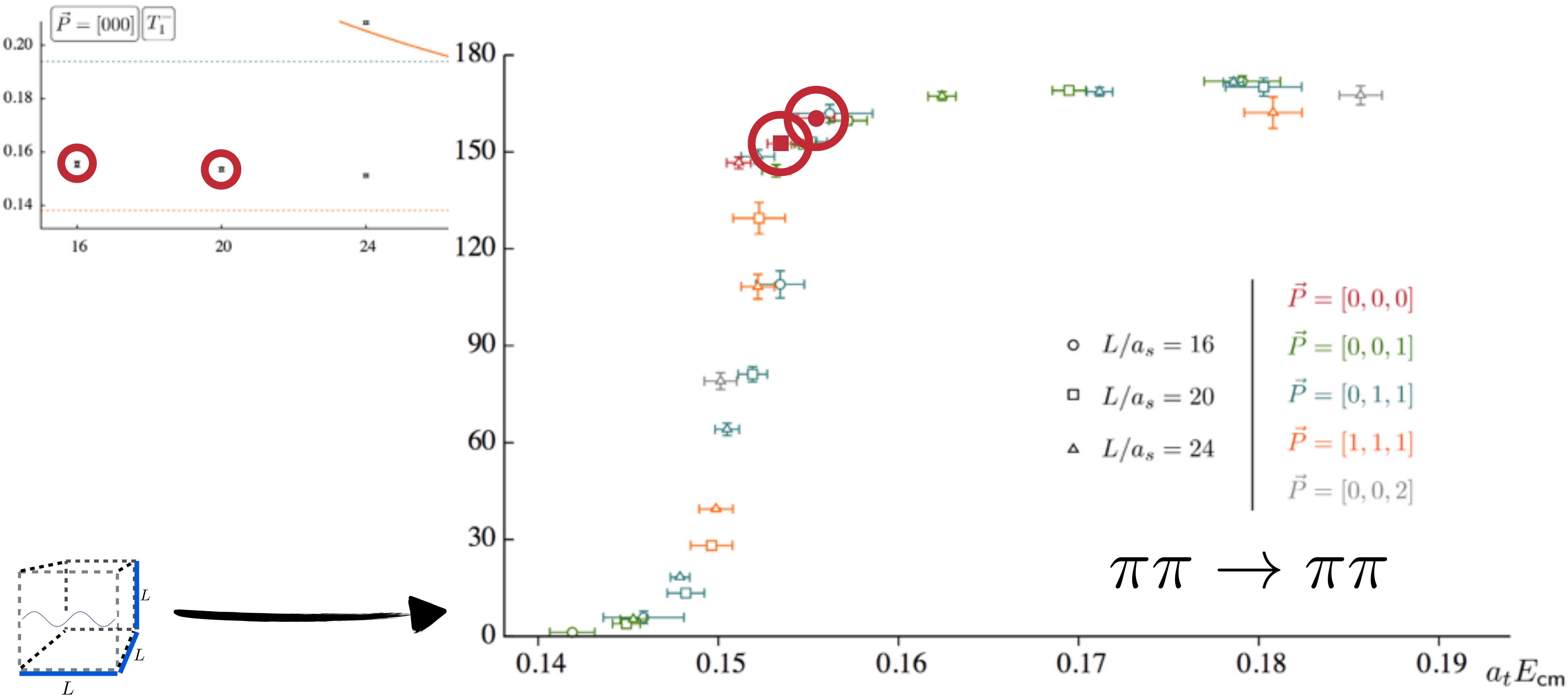

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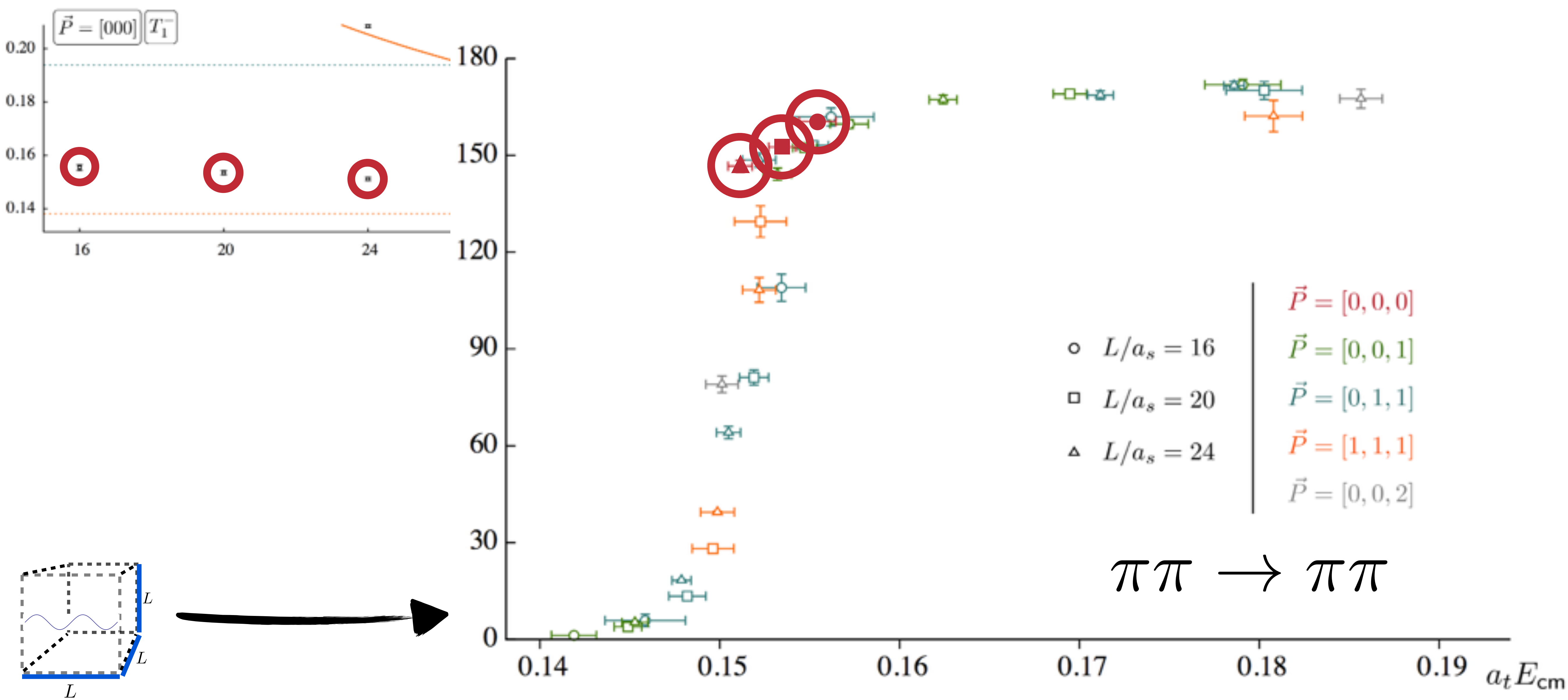

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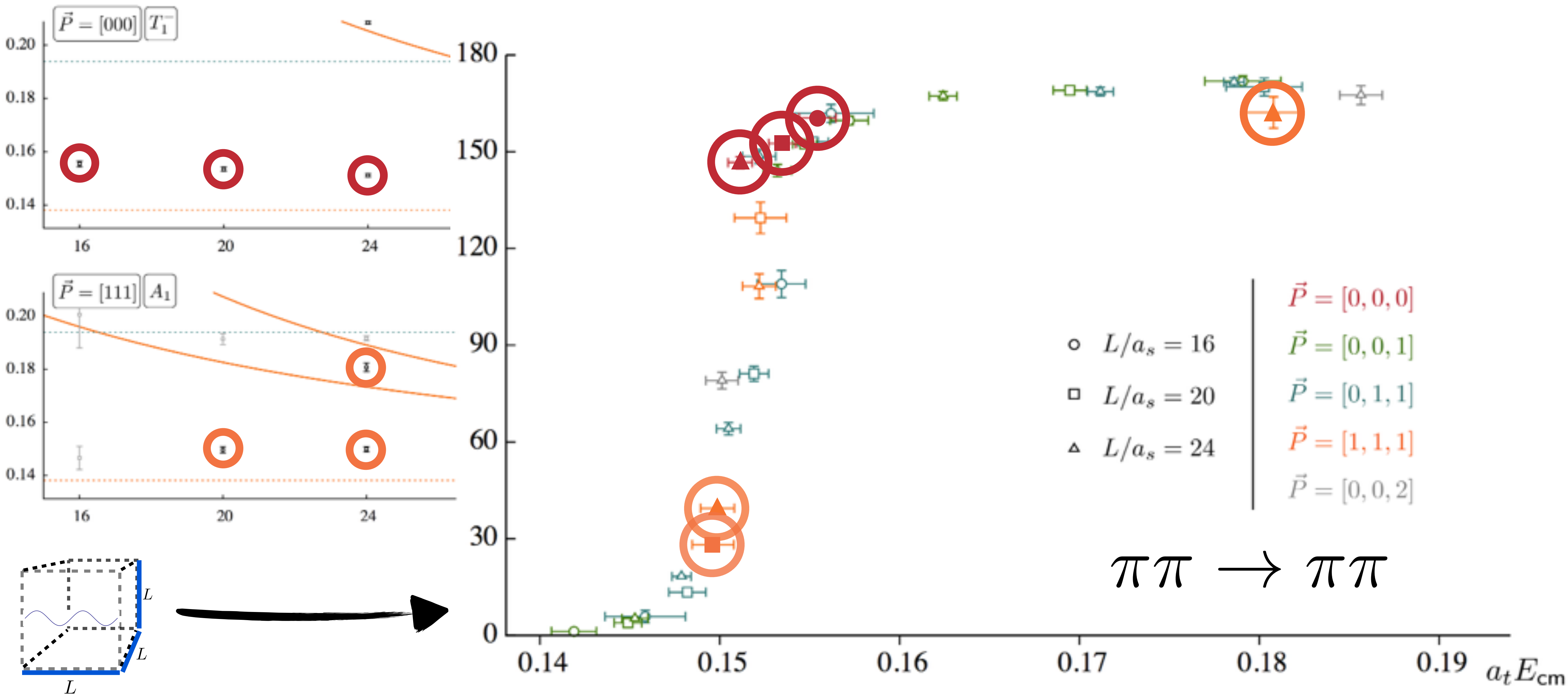

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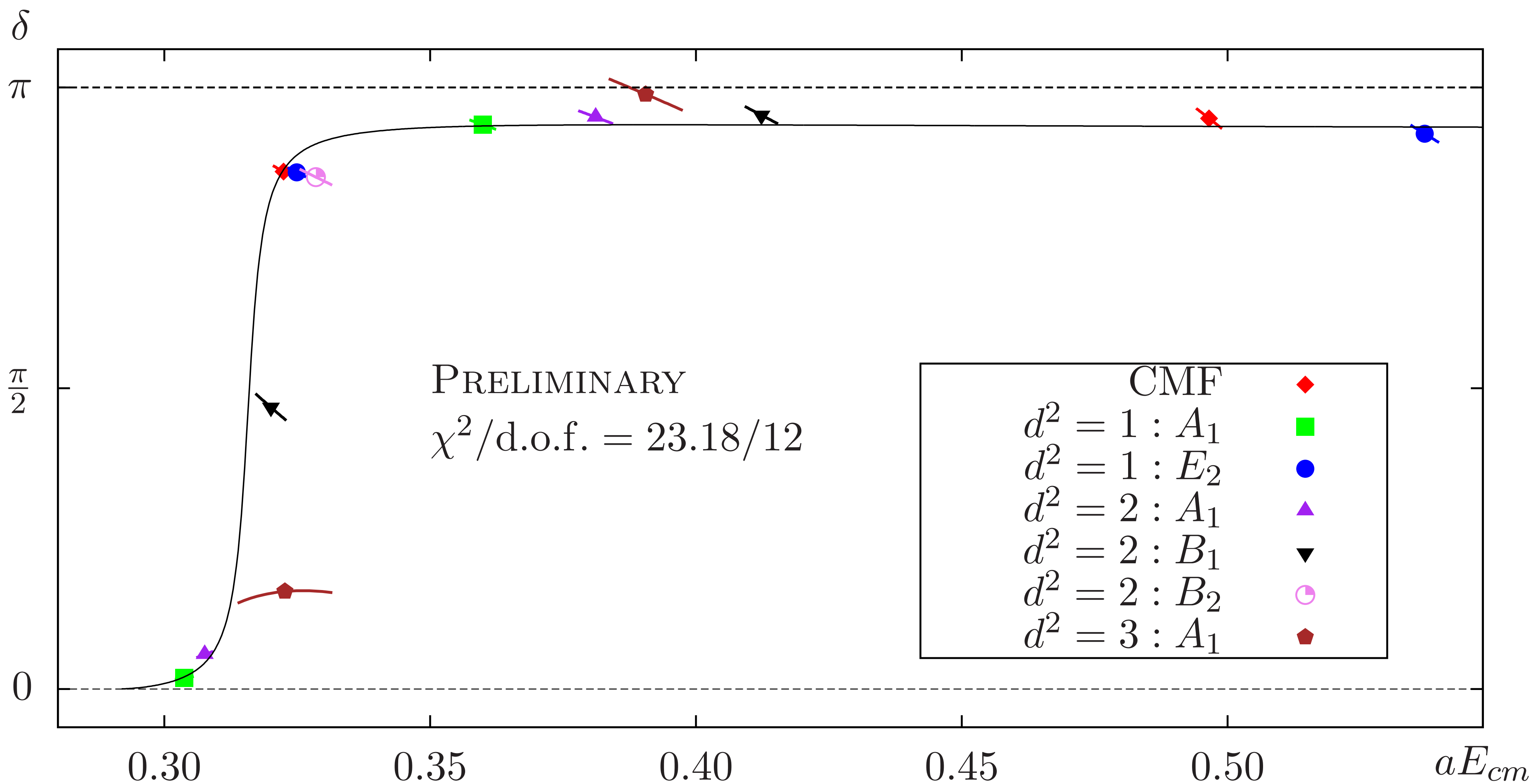
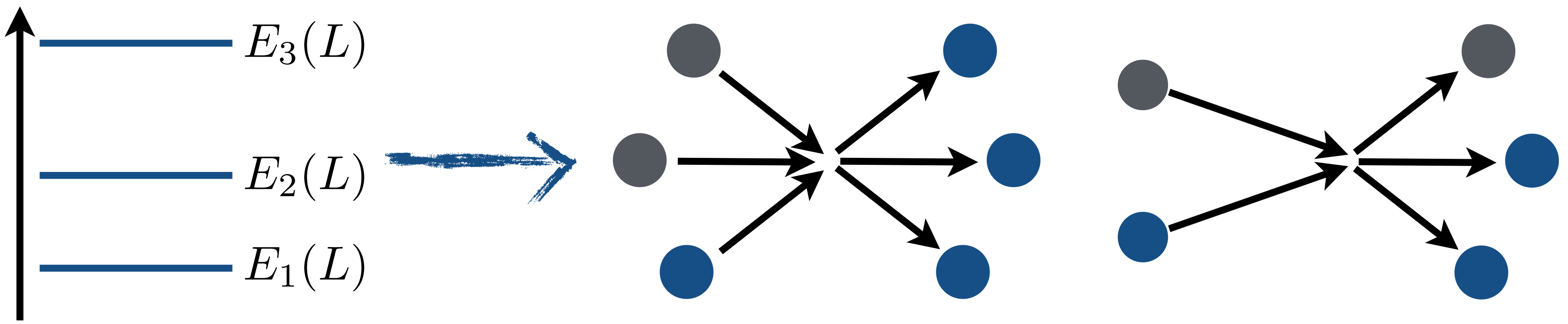
Many groups have looked this…
including ongoing work here in Mainz



F. Erben, J. Green, D. Mohler, H. Wittig, Lattice 2016 proceedings, 1611.06805

Our aim is to derive the generalization for arbitrary two- and three-particle systems



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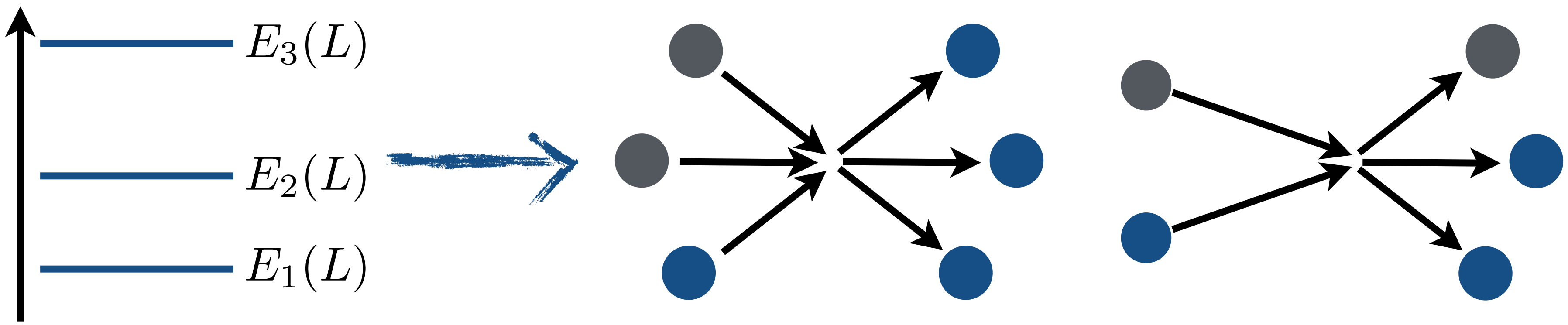
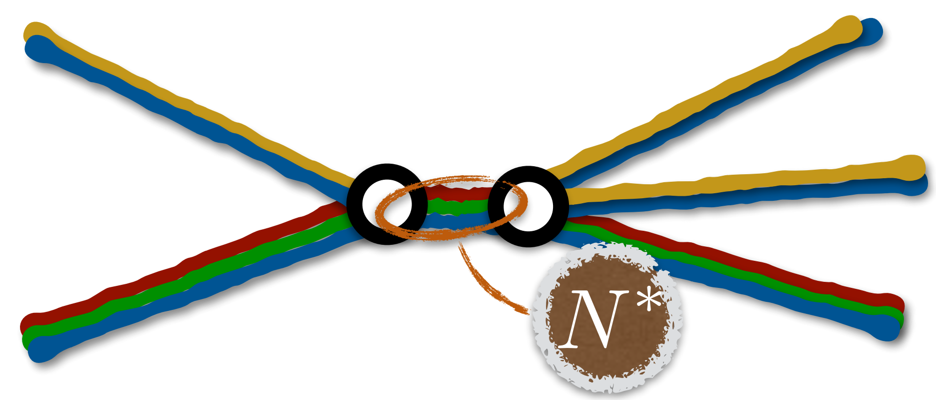


Potential applications…

Studying three-particle resonances

$$\omega(782) \to \pi\pi\pi$$

$$N(1440) \to N\pi, N\pi\pi$$

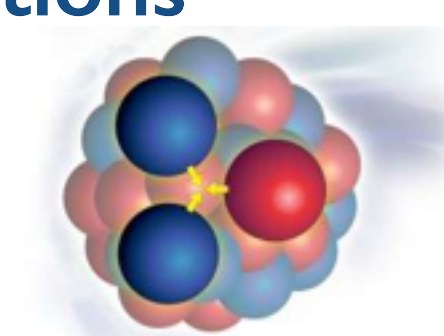


Calculating weak decay amplitudes and form factors

$$K \to \pi\pi\pi$$

Determining three-body interactions

NNN three-body forces needed as EFT input for studying larger nuclei and nuclear matter



Status of our three-particle formalism

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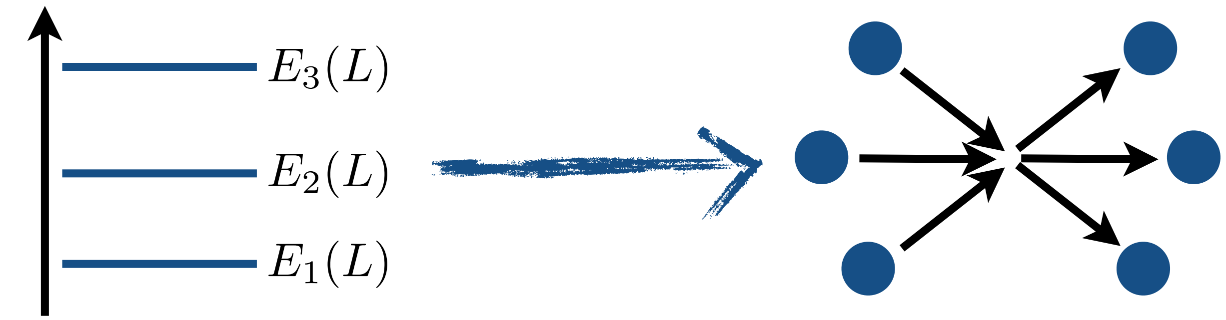
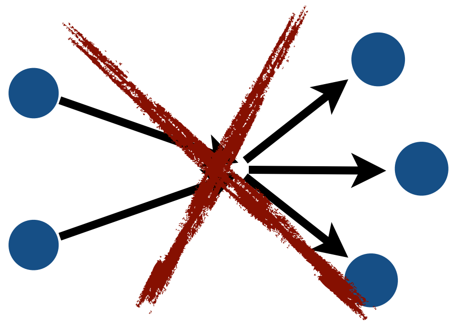
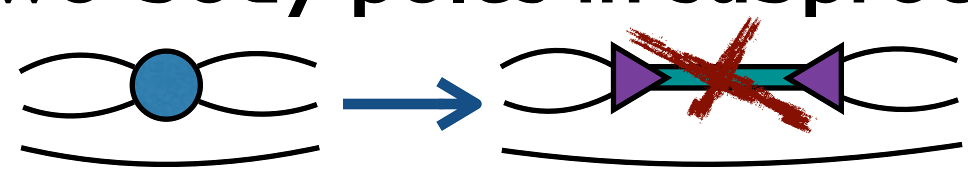


Complete for identical scalar particles with…

No two-to-three coupling



No two-body poles in subprocesses



K matrix poles generate finite-volume effects that need to be accommodated

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)
MTH and Sharpe, *Phys. Rev.* D92, 114509 (2015)

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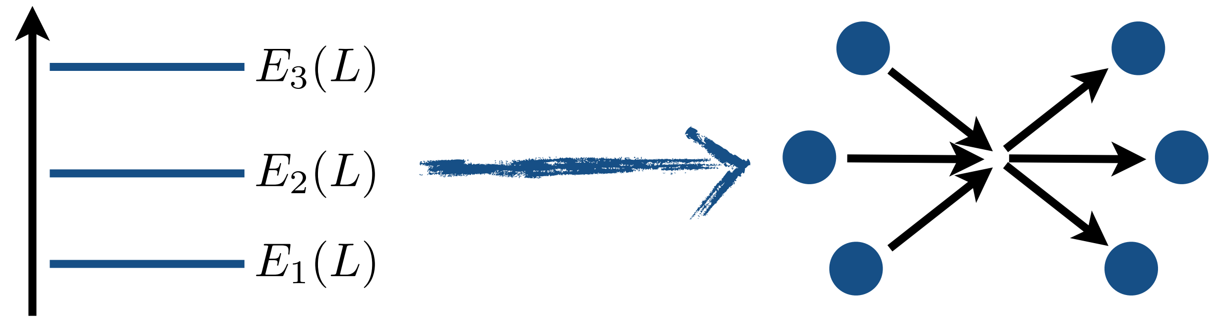

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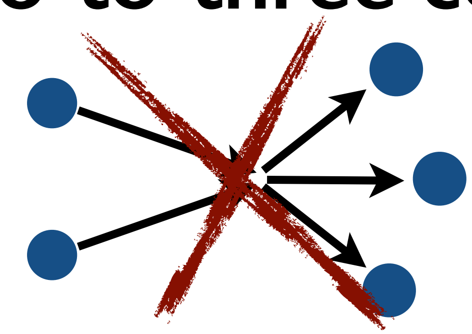

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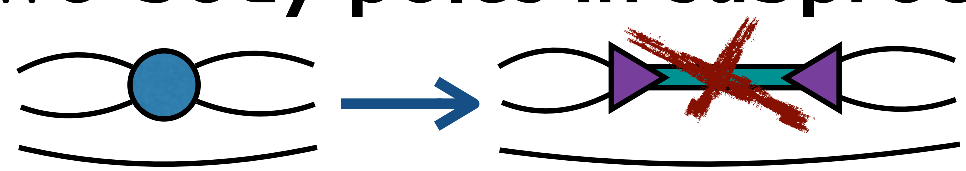

K matrix poles generate finite-volume effects that need to be accommodated

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)
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Total energy such that only three-particle states are on shell

$$3M_\pi < E^* < 5M_\pi$$

Ignores (drops) exponentially suppressed corrections $e^{-M_\pi L}$

Only useful if one truncates angular momentum space

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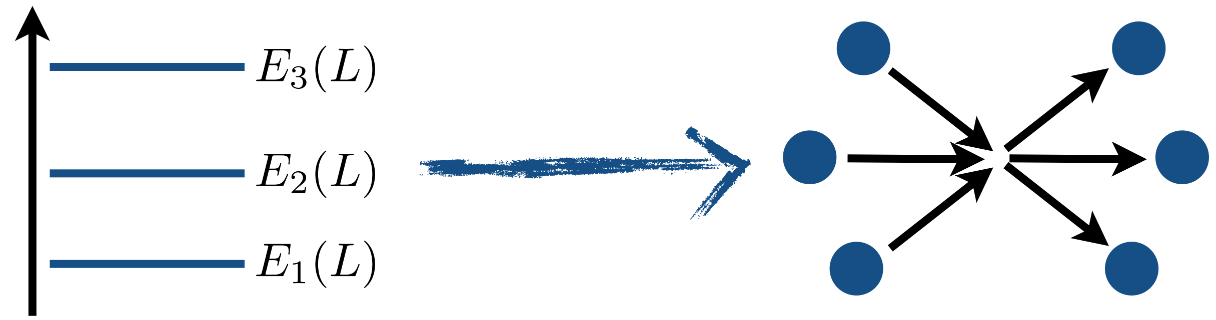

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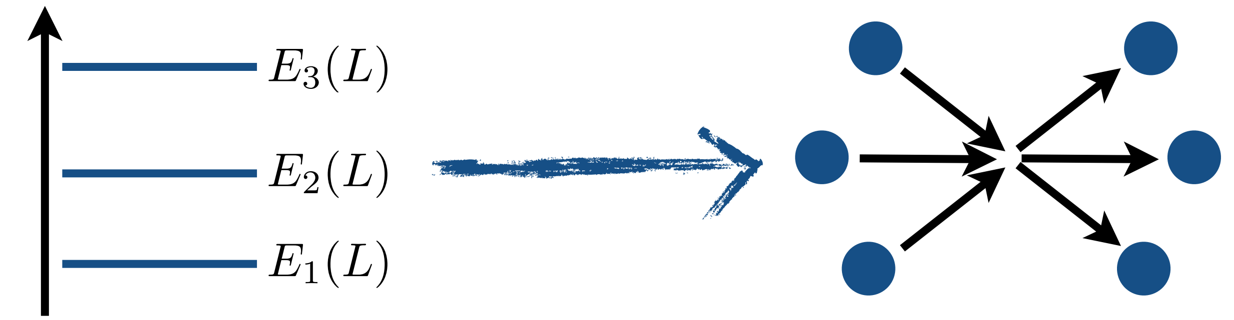

Result takes the form of a new quantization condition

$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(E_n^*) + F_3(E_n, \vec{P}, L)\right] = 0$$

infinite-volume three-to-three scattering quantity

new object built from geometric functions and two-to-two scattering

matrices in a new space, appropriate for three particles

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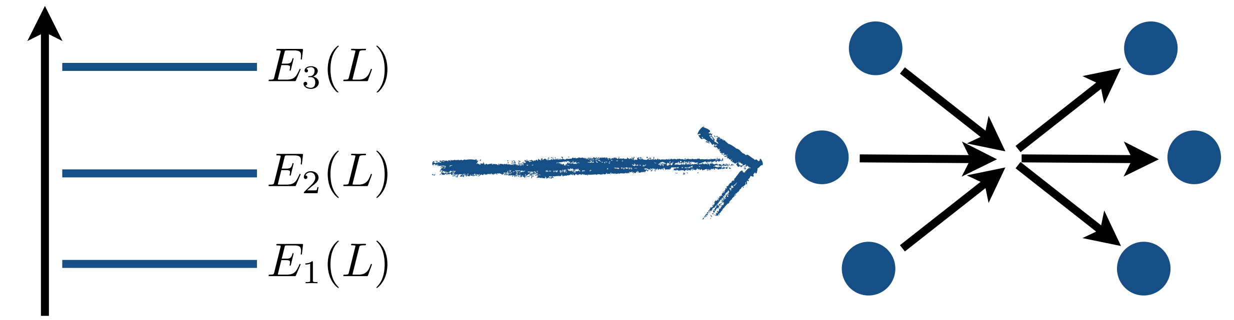

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$\mathcal{K}_{\mathrm{df},3}$ is a non standard object, similar to a K-matrix

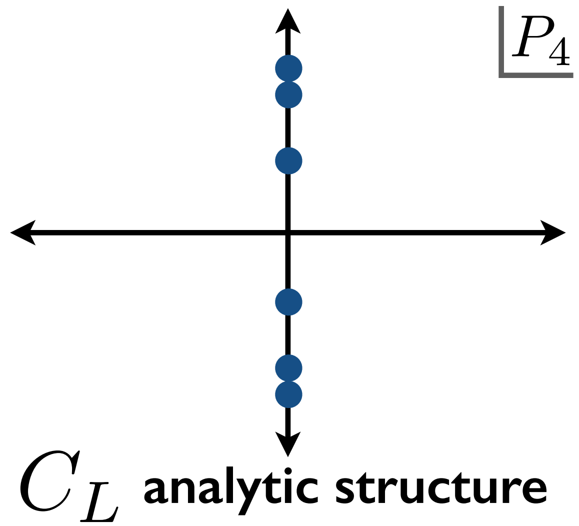
A known integral equation relates this modified K-matrix to the standard scattering amplitude

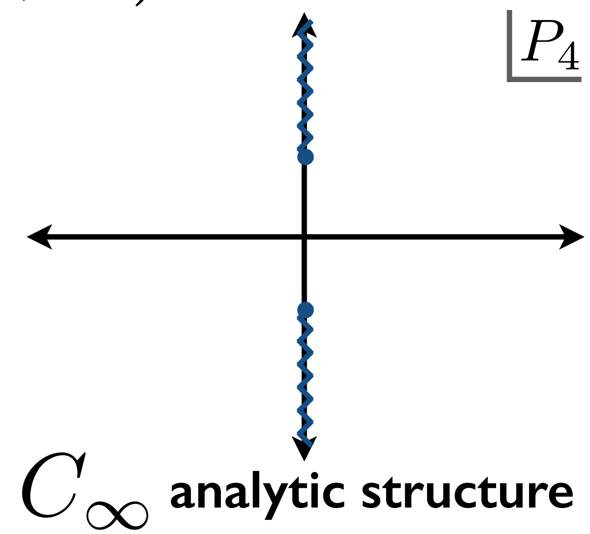
$$\mathcal{M}_{3\to3}(E^*) = \mathcal{IE}[\mathcal{K}_{\mathrm{df},3}(E^*)]$$

Sketch of the derivation

$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(E_n^*) + F_3(E_n, \vec{P}, L)\right] = 0$$

1. **Define three-to-three correlator $C_L(E, \vec{P})$ (poles at finite-volume energies)**





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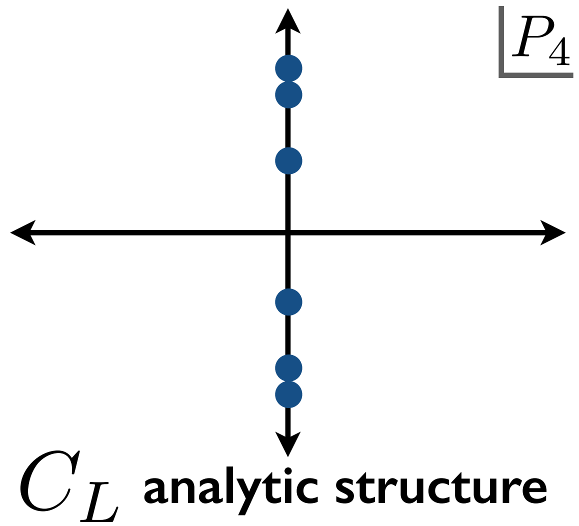
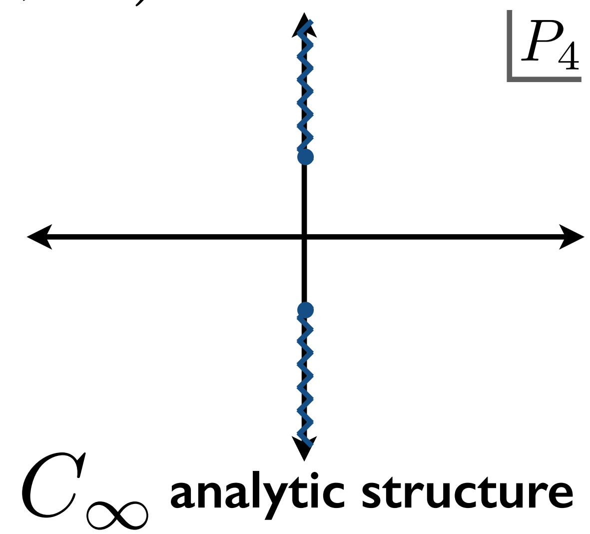
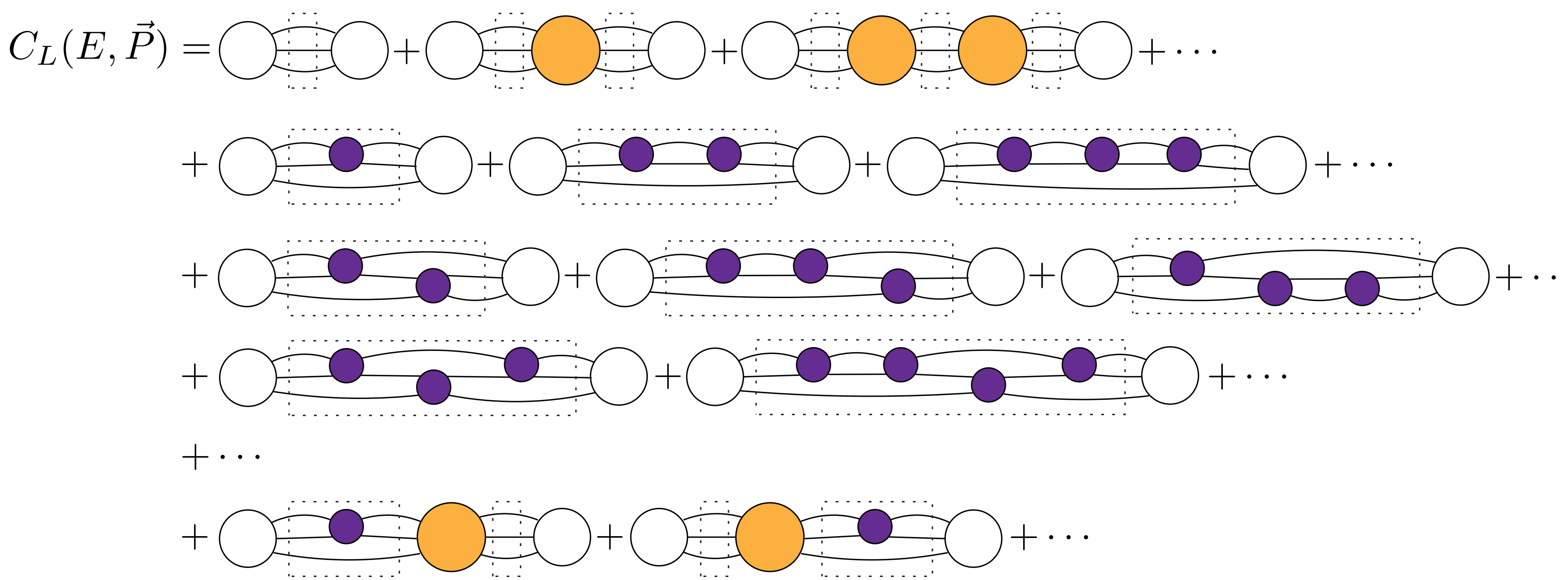




2. **Show that all** $1/L^n$ **finite-volume effects are captured by a skeleton expansion**



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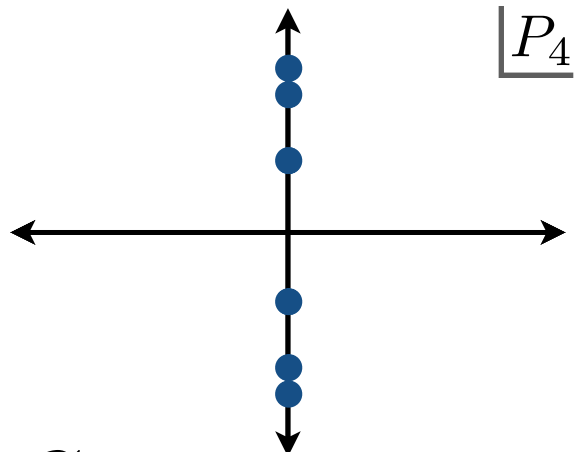
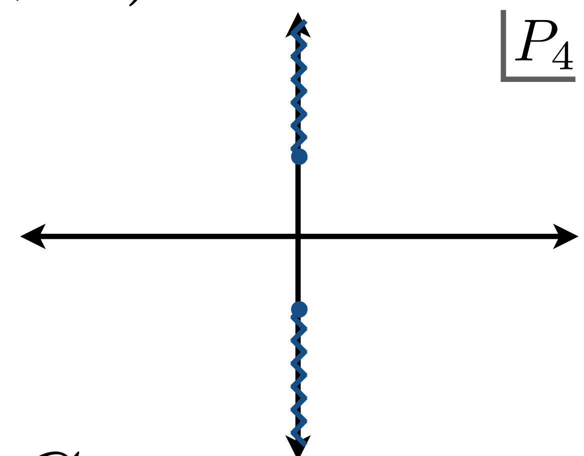


C_L **analytic structure**



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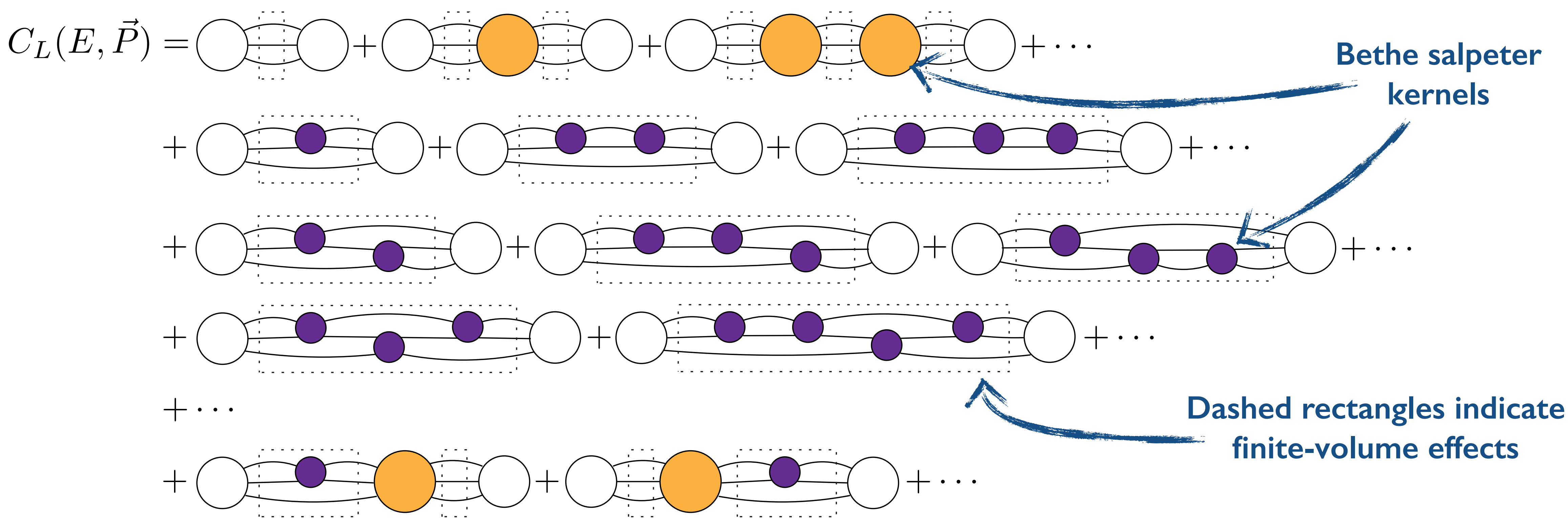

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3. Break all diagrams into finite- and infinite-volume parts and sum

$$C_L = C_\infty - A' F_3 \frac{1}{1 + \mathcal{K}_{\mathrm{df},3} F_3} A$$

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MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

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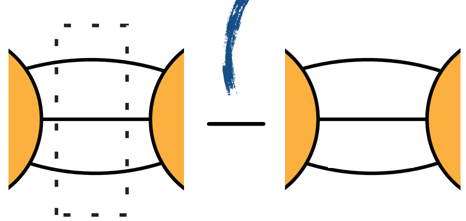

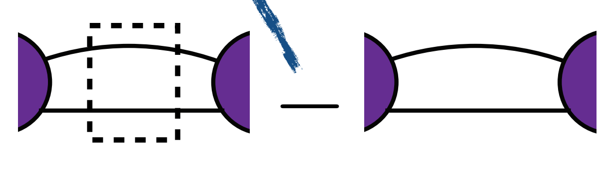

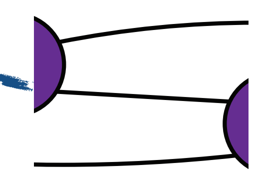

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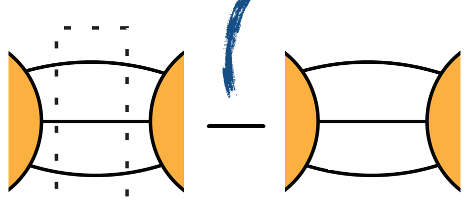

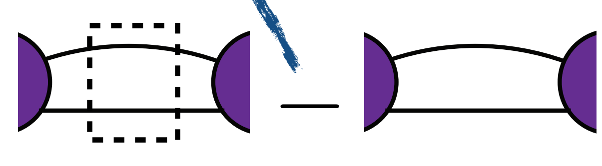

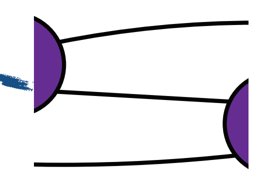

These are all matrices with indices

momentum of one particle $\otimes$ **angular momentum of the other two**

$$\vec{k} = \frac{2\pi\vec{n}}{L} \qquad \ell, m$$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

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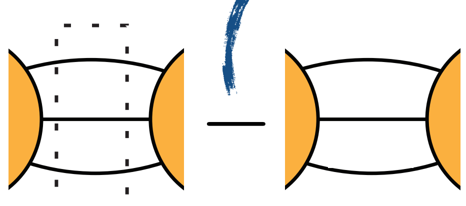
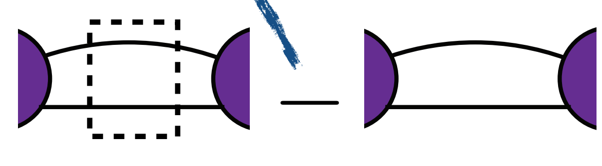
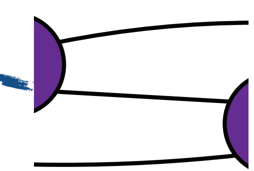






These are all matrices with indices

momentum of one particle $\vec{k} = \frac{2\pi\vec{n}}{L}$ $\otimes$ angular momentum of the other two ℓ, m

F and G are geometric functions

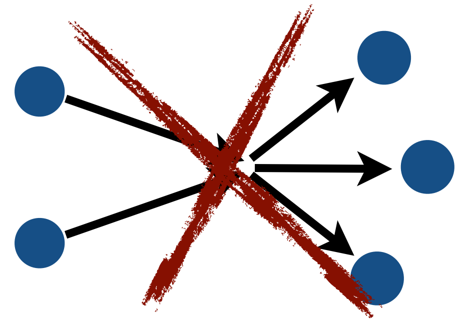
$\mathcal{M}_{2,L}$ depends on F and $\mathcal{M}_2$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

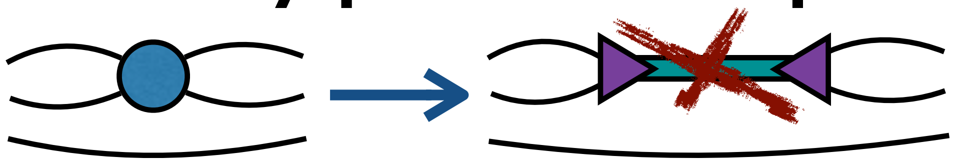
Status of our three-particle formalism

Complete for identical scalar particles with…

No two-to-three coupling



No two-body poles in subprocesses



K matrix poles generate finite-volume effects that need to be accommodated

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)
MTH and Sharpe, *Phys. Rev.* D92, 114509 (2015)

Our first goal is to lift these two restrictions

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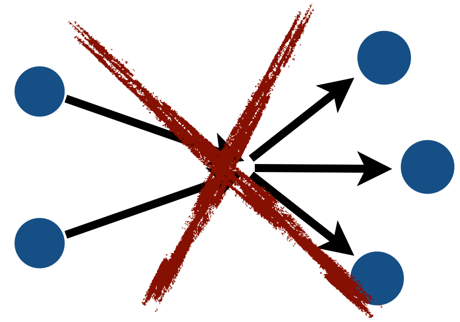

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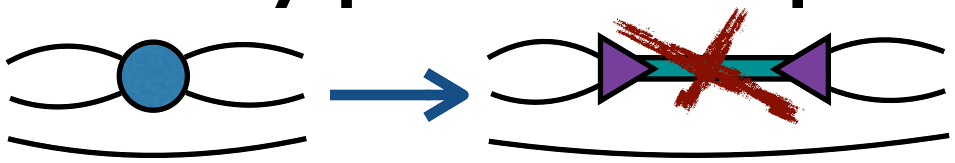

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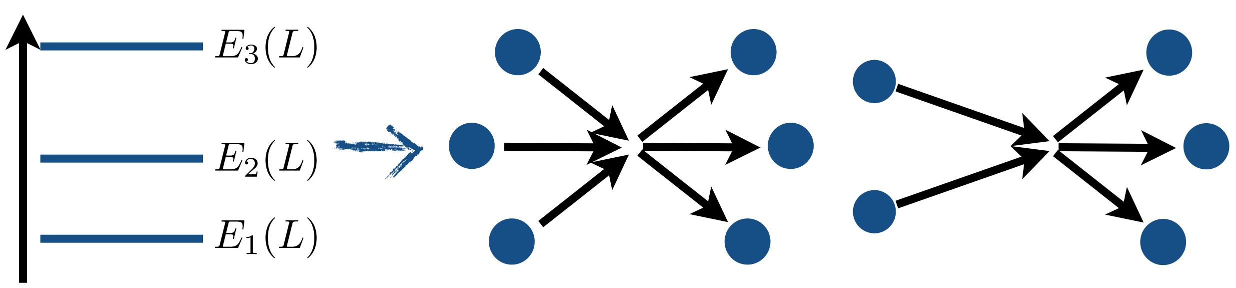

Inclusion of two-to-three coupling is complete, write-up to appear soon (with **Raùl Briceño** and **Steve Sharpe**)

The result is (yet another) quantization condition

$$\det\left[\begin{pmatrix} \mathcal{K}_2 & \mathcal{K}_{23} \\ \mathcal{K}_{32} & \mathcal{K}_{\mathrm{df},3} \end{pmatrix}^{-1} + \begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}\right] = 0$$

(Inclusion of two-body poles, spin and coupled-channels, in early stages)

Looking forward…

(1) Complete the formalism for any system of coupled two and three particle channels (any flavor, spin, any number of channels)

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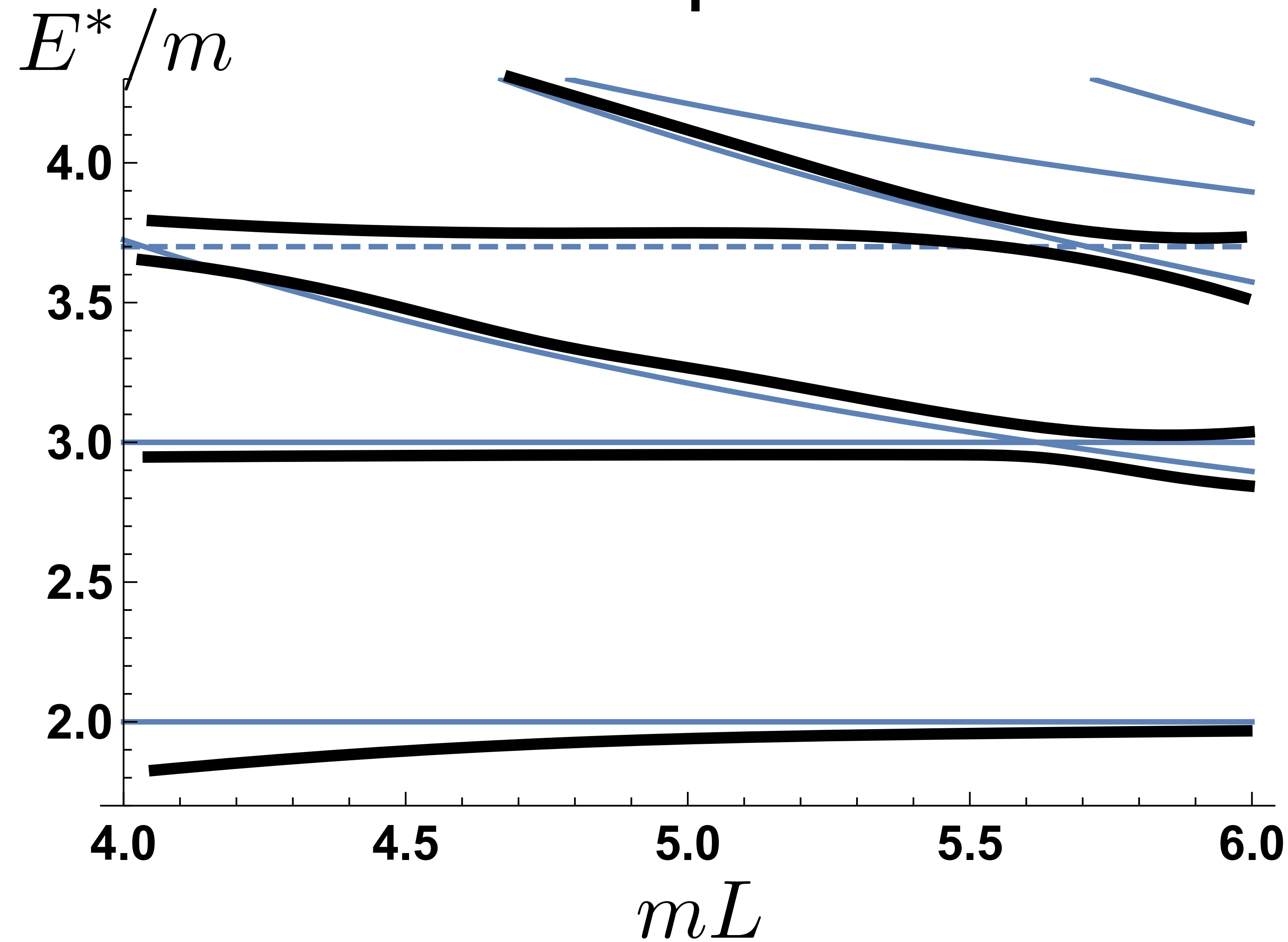
Cartoon spectrum…



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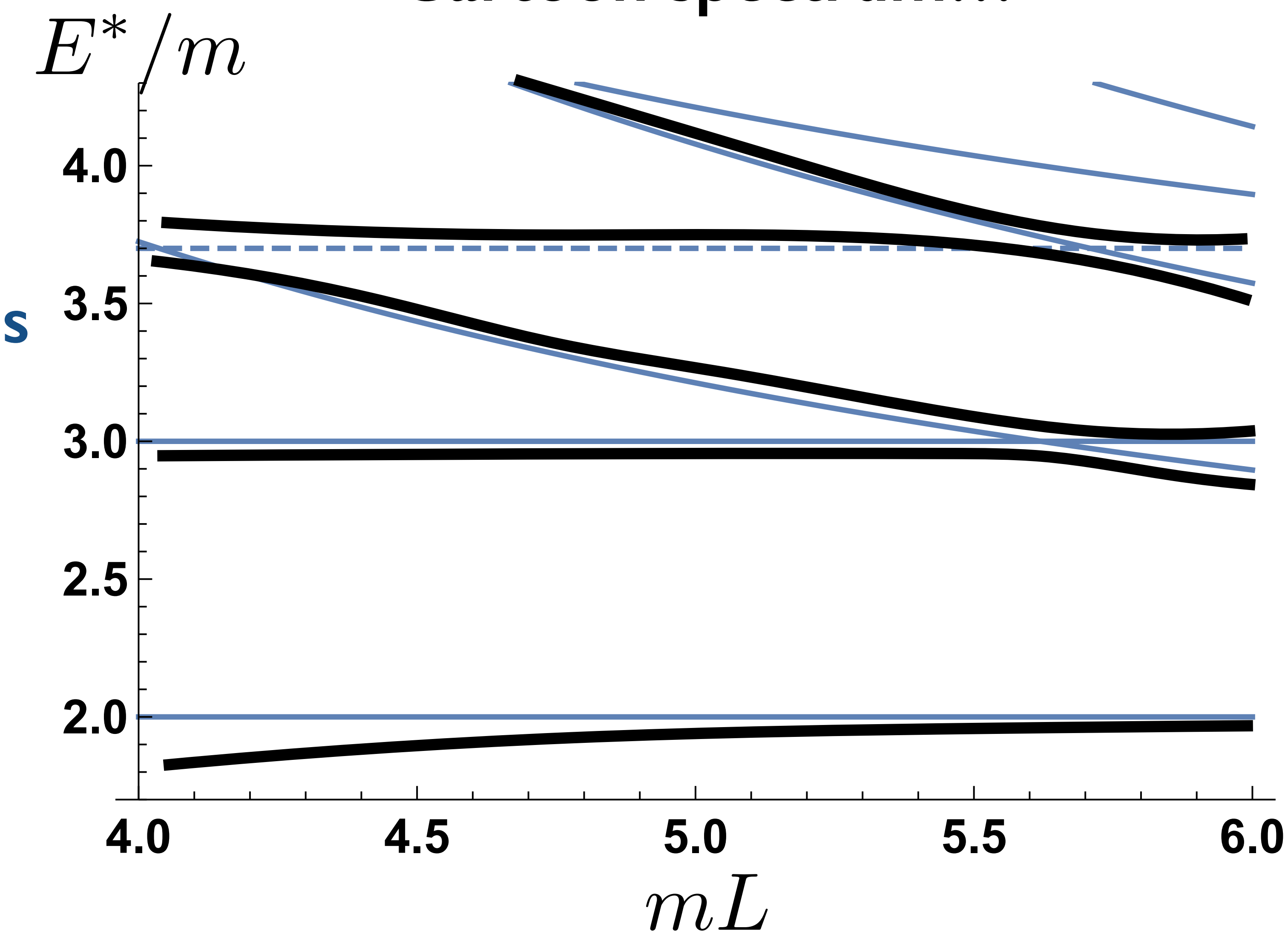
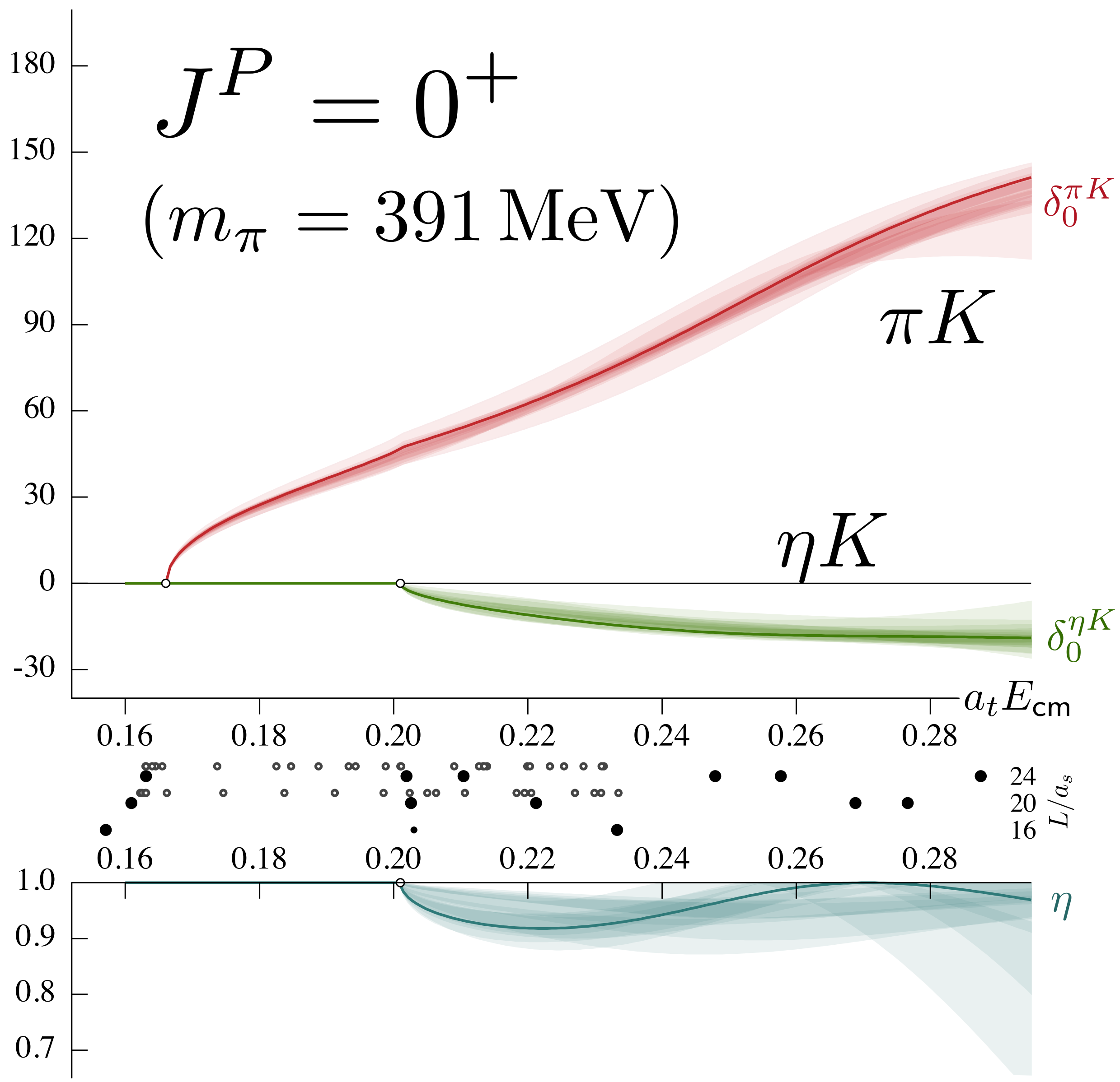


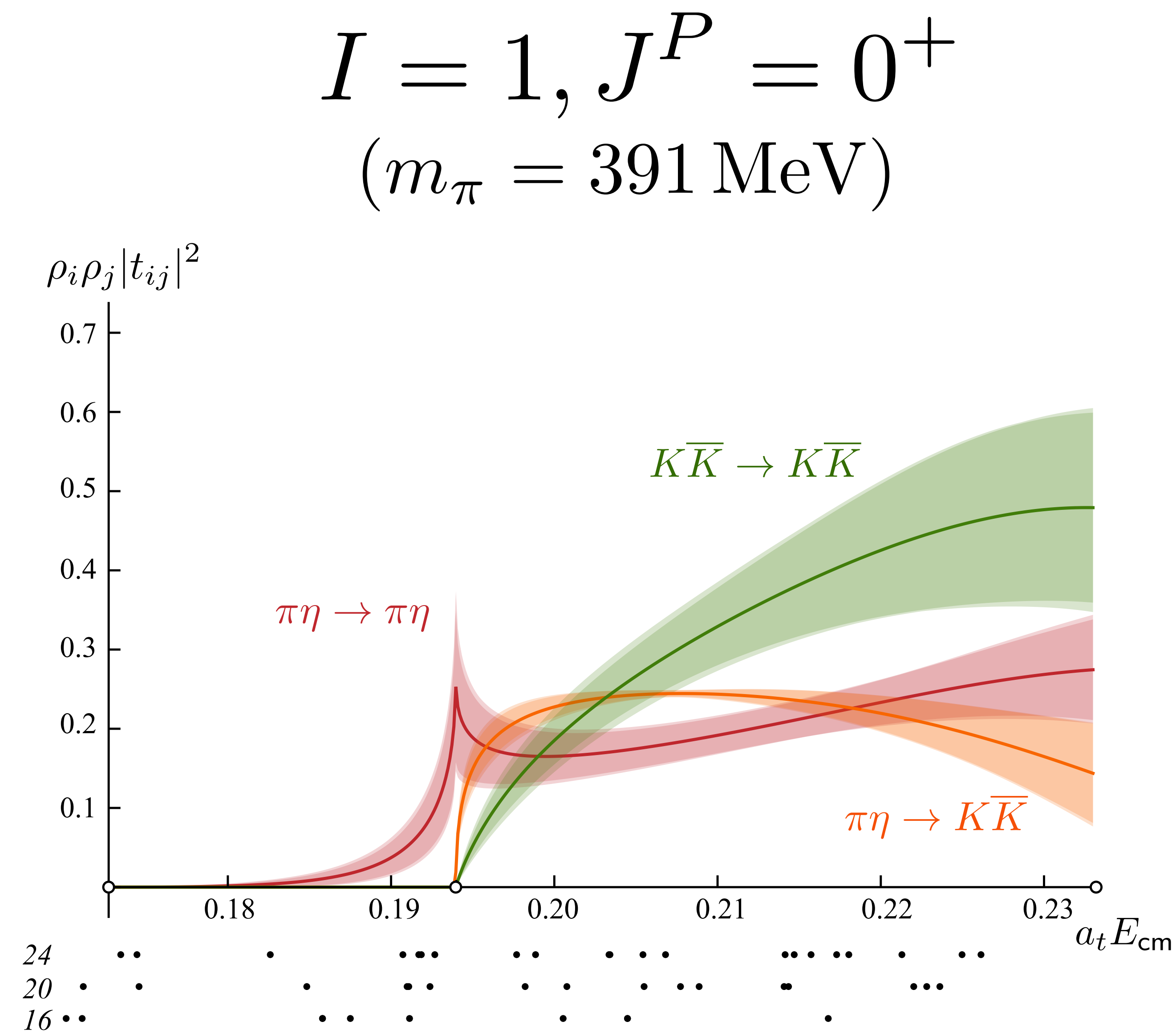
(3) **Fit** to finite-volume energies determined via LQCD to..

- rule out models
- constrain parameters
- determine scattering and resonance properties

This program as already seen great success with coupled two-particle channels



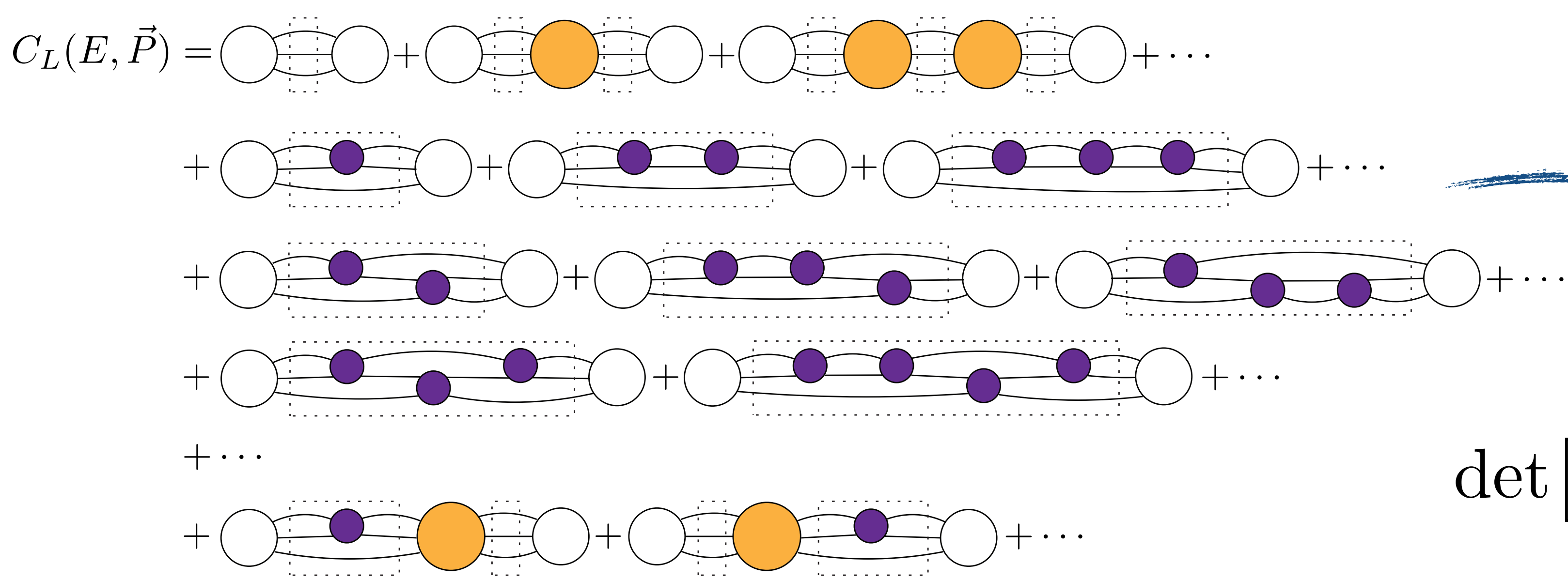
Wilson, Dudek, Edwards, Thomas, *Phys. Rev.* D 91, 054008 (2015)



Dudek, Edwards, Wilson *Phys. Rev.* D 93, 094506 (2016)

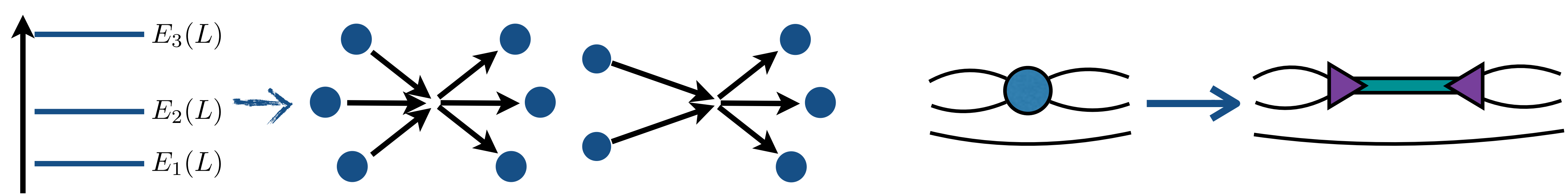
Summary and Conclusions

Formalism is complete for the simplest three-particle system



$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(E_n^*) + F_3(E_n, \vec{P}, L)\right] = 0$$

Extension to fully general systems is underway



Stay tuned for three-particle scattering and resonances from LQCD!

Testing the formalism

$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(E_n^*) + F_3(E_n, \vec{P}, L)\right] = 0$$

Result and derivation are complicated, so it is important to provide checks to demonstrate that result is correct (and useful)

(1) 1/L expansion $E = 3m + \mathcal{O}(1/L^3)$

Expanding our result about the non-interacting threshold energy, we find…

MTH and Sharpe, *Phys. Rev.* D 93, 096006 (2016)

$1/L^3, 1/L^4, 1/L^5$ **match NRQM predictions**

Huang and Yang (1957), Beane, Detmold, and Savage, *Phys. Rev.* D76 074507 (2007), Tan, *Phys. Rev.* A78, 013636 (2008)

$1/L^3 - 1/L^6$ **match prediction of relativistic ϕ^4 theory through order λ^3.**

MTH and Sharpe, *Phys. Rev.* D 93, 014506 (2016)

(2) Unitary three-particle boundstate

We reproduce and extend the NRQM prediction

Meißner, Rìos and Rusetsky, *Phys. Rev. Lett.* 114, 091602 (2015) + erratum

NRQM prediction

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The infinite-volume boundstate energy, $E_B \equiv 3m - \dfrac{\kappa^2}{m}$
is shifted in finite volume by an amount

$$\Delta E(L) = c|A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \cdots$$

$c = -96.351\cdots$ geometric constant from Effimov wavefunction

$|A|$: normalization correction factor (close to one)

Assumes two-body potential, unitary limit, P=0, s-wave only

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Our formalism gives a general relation between scattering amplitudes and energy levels. So we substitute…

$$\mathcal{M}_3 \sim -\frac{\Gamma\,\overline{\Gamma}}{E^2 - E_B^2} \qquad \mathcal{M}_2 = -\frac{16\pi E_2^*}{ip^*}$$

and study the lowest three-particle finite-volume level

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We aim to reproduce the exponent, leading power and overall constant using our relativistic formalism

Reproducing the result…

1. **Show that the relativistic quantization predicts (at leading order in 1/L)**

$$\Delta E(L) = -\frac{1}{2E_B}\left[\frac{1}{L^3}\sum_{\vec{k}} - \int_{\vec{k}}\right]\frac{\overline{\Gamma}^{(u)}(k)\Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(k)}$$

MTH and Sharpe, arXiv:1609.04317, (2016)

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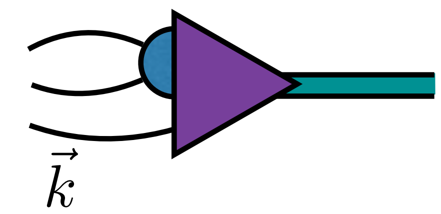
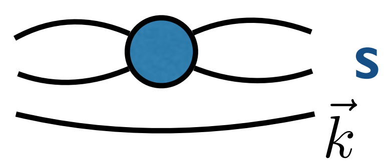
usymmetrized residue factor





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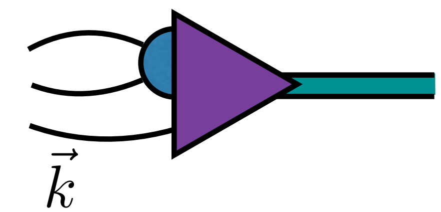

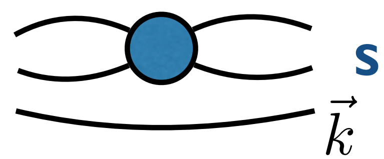

s-wave scattering amplitude

2. **Derive the functional forms of the infinite-volume quantities**

$$\Gamma^{(u)}(k) = \frac{3^{3/8}\pi^{1/4}}{4} A\sqrt{-c}\,\mathcal{M}_2(k) \qquad \mathcal{M}_2(k) = \frac{32\pi m}{\kappa}\left[1 + \frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

follows from matching to Effimov wavefunction

unitary amplitude with spectator "stealing" some momentum

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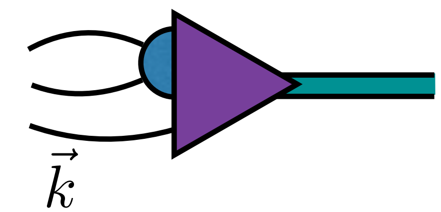

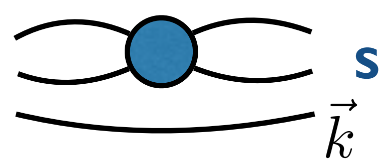

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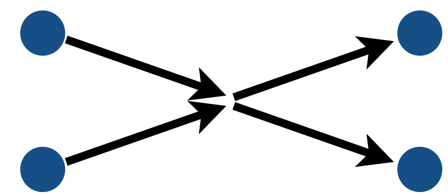
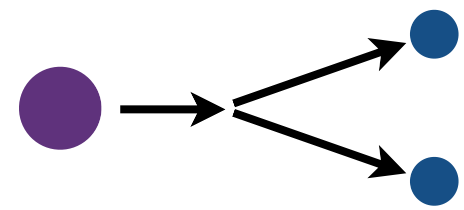
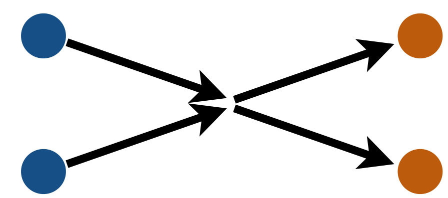
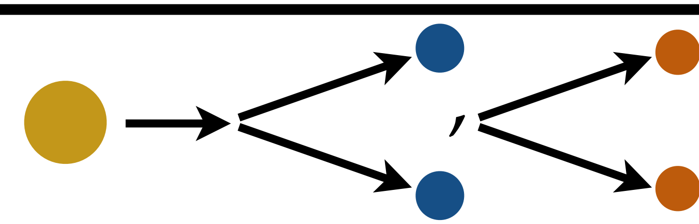
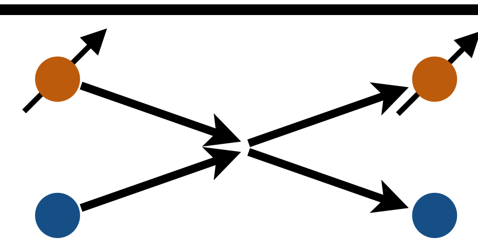
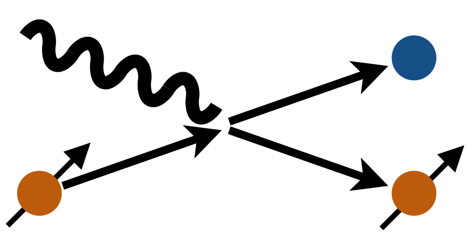
3. Evaluate the sum-integral difference with Poisson summation

$$\Delta E(L) = c|A|^2\frac{3^{3/4}\pi^{3/2}}{3\kappa}6\int_{\vec{k}} e^{iL\hat{x}\cdot\vec{k}}\frac{1}{2\omega_k}\left[1+\frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

$$= c|A|^2\frac{\kappa^2}{m}\frac{1}{(\kappa L)^{3/2}}e^{-2\kappa L/\sqrt{3}} + \cdots$$

MTH and Sharpe, arXiv:1609.04317, (2016)

Status of multi-hadron matrix elements in LQCD…

physical system		Method to get it from LQCD
elastic scattering of identical scalars		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)
decay into identical scalars (no other open decay channels)		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005), Christ, Kim and Yamazaki (2005)
non-identical scalars, multiple coupled channels*		Bernard et al. (2011), Fu (2012), Briceño and Davoudi (2012)
decay into multiple, coupled two-particle channels*		MTH and Sharpe (2012)
scattering of particles with intrinsic spin*		Detmold and Savage (2004) Göckeler et al. (2012) Briceño (2014)
particle production mediated by a generic local current*		Meyer (2011), Bernard et al. (2012), A. Agadjanov et al. (2014), Briceño, MTH and Walker-Loud (2014) Briceño and MTH (2015)

***(assumes no three or four-particle channels open)**